Marsden & Tromba VC

Ch 3.5

5/20/2003

(8) Frequent Calculations with $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $r = \sqrt{x^2+y^2+z^2} =: \rho^{1/2}$

<u>Lemma</u> $\frac{\partial}{\partial x_i} r = \frac{x_i}{r}$

Just show it for $\frac{\partial}{\partial x} r = \frac{1}{2}\rho^{-1/2}\frac{\partial \rho}{\partial x} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{\rho^{1/2}} = \frac{x}{\rho^{1/2}} = \frac{x}{r}$ □

(a) Show $\vec{\nabla}(r^n) = n r^{n-2}\vec{r}$ and $\nabla(\ln r) = \frac{1}{r^2}\vec{r}$

The special case that we care about is $n=-1$: $\boxed{\nabla(1/r) = \frac{-1}{r^3}\vec{r} \quad \text{for } r \neq 0}$

$\nabla(r^n) = \begin{bmatrix} \frac{\partial}{\partial x} r^n \\ \frac{\partial}{\partial y} r^n \\ \frac{\partial}{\partial z} r^n \end{bmatrix}$ and $\frac{\partial}{\partial x} r^n = n r^{n-1}\frac{\partial r}{\partial x} \overset{\text{lemma}}{=} n r^{n-1}\frac{x}{r} = n r^{n-2} x$

Thus we see that the answer is: $n r^{n-2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = n r^{n-2}\vec{r}$ □

▷ $\nabla(\ln r) = ?$ $\quad D_x(\ln r) = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\frac{x}{r} = \frac{x}{r^2}$

$\Rightarrow \nabla(\ln r) = \frac{1}{r^2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{r^2}\vec{r}$ □

(b) Show $\nabla^2(r^n) = n(n+1)r^{n-2}$ $\qquad \boxed{\text{Important Special case: } \nabla^2(1/r) \overset{n=-1}{=} 0}$

$$\nabla^2(r^n) = D_x^2 r^n + D_y^2 r^n + D_z^2 r^n$$

$$\frac{\partial}{\partial x} r^n = n r^{n-2} x$$

then $D_x^2 r^n = \frac{\partial}{\partial x}(n r^{n-2} x) = n(n-2) r^{n-3}\frac{\partial}{\partial x} r \cdot x + n r^{n-2}\cdot 1$

$$= n(n-2) r^{n-4} x^2 + n r^{n-2}$$

$$\Rightarrow \sum_{i=1}^{3} D_{x_i}^2 r^n = n(n-2) r^{n-4}\underbrace{(x^2+y^2+z^2)}_{r^2} + 3n r^{n-2}$$

$$= (n(n-2)+3n) r^{n-2}$$

$$= n(n+1) r^{n-2}$$ □

(c) Show $\nabla\cdot(r^n\vec{r}) = (n+3)r^n$ $\qquad \boxed{\text{Important Special case: } \nabla\cdot(\tfrac{1}{r^3}\vec{r}) \overset{n=-3}{=} 0}$

$$\nabla\cdot\Big(\underbrace{r^n\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{f}\Big) = D_x f^{(1)} + D_y f^{(2)} + D_z f^{(3)}$$

$$\frac{\partial}{\partial x}(r^n x) = n r^{n-1}\frac{x^2}{r} + r^n$$

Thus it is plain to see:

$\nabla\cdot(r^n\vec{r}) = (n r^{n-2}x^2 + r^n) + (n r^{n-2}y^2 + r^n) + (n r^{n-2}z^2 + r^n)$

$= n r^{n-2}(x^2+y^2+z^2) + 3r^n$

$= n r^n + 3r^n \quad = (n+3)r^n$ ▨ QED