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The real Lemma is
$$C(a\times b) = \frac{1}{\det(C^{-1})^T}\left[(C^{-T}a)\times(C^{-T}b)\right] = (\det C)\left[(C^{-1})^T a \times (C^{-1})^T b\right]$$

want to show: $\boxed{Ma \times Mb = (\det M)(M^{-1})^T(a\times b)}$

$$M^{-1} = \frac{1}{\det M} M_{cof}$$

where
$$M_{cof} = \begin{bmatrix} |M|_{11} & |M|_{21} & |M|_{31} \\ |M|_{12} & |M|_{22} & |M|_{32} \\ |M|_{13} & |M|_{23} & |M|_{33} \end{bmatrix}$$

$$= \cancel{(\det M)}\frac{1}{\cancel{\det M}} M_{cof}^T(a\times b)$$

$$(m_1 a_1 + m_2 a_2 + m_3 a_3)\times(m_1 b_1 + m_2 b^2 + m_3 b^3)$$

$$= \cancel{a^1b^2(m_1\times m_2)} + a^1b^3\cancel{m_1}\times m_3 +$$
$$\cancel{a^2b^1(m_2\times m_1)} + a^2b^3(m_2\times m_3) +$$
$$a^3b^1(\cancel{m_3}\times m_1) + a^3b^2(m_3\times m_2)$$

$$= \underbrace{(a^1b^2 - a^2b^1)}_{z^3}(\vec{m}_1\times\vec{m}_2) + \underbrace{(a^1b^3 - a^3b^1)}_{-z^2}(\vec{m}_1\times\vec{m}_3) + \underbrace{(a^2b^3 - a^3b^2)}_{z^1}(m_2\times m_3)$$

Note
$$a\times b = \begin{vmatrix} i & j & k \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = \begin{bmatrix} a^2b^3 - a^3b^2 \\ -(a^1b^3 - a^3b^1) \\ (a^1b^2 - a^2b^1) \end{bmatrix}$$

$$\vec{m}_1\times\vec{m}_2 =$$

Let

$$\vec{m}_1 = \begin{bmatrix} m^1 \\ m^2 \\ m^3 \end{bmatrix} \quad \vec{m}_2 = \begin{bmatrix} n^1 \\ n^2 \\ n^3 \end{bmatrix} \quad \vec{m}_3 = \begin{bmatrix} p^1 \\ p^2 \\ p^3 \end{bmatrix}$$

$$a_{13} = a_3^1$$

$$\begin{bmatrix} | & | & | \\ m_1 & m_2 & m_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{bmatrix}$$

$$\vec{m}_1\times\vec{m}_2 = \begin{vmatrix} i & j & k \\ m^1 & m^2 & m^3 \\ n^1 & n^2 & n^3 \end{vmatrix} = \begin{bmatrix} m^2n^3 - m^3n^2 \\ -(m^1n^3 - m^3n^1) \\ m^1n^2 - m^2n^1 \end{bmatrix} \begin{matrix} = M_{13} \\ = M_{23} \\ = M_{33} \end{matrix}$$

$$\vec{m}_1\times\vec{m}_3 = \begin{vmatrix} i & j & k \\ m^1 & m^2 & m^3 \\ p^1 & p^2 & p^3 \end{vmatrix} = \begin{bmatrix} m^2p^3 - m^3p^2 \\ -(m^1p^3 - m^3p^1) \\ m^1p^2 - m^2p^1 \end{bmatrix} \begin{matrix} = M_{12} \\ = M_{22} \\ = M_{32} \end{matrix}$$

$$\vec{m}_2\times\vec{m}_3 = \begin{vmatrix} i & j & k \\ n^1 & n^2 & n^3 \\ p^1 & p^2 & p^3 \end{vmatrix} = \begin{bmatrix} n^2p^3 - n^3p^2 \\ -(n^1p^3 - n^3p^1) \\ n^1p^2 - n^2p^1 \end{bmatrix} \begin{matrix} = M_{11} \\ = M_{21} \\ = M_{31} \end{matrix}$$

$$z^3\begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \end{bmatrix} + -z^2\begin{bmatrix} M_{12} \\ M_{22} \\ M_{32} \end{bmatrix} + z^1\begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix} = \underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}}_{M_{cof}^T}\begin{bmatrix} z^1 \\ -z^2 \\ z^3 \end{bmatrix} = M_{cof}^T(\vec{a}\times\vec{b})$$

$\boxed{\text{QED}}$