Pars ~~ATOAD~~ P.91
Whittaker P.2
McCauley P.223
Goldstein P.158

Kinematics: Euler's Thm, Chasles' Thm

5/25/2024

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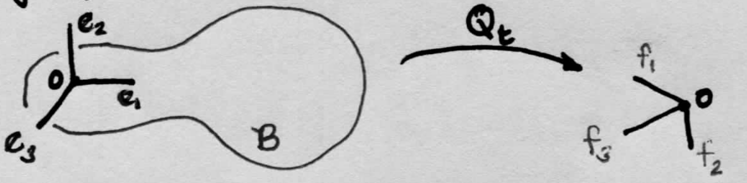
Euler's Thm :
- The resultant of any general motion of a RB with 1 pt held fixed is a rotation about some axis thru that pt, call it O.

Restated several ways
- [The initial and final positions of RB are connected by a simple rotation about an axis thru O]
- The resultant position of any motion of the RB that leaves one pt O fixed must also leave a line thru O fixed.
- Let RB $\mathcal{B}$ undergo any spinning, twisting, tumbling motion as long as one pt (origin $O \in \mathcal{B}$) remains fixed $\Rightarrow$ The final position of $\mathcal{B}$ can always be achieved from the initial position by a simple rotation about some axis thru O

Pf. Pars and Whittaker give geometrical pfs, Goldstein sort of gives a linear algebra pf. I will give a real linear algebra pf.

1. Take the FP in the body $\mathcal{B}$ to be origin O of both the fixed system axes and the body system axes. (OG triad, O.N. basis vectors for $\mathbb{R}^3$).



I won't try to draw $\mathcal{B}$, but the axes end up in some other configuration

origin is fixed, distances and angles are preserved a priori so we know this is possible to be obtained by a O.N. linear map Q. In fact, it is a change of basis $\{f_1 f_2 f_3\} = \{e_1 e_2 e_3\} Q$.
Then the co-ords are related by $v_e = Q v_f$ or $Q^T v_e = v_f$ for a vector v expressed in each co-ord sys.

The arbitrary twisting turning motion takes place continuously for $t \in [0, b]$

2. If we stop at time $t = b$, how do we know the resulting matrix Q_b is a simple rotation and not some potentially more general ON map with inversions, etc...?

$\triangleright$ $\boxed{\text{For physical motions of a RB, } \det Q_t = +1}$

For any ON Q, $\det Q = \pm 1$ and here $\det Q_{t=0} = \det I = +1$

How do we know $\det Q_t \neq -1$ for some t?

Let $f(t) := \det Q_t$ then $f : [0,b] \longrightarrow \{-1\} \cup \{1\}$ and f is cont.
$t \longmapsto \det Q_t$

Cont fcn f(conn set) = conn set and [0,b] is a conn set

$a \cap c = \emptyset$
disjoint

Classic Disconn arg: $a := f^{-1}(\{1\})$ and $c := f^{-1}(\{-1\})$.

$\{1\}$ is an open set in the inherited topology from $\mathbb{R}$
$\{1\} = \{1\} \cap (1-\varepsilon, 1+\varepsilon)$ open set
Same for $\{-1\}$

$f^{-1}(\text{open}) = \text{open}$ so a open in [0,b], c open in [0,b]

a not empty because $0 \in a$

In order not to disconnect [0,b] and get a contradiction, we must have $c = \emptyset \Rightarrow \det Q_t = +1 \;\forall\, t \in [0,b]$

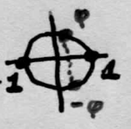
Euler cont'd

3 we will show: $\boxed{\text{Given } Q: \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ Q^{-1} = Q^T,\ \det Q = +1,\ n \text{ odd} \Rightarrow Q \text{ has at least one EW } \lambda = +1 \text{ [and thus } Qv = v \text{ for some } v\text{; a line is fixed]}}$

(a) To do this, we will show the set of EWs must break up into pairs — If λ is in the set, so is $\frac{1}{\lambda}$

(b) $\det Q = \lambda_1 \lambda_2 \cdots \lambda_n \stackrel{!}{=} 1$

$|\lambda| = 1$ if $Qx = \lambda x$ because $\|x\| = \|Qx\| = \|\lambda x\| = |\lambda|\,\|x\| \Rightarrow |\lambda| = 1$
$\Rightarrow \lambda = e^{i\varphi}$



▷ $\boxed{\text{Q and } Q^{-1} \text{ must have the same set of EWs; always } \det A = \det A^T}$

This was the beginning of an argument that I don't need.

$\det(Q - \lambda I) = 0 \Rightarrow$ poly with Real coeffs $\Rightarrow$ if λ is soln, so is $\bar{\lambda}$

on unit circle $\Delta(0,1)$ in $\mathbb{C}$, $\bar{\lambda} = \frac{1}{\lambda}$

Thus the possibilities are $\lambda = 1$ $\quad \frac{1}{\lambda} = \frac{1}{1} = 1$ so 1 pairs with itself

$\lambda = -1$ $\quad \frac{1}{\lambda} = \frac{1}{-1} = -1$ so -1 pairs with itself

$\varphi \neq 0, \pi$ $\quad \lambda = e^{i\varphi}$ $\quad \frac{1}{\lambda} = e^{-i\varphi}$ so each $\lambda = e^{i\varphi}$ must pair with distinct $\lambda = e^{-i\varphi}$
$\Rightarrow$ even number of these

▷ Now consider $\det Q = \lambda_1 \lambda_2 \cdots \lambda_n = +1$

$$\underbrace{(e^{i\varphi_1})(e^{-i\varphi_1})}_{\text{Pair — mult to 1}}\underbrace{(e^{i\varphi_2})(e^{-i\varphi_2})}_{\text{Pair — mult to 1}} \cdots \underbrace{(-1)(-1)}_{\text{must be in pairs to mult to 1}} \cdots \underbrace{(1)(1)(1)}_{\text{can be arb number of +1s}}$$

If n is odd, there must be at least one $\lambda = +1$

□

Remove the restriction on 1 pt being fixed

<u>Chasle's Thm</u> an arb motion of a RB can be realized as a translation and a rotation.

There is more, but no need for it at this time.