① Given linear map $T: V \to V$ and basis $\{e_1 \ldots e_n\}$, derive matrix rep $[T]_e = A$

$$\begin{array}{ccc} (V, \{e\}) & \xrightarrow{T} & (V, \{e\}) \\ \downarrow & & \downarrow \\ \mathbb{R}^n & \xrightarrow{[T]_e = A} & \mathbb{R}^n \\ [v]_e & \longrightarrow & [Tv]_e = Av_e \end{array}$$

Step 1 $Te_i = \{e_1 \ldots e_n\} \underbrace{\begin{bmatrix} a_i^{1} \\ \vdots \\ a_i^{n} \end{bmatrix}}_{\vec{A}_i} = \sum_{k=1}^{n} a_i^{k} e_k$ ⊛

Thus

$T(\{e_1 \ldots e_n\}) = \{e_1 \ldots e_n\}\begin{bmatrix} A \end{bmatrix}$

Step 2 $v = \{e_1 \ldots e_n\}\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$

$Tv = T(\{e\}\vec{v}) = \sum v^i Te_i$

$= v^1\{e_1 \ldots e_n\}\vec{A}_1 + \ldots + v^n\{e\}\vec{A}_n$

$= \{e_1 \ldots e_n\}[v^1 A_1 + \ldots + v^n A_n] = \{e\}\underbrace{\begin{bmatrix} A \end{bmatrix}}_{[T]_e}\underbrace{\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}}_{[v]_e}$

② Given an inner prod, show $a_{ij} = \langle Te_j, e_i \rangle$
Provided $\{e\}$ is O.N. [note: generally $\{X_u, X_v\}$ NOT O.N.]

From ⊛ $Te_j = \sum_k a_j^{k} e_k$

Fix i

To isolate a_j^i $\quad \langle Te_j, e_i \rangle = \sum_k a_j^k \overbrace{\langle e_k, e_i \rangle}^{\delta_{ik} \text{ here we use O.N.}} = a_j^{i} = a_{ij}$ ✓

③ T has an adjoint $\langle Tx, y \rangle = \langle x, T^* y \rangle$. show matrix rep $B = [T^*]_e$ satisfies $b_{ij} = \overline{a}_{ji}$ Thus $B = A^H$ but since we are Real here, $B = A^T$

$b_{ij} \overset{\text{using step 2}}{=} \langle T^* e_j, e_i \rangle \overset{\text{switch means conj}}{=} \overline{\langle e_i, T^* e_j \rangle}$

$\overset{\text{def of adj}}{=} \overline{\langle Te_i, e_j \rangle} = \overline{a}_{ji}$ Thus $B = A^H$ but since all elts $\mathbb{R}$, $B = A^T$

④ Now specify T is self-adjoint $\langle Tx, y \rangle = \langle x, Ty \rangle$ i.e. $T = T^*$
show $A = A^T$

From step 3, $b_{ij} = \langle T^* e_j, e_i \rangle = a_{ji}$

$\overset{T=T^*}{\parallel}$

$\langle Te_j, e_i \rangle$

$\parallel$

a_{ij}

$\Rightarrow a_{ij} = a_{ji} \Rightarrow A = A^T$ sym.

T self adj

⑤ Define bilinear $B(x,y) := \langle Tx, y \rangle$ and show this is $y^T A x$

$$\begin{array}{ccc} \overset{x}{V} \times \overset{y}{V} & \xrightarrow{B} & \mathbb{R} \\ | \quad | & \nearrow_{(\cdot)[A](\cdot)} & \\ \mathbb{R}^n \times \mathbb{R}^n & & \\ \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} & & \end{array}$$

$x = \sum x^i e_i$
$y = \sum y^j e_j$

$B(x,y) = B(\sum x^i e_i, \sum y^j e_j)$

$= \sum_i^n \sum_j^n x^i y^j B(e_j, e_i)$

$B(e_j, e_i) = \langle Te_j, e_i \rangle = a_{ij}$

$= [y^1 \ldots y^n]\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} x \end{bmatrix}$

B satisfies $B(x,y) = B(y,x)$ since T self-adj
Thus $B(x,y) = y^T A x = (y^T A x)^T$ since scalar $= x^T A^T y$
$\parallel$
$B(y,x) = x^T A y \Rightarrow$ Thus again $A = A^T$

Quadratic form

6 Define $Q(x) = B(x,x) = \langle Tx, x\rangle$ again we are saying T self-adj.

$$\begin{tikzcd} \underset{x}{V} \arrow[r, "Q"] \arrow[d, "\alpha"'] & \underset{\langle Tx,x\rangle}{\mathbb{R}} \\ \underset{[x]}{\mathbb{R}^n} \arrow[ur, "{(\cdot)^T[A](\cdot)}"', "Q_\alpha"] & \end{tikzcd}$$

Thus $Q(x) = [x\ y]\begin{bmatrix} a & b \\ b & c\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}$ in the 2-D case we are explicitly needing in this appendix

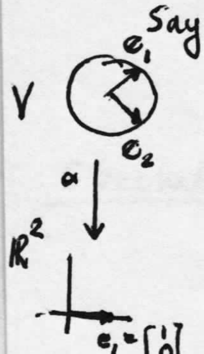
Prop $Q: V \to \mathbb{R}$ quadratic form $\Rightarrow \exists$ O.N. basis $\{e_1\, e_2\} \ni$
(Just taking V 2-dim here)
if $v = \{e_1\, e_2\}\begin{bmatrix} x \\ y\end{bmatrix}$

$$\Rightarrow Q(v) = [x\ y]\begin{bmatrix} \lambda_1 & \\ & \lambda_2\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \lambda_1 x^2 + \lambda_2 y^2$$

where $\lambda_1 = \max\limits_{v\in S^1} Q(v)$

$\lambda_2 = \min\limits_{v\in S^1} Q(v)$

Pf. The cont fcn Q has max on cpt set S^1 (not nec uniq)
Say max value is λ_1 and it occurs fr $e_1 \in V$, $\|e_1\| = 1$ $\quad Q(e_1) = \lambda_1$



Let $e_2 \perp e_1$ that is $\langle e_1, e_2\rangle = 0$ $\quad$ Define $\lambda_2 = Q(e_2)$

Then $\{e_1\, e_2\}$ is an O.N. basis fr V $\quad v = \{e_1\, e_2\}\begin{bmatrix} x \\ y\end{bmatrix}$ $\quad \lambda_2 \le \lambda_1$ maybe equal

$$Q(v) = B(v,v) = B(xe_1 + ye_2,\ xe_1 + ye_2)$$

$$= [x\ y]\begin{bmatrix} B(e_1,e_2) & B(e_2,e_1) \\ B(e_1,e_2) & B(e_2,e_2)\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}$$

we needed $\langle e_1, e_2\rangle = 0$ O.N. basis so this matrix would be sym

Observe $B(e_1,e_1) = Q(e_1) = \lambda_1$ and $B(e_2,e_2) = Q(e_2) = \lambda_2$

Define $b = B(e_2,e_1) = B(e_1,e_2)$

$$\Rightarrow Q(v) = \lambda_1 x^2 + 2bxy + \lambda_2 y^2 = Q_\alpha(x,y)$$

From the <u>Lemma</u> (to be proved next) $\quad Q_\alpha(1,0) = \max \Rightarrow b = 0$

$$\Rightarrow Q = \lambda_1 x^2 + \lambda_2 y^2$$

$\triangleright$ Now we must show λ_2 is the min value of Q on S^1

For any $\begin{bmatrix} x \\ y\end{bmatrix} \in S^1$ $\quad Q_\alpha(x,y) = [x,\ y]\begin{bmatrix} \lambda_1 & \\ & \lambda_2\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \lambda_1 x^2 + \lambda_2 y^2 \ge \lambda_2 x_2 + \lambda_2 y^2$ $\quad$ ($\lambda_2 \le \lambda_1 \downarrow$)

$= \lambda_2(x^2 + y^2)$

$= \lambda_2 \cdot 1$

$= Q(e_2)$

$= Q(0,1)$ or $Q(0,-1)$ both work $\pm\hat{e}_2$

Thus $Q(v) \ge Q(e_2)\ \forall v$

so e_2 is a min pt, λ_2 min value.

$\square$

Observe
$$[0\ -1]\begin{bmatrix} \lambda_1 & \\ & \lambda_2\end{bmatrix}\begin{bmatrix} 0 \\ -1\end{bmatrix} = [0\ -1]\begin{bmatrix} 0 \\ -\lambda_2\end{bmatrix} = +\lambda_2$$

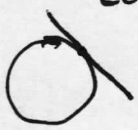
Lemma $Q: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$\langle x,y\rangle \longmapsto [x\ y]\begin{bmatrix} a & b \\ b & c\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}$
x^TAx

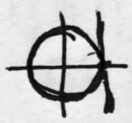
$Q\big|_{S^1}$ has local max at $\langle 1,0\rangle$ (extrema) $\Rightarrow$ $b=0$ Thus $A=\begin{bmatrix} a & \\ & c\end{bmatrix}$ and is diagonal.

Pf. $Q(x) = x^TAx$ and A sym: $A=A^T$

$DQ_x(h) = x^TAh = (x^TAh)^T = h^TA^Tx = h^TAx$ Avez DC ch 10

Let $h\in T_xS^1$ If Q attains extrema at $\begin{bmatrix}1\\0\end{bmatrix}=e_1$ then $DQ_{e_1}(h)=0 \quad \forall\, h\in T_{e_1}S^1$





That is $DQ_{\hat{e}_1}(\lambda\hat{e}_2) \overset{!}{=} 0 \quad \forall\, \lambda\in\mathbb{R}$

$\Rightarrow [1\ 0]\begin{bmatrix} a & b \\ b & c\end{bmatrix}\begin{bmatrix} 0 \\ 1\end{bmatrix} \overset{!}{=} 0$

$[1\ 0]\begin{bmatrix} b \\ c\end{bmatrix} = 0$

$b=0$ $\square$

Spectral Thm $T: V\to T$ self-adj $\Longrightarrow$ $\exists$ ON basis $\{e_1, e_2\} \ni$
$Te_1=\lambda_1 e_1 \quad Te_2 = \lambda_2 e_2$
i.e. $T\{e_1\ e_2\} = \{e_1\ e_2\}\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}$
A diagonal
where $\lambda_1 = \max Q$ on S^1
$\lambda_2 = \min Q$ on S^1

Pf. Let $Q(v) = \langle Tv, v\rangle$

By previous Prop, $\exists$ O.N. $\{e_1\ e_2\}$ where $Q(e_1)=\lambda_1 \quad Q(e_2)=\lambda_2$

From pf of Prop, $b := B(e_1,e_2) = \langle Te_1, e_2\rangle = 0$ by Lemma

and since $e_2\neq 0$ we must have $Te_1\perp e_2$ that is $Te_1 = \mu e_1$ for some $\mu\in\mathbb{R}$
maybe $\mu=0$

If $\mu\neq 0$,

$\langle Te_1, e_1\rangle = Q(e_1) = \lambda_1$ also $\langle Te_1,e_1\rangle = \langle \mu e_1, e_1\rangle = \mu\langle e_1,e_1\rangle = \mu$

$\Rightarrow \lambda_1=\mu$ so $Te_1 = \lambda_1 e_1$

This holds also if $\mu=0$

▷ To show $Te_2 = \lambda_2 e_2$ $\quad b = B(e_1,e_2) = B(e_2,e_1) = \langle Te_2, e_1\rangle = 0$

Thus $Te_2 = \mu e_2$

Same arg $\langle Te_2, e_2\rangle = Q(e_2) = \lambda_2$
$\langle Te_2,e_2\rangle = \mu\langle e_2,e_2\rangle = \mu$ $\Rightarrow Te_2 = \lambda e_2$ $\square$