Do Carmo DGOCAS
Ch 2

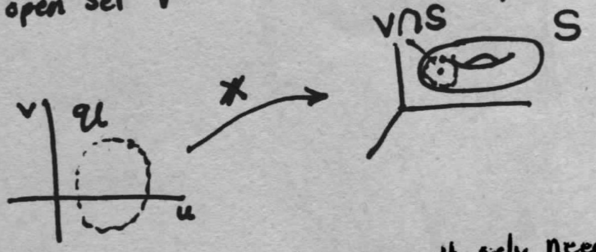
Summary and Extensions

11/6/2015
①

p.52 **Regular Surface** [submfd of $\mathbb{R}^3$]

a set $S \subset \mathbb{R}^3$ is a <u>regular surface</u> if it can be covered by co-ord charts $\{(\mathbf{X}, \mathcal{U})\}$.

That is to say, for each $p \in S$, $\exists$ an open set V in $\mathbb{R}^3$ and a map $\mathbf{X} : \mathcal{U} \longrightarrow V \cap S$ such that



① $\mathbf{X} : \mathcal{U} \longrightarrow \mathbb{R}^3$ is dif'b i.e

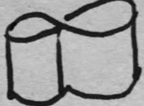
$$d\mathbf{X}_u = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$ all partials are cont

② $\mathbf{X}$ is a homeo (so $\mathbf{X}^{-1}$ exists and is cont) ⟶ we really only need $\mathbf{X}$ is One-to-One we get that $\mathbf{X}$ is a diffeo from Inv Fcn Thm

③ [regularity cond] $d\mathbf{X}_u : \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ is One-to-One $\forall u \in \mathcal{U}$.

Cond (2) prevents self-intersections like  or even  see G&P

p.54

Cond (3): Let $A := d\mathbf{X}_u = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$ surface 

▷ A is One-to-One $\Longleftrightarrow$ cols of A are LI

($\Rightarrow$) If $\exists$ scalars x_1, x_2 (not both 0) $\ni x_1\vec{a}_1 + x_2\vec{a}_2 = 0$

Then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ and $A\vec{0} = \vec{0}$ $\Rightarrow\Leftarrow$

($\Leftarrow$) Let $Ax = Ay$ for $x \neq y$ Then $x_1\vec{a}_1 + \vec{a}_2 x_2 = y_1\vec{a}_1 + y_2\vec{a}_2$

$\Leftrightarrow (x_1 - y_1)a_1 + (x_2 - y_2)a_2 = 0$

$\Leftrightarrow x_1 - y_1 = 0$ & $x_2 - y_2 = 0$ since $\{a_1, a_2\}$ LI

$\Leftrightarrow x = y$ $\Rightarrow\Leftarrow$

▷ Cols of A LI $\Longleftrightarrow$ $\mathbf{X}_u \times \mathbf{X}_v \neq 0$

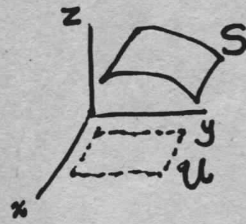
$$\mathbf{X}_u \times \mathbf{X}_v = \begin{bmatrix} \det\begin{bmatrix} y_u & z_u \\ y_v & z_v \end{bmatrix} \\ -\det\begin{bmatrix} x_u & z_u \\ x_v & z_v \end{bmatrix} \\ \det\begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} \end{bmatrix}$$

we know from linear alg that since A has rank 2, at least one of these subdets must not be 0

row rank = col rank.

Prop 1 $f : \overset{\mathbb{R}^2}{\mathcal{U}}^{open} \longrightarrow \mathbb{R}$ dif'b $\Longrightarrow S = \text{graph}(f)$ is regular surf.

C^1?



$$\mathbf{X}(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$$

$$d\mathbf{X}_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{bmatrix}$$

<u>Cond 1</u> satisfied by single chart and obviously partials cont

<u>Cond 3</u> obviously top subdet $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$

<u>Cond 2</u> $\mathbf{X}$ is obviously One-to-One by vertical line test.
$\mathbf{X}^{-1}$ = projection $\pi|_S$ obviously cont.

□

(p. 58) For any map $F: \overset{\mathbb{R}^n}{\mathcal{U}} \longrightarrow \mathbb{R}^m$

We say $x \in \mathcal{U}$ is a critical pt of F if $dF_x: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is not Onto.

$y = F(x)$ is a critical value [If y not a critical value, it is a regular value]

If $x \in \mathcal{U}$ has dF_x Onto, it is a regular pt.

[Sard's Thm Spivak COM p.72
The set of critical values
has ⓜ 0]

Prop 2 [G&P's Pre-image Thm]

$$\left.\begin{array}{l} f: \overset{\mathbb{R}^3}{\mathcal{U}} \longrightarrow \mathbb{R} \text{ smooth} \\ a \in \mathcal{U} \text{ is regular value } [\, df_x \text{ Onto } \mathbb{R} \ \forall x \in f^{-1}(a)\,] \end{array}\right\} \Longrightarrow f^{-1}(a) \text{ is a regular surf}$$

Pf. Let $S = f^{-1}(a)$

Fix $p \in S$. We want to show $S = \text{graph}(h)$ for some fcn h in a nbhd $\mathcal{V}_p$ of p. Then we are done by Prop 1.

That is to say, we want $\mathbf{X}: (x,y) \longmapsto \begin{bmatrix} x \\ y \\ h(x,y) \end{bmatrix}$

We know $df_x = [f_x \ f_y \ f_z] \neq [0, 0, 0]$ so (by relabeling axes if necessary) assume $f_z \neq 0$

Define $F: \overset{\mathbb{R}^3}{\mathcal{U}} \longrightarrow \mathbb{R}^3$ $\quad$ F = Id on first 2 components $\quad DF_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix}$ $\quad$ $\det DF_p \neq 0$ by hypoth

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ f(x,y,z) \end{bmatrix}$$

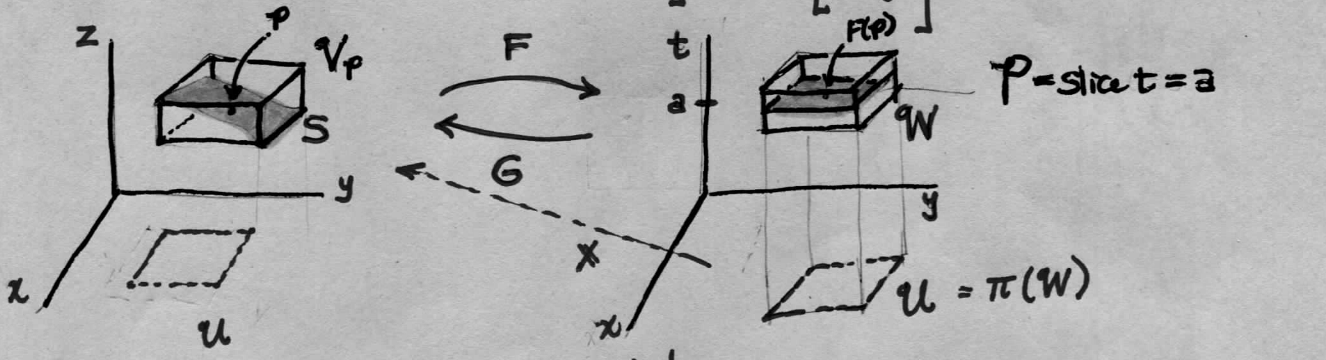
Apply Inv Fcn Thm: $\exists$ open set $\mathcal{V}_p$ around p, and $\mathcal{W}_{F(p)}$ around $F(p)$ where F is a diffeo

Thus a smooth F^{-1} exists, call it $G: \mathcal{W}_{F(p)} \longrightarrow \mathcal{V}_p$

Thus $G(\mathcal{P}) = S \cap \mathcal{V}_p$

$$G\Big|_{\mathcal{P}} = \underset{\text{for } (x,y,\cdot) \in \mathcal{W}}{G(x,y,a)} = \begin{bmatrix} x \\ y \\ G^{3}(x,y,a) \end{bmatrix} =: \begin{bmatrix} x \\ y \\ h(x,y) \end{bmatrix} = \mathbf{X} \quad \text{The graph chart we seek.}$$

p.58



$\mathbf{X} = \text{Id} \times h$

③

1.61 (ex 2) Ellipsoid $E: \frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 + \frac{1}{c^2}z^2 = 1$ is a Reg Surf because

$E = f^{-1}(0)$ for $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$df_x = \left[\frac{2}{a^2}x \;\; \frac{2}{b^2}y \;\; \frac{2}{c^2}z\right]$ and this is $[0\ 0\ 0]$ only at origin, but origin $\notin E$.

(ex 3) Hyperboloid of 2 sheets $H: -x^2 - y^2 + z^2 = 1$

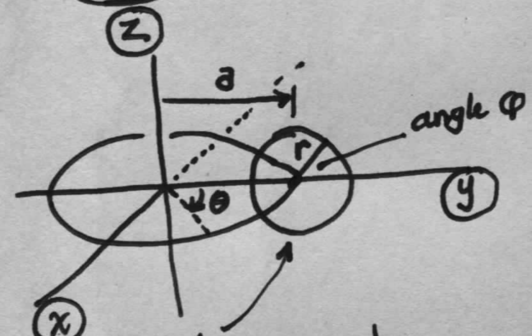
$H = f^{-1}(0)$ for $f(x,y,z) = -x^2 - y^2 + z^2 - 1$

$df_x = [-2x \;\; -2y \;\; 2z] \neq [0\ 0\ 0]$ on H

Observe H is not Conn: Rewrite it $z = \pm\sqrt{1 + x^2 + y^2}$ For no value of x, y can $z = 0$ so the surfs are dis conn.

[see my sheets on Quadric Surfs]

(ex 4) The torus T^2 is a regular surface.

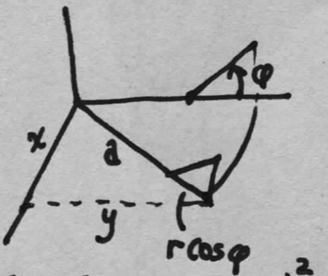


This circle S^1 is given by

$$\sigma : \varphi \longmapsto \begin{bmatrix} 0 \\ r\cos\varphi + a \\ r\sin\varphi \end{bmatrix}$$

To rotate any pt (x,y,z) in the image of σ around the z axis:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ rC_\varphi + a \\ rS_\varphi \end{bmatrix} = \begin{bmatrix} -S_\theta(rC_\varphi + a) \\ C_\theta(rC_\varphi + a) \\ rS_\varphi \end{bmatrix}$$



$x^2 + y^2 = (a + rC_\varphi)^2$

$\frac{\sqrt{x^2+y^2} - a}{r} = C_\varphi$

To derive the eq for T^2 without params, use norm squared:

$$\begin{aligned} x^2 + y^2 + z^2 &= S_\theta^2(rC_\varphi + a)^2 + C_\theta^2(rC_\varphi + a)^2 + r^2 S_\varphi^2 \\ &= (S_\theta^2 + C_\theta^2)(rC_\varphi + a)^2 + r^2 S_\varphi^2 \\ &= (rC_\varphi + a)^2 + r^2 S_\varphi^2 \\ &= r^2 C_\varphi^2 + 2arC_\varphi + a^2 + r^2 S_\varphi^2 \\ &= r^2(C_\varphi^2 + S_\varphi^2) + 2arC_\varphi + a^2 \\ &= r^2 + 2a\cancel{r}\left(\frac{\sqrt{x^2+y^2} - a}{\cancel{r}}\right) + a^2 \end{aligned}$$

$$\Rightarrow z^2 = r^2 - (x^2 + y^2) + 2a\sqrt{x^2+y^2} - 2a^2 + a^2$$

$$= r^2 - \left(\sqrt{x^2+y^2} - a\right)^2$$

$$\Rightarrow \underbrace{\left(\sqrt{x^2+y^2} - a\right)^2 + z^2}_{f(x,y,z)} = r^2$$

f is smooth for $(x,y) \neq (0,0)$

$T^2 = f^{-1}(r^2)$

$$df_x = \left[\frac{2x\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}} \quad \frac{2y\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}} \quad 2z \right] \neq [0,0,0] \text{ on } T^2$$

$\Rightarrow T^2$ is a Regular Surf

□

p.63

Prop 3 | Locally Every Reg Surf is a Graph

Let S be Regular Surf $\Longrightarrow$ Every pt $\underset{\sim}{p} \in S$ has a nbhd $V_{\underset{\sim}{p}}$ where $S \cap V_{\underset{\sim}{p}}$ is the graph of a fcn (It could be over the xy plane, or xz or yz)

Pf Let $\mathbb{X} : \mathcal{U} \longrightarrow S$ be a co-ord chart containing $\underset{\sim}{p}$.
Since $d\mathbb{X}_u$ is One-to-One by def of Reg Surf, at least one Jacobian subdet is nonzero, wlog lets consider the case where $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$

Consider the proj map $F := \pi_{xy}\mathbb{X}$ where $\pi_{xy} = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$

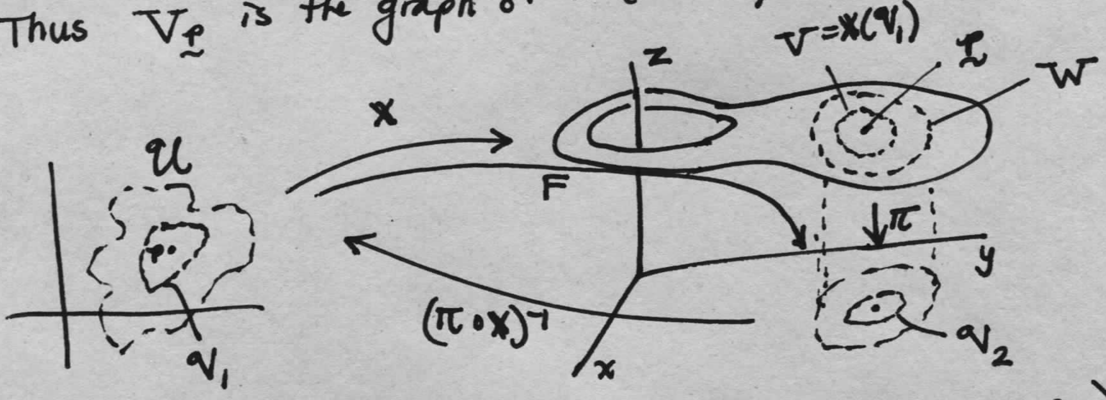
Then $DF = \pi \circ d\mathbb{X} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$ and we know $\det DF_x \neq 0$ by hypoth

By *Inv Fcn Thm* $\exists$ nbhds $\mathcal{V}_1, \mathcal{V}_2 \ni F : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is a diffeo.

Then the map we seek is $\mathbb{X} \circ F^{-1} = \mathbb{X} \circ (\pi \circ \mathbb{X})^{-1} : \mathcal{V}_2 \longrightarrow V$

$V = V_{\underset{\sim}{p}} = \mathbb{X}(\mathcal{V}_1)$

Thus $V_{\underset{\sim}{p}}$ is the graph of $\mathbb{X} \circ (\pi \circ \mathbb{X})^{-1}$, that is to say $\mathbb{X} \circ (\pi \circ \mathbb{X})^{-1} = \begin{bmatrix} x \\ y \\ h(x,y) \end{bmatrix}$ □



Observe: If we *know* S is a graph (and $\mathbb{X}$ is One-to-One) then this *same argument* gives us that $\mathbb{X}^{-1}$ is smooth, because $\mathbb{X}^{-1}\big|_V = (\pi \circ \mathbb{X}^{-1}) \circ \pi$ composition of smooth fcns.

Prop 4

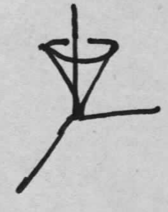
If S is known to be a Reg Surf
we have a Candidate $\mathbb{X} : \mathcal{U} \to V$ for a param
satisfying (1) $\mathbb{X}$ has smooth partials
(2) $\mathbb{X}$ is One-to-One
(3) $d\mathbb{X}_u$ is One-to-One

$\Longrightarrow$ $\mathbb{X}^{-1}$ is smooth.

This weakens what we needed in def of Reg Surf (mfd) p.52 → (2)

This saves us having to check more conditions on a candidate chart for a known mfd. □

example 5



Let's show that the cone $z = \sqrt{x^2+y^2}$ is <u>not</u> a Reg Surf.

The obvious param $\mathbb{X} : (x,y) \longmapsto \begin{bmatrix} x \\ y \\ \sqrt{x^2+y^2} \end{bmatrix}$ has $d\mathbb{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-2x}{\sqrt{x^2+y^2}} & \frac{-2y}{\sqrt{x^2+y^2}} \end{bmatrix}$

not even defined at origin.

But could $\exists$ some other param that works? No.

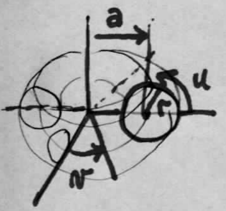
By Prop 3, S must be a graph at $(0,0,0)$

It can't be a graph over xz or yz planes, not One-to-One.

what about graph over xy plane? any param would have to agree with $z = \sqrt{x^2+y^2}$ in nbhd of origin; this is not smooth.

example 6

a param of torus T^2



$$\mathbb{X}(u,v) = \begin{bmatrix} (r\cos u + a)\cos v \\ (r\cos u + a)\sin v \\ r\sin u \end{bmatrix} \qquad d\mathbb{X}_u = \begin{bmatrix} -r\sin u\cos v & -[r(\cos u)+a]\sin v \\ -r\sin u \sin v & (r\cos u + a)\cos v \\ r\sin u & 0 \end{bmatrix}$$

$0 < u < 2\pi$
$0 < v < 2\pi$

<u>check conds for prop 4</u> : ① obviously $d\mathbb{X}_u$ is smooth

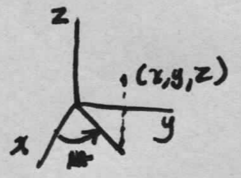
③ we must show $\mathbb{X}_u \times \mathbb{X}_v \neq \vec{0}$

$$\det \begin{vmatrix} i & j & k \\ rS_uC_v & rS_uS_v & rC_u \\ -(rC_u+a)S_v & (rC_u+a)C_v & 0 \end{vmatrix} = \begin{bmatrix} -rC_u(rC_u+a)C_v \\ -(+rC_u(rC_u+a)S_v \\ rS_uC_v(rC_u+a)C_v + rS_uS_v(r+C_u+a)S_v \end{bmatrix} \neq \vec{0}$$

we previously showed $T^2 = f^{-1}(r^2)$ so T^2 is a Reg Surf.

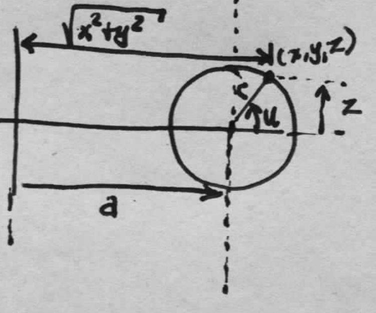
② show $\mathbb{X}$ is One-to-One:

Given $(x,y,z) \in T^2$, we must show this corresponds to One, and only One, angles (u,v)



There is only one angle v that gives a line that corresponds to x,y,z

For any v, we have the following diagram to find u:



$\sin u = \frac{z}{r}$

If $\sqrt{x^2+y^2} \leq a$ we are on left half of circle and $\frac{\pi}{2} \leq u \leq \frac{3\pi}{2}$

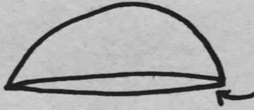
If $\sqrt{x^2+y^2} \geq a$ we on on right half $0 < u \leq \pi/2$ or $\frac{3\pi}{2} \leq u < 2\pi$

Thus (x,y,z) uniquely determines (u,v) and $\mathbb{X}$ is One-to-One ☐

Let's take a moment and list several co-ord systems on S^2

p. 55-56 $\mathbb{X}_1 : \overset{\mathbb{R}^2}{\mathring{B}}(0,1) \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto \begin{bmatrix} x \\ y \\ \sqrt{1-(x^2+y^2)} \end{bmatrix}$$

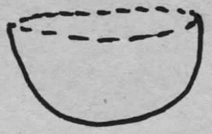


Open edge - missing equator

Top half of Sphere

$\mathbb{X}_2 : \mathring{B}(0,1) \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto \begin{bmatrix} x \\ y \\ -\sqrt{1-(x^2+y^2)} \end{bmatrix}$$



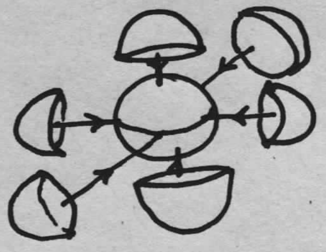
Bot half minus equator

$$\mathbb{X}_3(x,z) = \left[x, \sqrt{1-(x^2+z^2)}, z\right]^T$$

$$\mathbb{X}_4(x,z) = \left[x - \sqrt{}, z\right]^T$$

$$\mathbb{X}_5(y,z) = \left[\sqrt{1-(y^2+z^2)}, y, z\right]^T$$

$$\mathbb{X}_6(y,z) = \left[-\sqrt{}, y, z\right]^T$$



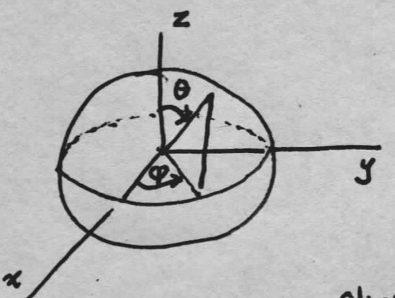
spherical co-ords

This param doesn't cover the poles and the line at $\varphi = 0$ joining them.

$$\mathbb{X} : \overset{\mathbb{R}^2}{(0,\pi)\times(0,2\pi)} \longrightarrow \mathbb{R}^3$$

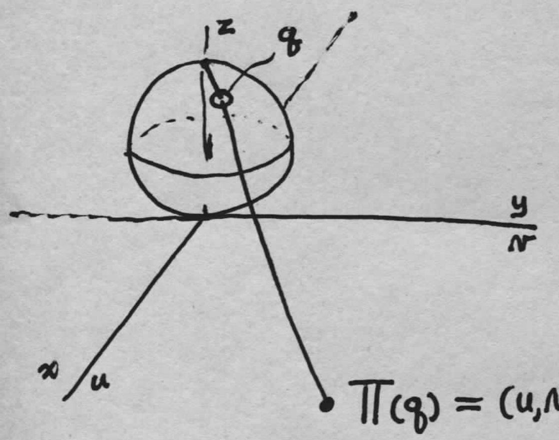
$$(\theta, \varphi) \longmapsto \begin{bmatrix} R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \\ R\cos\theta \end{bmatrix}$$

$$d\mathbb{X}_{(\theta,\varphi)} = \begin{bmatrix} R\cos\theta\cos\varphi & -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi & R\sin\theta\cos\varphi \\ -R\sin\theta & 0 \end{bmatrix}$$



Note how DoCarmo flips θ and φ from Marsden & Tromba's convention.

p. 67 Stereographic projection



$\Pi(q) = (u,v)$

Not the "π" of standard co-ord projection

DoCarmo puts center of sphere at $z=1$, Palka AITCFT p.351 puts it at origin

$$x^2+y^2+(z-1)^2 = 1$$

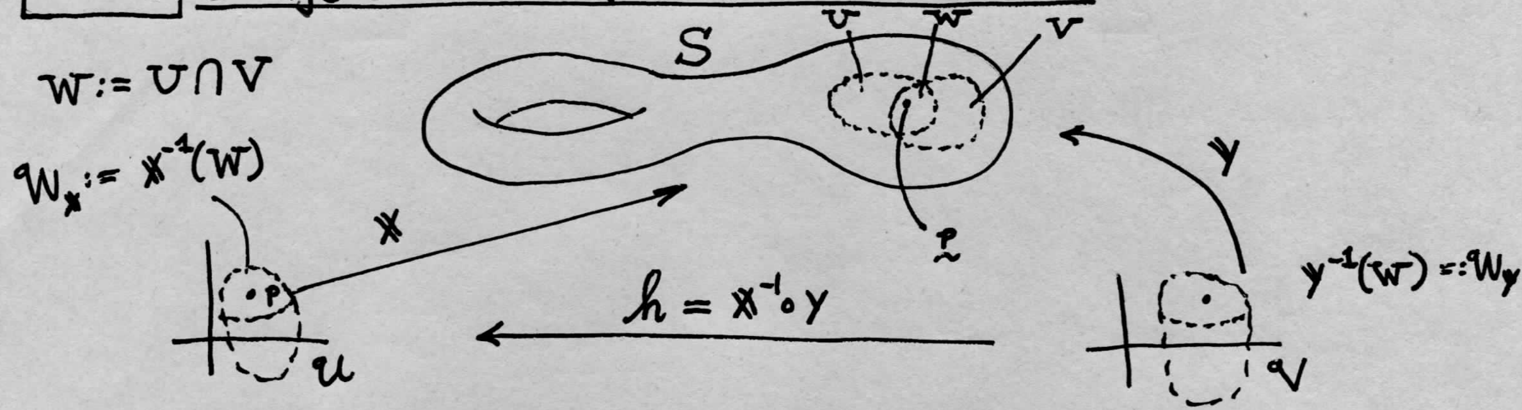
The map is defined

$$\Pi^{-1} : \mathbb{R}^2 \longrightarrow \overset{\mathbb{R}^3}{S^2}$$

$$(u,v) \longmapsto \begin{bmatrix} \dfrac{4u}{u^2+v^2+4} \\ \dfrac{4v}{u^2+v^2+4} \\ \dfrac{2(u^2+v^2)}{u^2+v^2+4} \end{bmatrix}$$

$\boxed{\text{ch 2.3}}$ Change of Parameters; Smooth Fcns on Surfs

$W := U \cap V$



p.70

Prop 1 Let W be the overlap of 2 co-ord charts containing pt $\underset{\sim}{p}$. Define Change-of-Coords fcn

$$\left.\begin{array}{l} h : W_y \longrightarrow W_x \\ v \longmapsto x^{-1} \circ y(v) \end{array}\right\} \Rightarrow h \text{ is a diffeo}$$

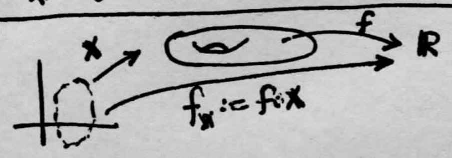
DoCarmo gives a pf based on his definitions and the Inv Fcn Thm but I'M JUST GOING TO ACCEPT THIS A PRIORI

DoCarmo wants to avoid referencing the embedding space $\mathbb{R}^3$ so he must refer everything to $\mathbb{R}^2$ in charts. For abstract mfds, we <u>require</u> h to be a diffeo.

G&P just make use of the fact that the embedding space is $\mathbb{R}^n$ and a smooth fcn f on our mfd is a smooth fcn f in an open set in $\mathbb{R}^n$ and we restrict f to our mfd. I will think this way. DoCarmo admits this is equivalent p.75 and p.82 #13

$\boxed{x : \mathcal{U} \to U \text{ is a diffeo}}$

p.72 **Def 1**



f is <u>smooth</u> if downstairs map is smooth.