Cheney Applied Math notes

6/18/2017

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Ch 3 Hilbert Space

Let X be a VS over $\mathbb{C}$ with inner prod $\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{C}$ which satisfies

(a) $\langle x,y \rangle = \overline{\langle y,x \rangle}$ $\mathbb{C}$-conj

(b) $\langle \alpha x + y, z \rangle = \alpha \langle x,z \rangle + \langle y,z \rangle$ Linear in 1st position

(c) $\langle x,x \rangle > 0$ if $x \neq 0$

[observe $\langle \vec{0},\vec{0} \rangle = \langle 0\vec{0},\vec{0} \rangle = 0\langle \vec{0},\vec{0} \rangle = 0$ linearity]

This def agrees with Rudin RACA p.76 and Schaum's LA ch 13

example $X = \mathbb{C}^n$ $\quad \langle x,y \rangle := \sum_{i=1}^{n} x_i \overline{y_i} = y^H x$

This is not like $x^T y$ in $\mathbb{R}^n$ because $x^T y = y^T x$ but in $\mathbb{C}^n$ $x^H y = \overline{y^H x}$.

This is a bit tricky because you can't do it the naive way of saying $z = \vec{a} + i\vec{b}$, $w = \vec{c} + i\vec{d}$ (Real components)
and do $z \cdot w = (a+ib)^T (c+id) = (a^T c - b^T d) + i(a^T d + b^T c)$

But here is what happens if we follow the defn:

$$
\begin{aligned}
\langle z,w \rangle = \langle a+ib, c+id \rangle &= \langle a, c+id \rangle + i \langle b, c+id \rangle \\
&= \overline{\langle c+id, a \rangle} + i\, \overline{\langle c+id, b \rangle} \\
&= \overline{\langle c,a \rangle + i \langle d,a \rangle} + i\left(\overline{\langle c,b \rangle + i \langle d,b \rangle}\right) \\
&= \langle a,c \rangle - i \langle a,d \rangle + i \langle b,c \rangle - i^2 \langle b,d \rangle \\
&= a^T c + b^T d + i(b^T c - a^T d)
\end{aligned}
$$

NOT the same

▷ why can't we just define $\langle z,w \rangle = z^T w$?

Because we want $\langle z,z \rangle = \|z\|_2^2$

$$
\begin{aligned}
z^H z = (a^T - ib^T)(a+ib) &= a^T a + b^T b + \underbrace{i a^T b - i b^T a}_{0} \\
&= \|a\|^2 + \|b\|^2 \\
&= \|z\|^2
\end{aligned}
$$

but $z^T z$ is not even Real.

▷ Scalar multiplication does not preserve "colinearity" in an intuitive sense:

$(\alpha + i\beta)(\vec{a} + i\vec{b}) = \alpha\vec{a} - \beta\vec{b} + i(\beta\vec{a} + \alpha\vec{b})$

Real parts not colinear in the $\mathbb{R}^n$ sense (neither are imag parts)

Define a norm from an inner prod

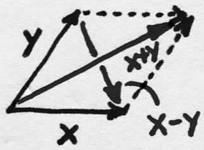
$$
\begin{aligned}
\|\cdot\| : X &\longrightarrow [0,\infty) \\
x &\longmapsto \sqrt{\langle x,x \rangle}
\end{aligned}
$$

Properties of norm:

(i) $\|x\| > 0$ if $x \neq 0$
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$
(iii) $\|x+y\| \leq \|x\| + \|y\|$

} any def of norm

(iv) $|\langle x,y\rangle| \leq \|x\|\,\|y\|$ Cauchy-Schwartz



(v) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ Parallelogram law

(vi) $\langle x,y\rangle = 0 \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$ Pythag

pf (i) follows from def (c)

(ii) $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\bar{\alpha}\langle x,x\rangle} = \sqrt{|\alpha|^2\langle x,x\rangle} = |\alpha|\,\|x\|$.

(vi) (Pythag) If $\langle x,y\rangle = 0$, $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \overset{0}{\cancel{\langle x,y\rangle}} + \overset{0}{\cancel{\langle y,x\rangle}} + \langle y,y\rangle$
$= \|x\|^2 + \|y\|^2$

(v) $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$
$= \langle x,x\rangle + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} + \langle y,y\rangle + \langle x,x\rangle - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle} + \langle y,y\rangle$
$= 2\langle x,x\rangle + 2\langle y,y\rangle$

(iv) Do Cauchy-Schwartz first, so we can use it for (iii) Triang Ineq.

Equality in CS $\Leftrightarrow$ $\vec{u} = k\vec{v}$

I'm giving the pf from wikipedia.

Want to show: $|\langle u,v\rangle| \leq \|u\|\,\|v\|$ Trivial if u or $v = 0$, so assume $u, v \neq 0$

Even though we haven't defined projections yet, form the vector

$$z := u - \frac{\langle u,v\rangle}{\langle v,v\rangle} v = u - P_v(u)$$ ← later we see this is O.G. proj of u onto v

$\Rightarrow u = z + P_v(u)$

▷ Now show $z \perp v$:

$$\langle z,v\rangle = \langle u - P_v(u), v\rangle = \langle u,v\rangle - \langle P_v(u), v\rangle = \langle u,v\rangle - \left\langle \frac{\langle u,v\rangle}{\langle v,v\rangle} v, v \right\rangle$$
$$= \langle u,v\rangle - \frac{\langle u,v\rangle}{\cancel{\langle v,v\rangle}} \cancel{\langle v,v\rangle} = 0$$

▷ Now use Pythag: $\vec{u} = \underbrace{\overrightarrow{P_v(u)}}_{\lambda\vec{v}} + \vec{z}$

$$\|u\|^2 = \|P_v(u)\|^2 + \|z\|^2$$
$$= \left|\frac{\langle u,v\rangle}{\langle v,v\rangle}\right|^2 \|v\|^2 + \|z\|^2$$
$$= \frac{|\langle u,v\rangle|^2}{\|v\|^{\cancel{4}2}} \cancel{\|v\|^2} + \|z\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2} \text{ since } \|z\| \geq 0$$

$\Rightarrow \|u\|^2\|v\|^2 \geq |\langle u,v\rangle|^2$ or $\boxed{|\langle u,v\rangle| \leq \|u\|\,\|v\|}$

▷ what about equality?

$z := u - P_v(u)$ so $z = 0 \Leftrightarrow u = P_v(u)$. Then $\|u\| = \|P_v(u)\| = \frac{|\langle u,v\rangle|}{\|v\|^2}\|v\| \Rightarrow \|u\|\,\|v\| = |\langle u,v\rangle|$

and $u = P_v(u) = \frac{\langle u,v\rangle}{\langle v,v\rangle} v = k\vec{v}$ scalar multiple

CONVERSELY, $u = kv \Rightarrow |\langle u,v\rangle| = |\langle kv,v\rangle| = |k|\,\|v\|^2 = |k|\,\|v\|\,\|v\| = \|kv\|\,\|v\| = \|u\|\,\|v\|$

Thus we have equality in CS $\Leftrightarrow \vec{u} = k\vec{v}$ □

cont'd

(iii) Triang Ineq $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$$= \|x\|^2 + \underbrace{2\,\mathrm{Re}(\langle x,y\rangle)}_{\leq \underbrace{2|\langle x,y\rangle|}_{\leq 2\|x\|\|y\|}} + \|y\|^2$$

$\leq 2\|x\|\|y\|$ by Cauch-Schwartz with strict ineq ($< 2\|x\|\|y\|$) iff $y \neq kx$

Thus $\|x+y\|^2 \leq \underbrace{\|x\|^2 + 2\|x\|\|y\| + \|y\|^2}_{(\|x\|+\|y\|)^2}$

Take sq root: $\boxed{\|x+y\| \leq \|x\| + \|y\|}$ and $\boxed{\|x+y\| < \|x\| + \|y\| \text{ iff } y \neq kx}$ □

Lemma p.2
(a) $x = 0 \iff \langle x,v\rangle = 0\ \forall v$
(b) $x = y \iff \langle x,v\rangle = \langle y,v\rangle\ \forall v$
(c) $\|x\| = \sup_{\|v\|=1} |\langle x,v\rangle|$

pf: (a) ($\Rightarrow$) trivial
($\Leftarrow$) $\langle x,v\rangle \overset{!}{=} 0\ \forall v$. Take $v = x$. Then $\langle x,x\rangle = 0$ which we know means $x=0$.
(b) $x = y \iff x-y=0$. Now apply (a)
(c) Take $\|v\|=1$ and apply Cauchy-Schwartz: $|\langle x,\hat{v}\rangle| \leq \|x\|\cdot 1$ and equality holds if $\hat{v} = \frac{1}{\|x\|}x$

Take $\sup_{\|v\|=1}$ of both sides: $\sup_{\|v\|=1} |\langle x,v\rangle| = \|x\|$ □

▷ **Def** Hilbert Space is an inner prod space X which is complete (any Cauchy seq converges to an elt of X).

(ex) Consider $C([0,1] \to \mathbb{C})$ with IP $\langle f,g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$

$$\|f\|_2 = \int_0^1 |f(t)|^2\,dt$$

This space is NOT complete.
But if we replace the requirement of continuous fcns by (m)able fcns, then it IS complete. It is $L^2[0,1]$
This is the Reisz-Fisher Thm.

Really EQ classes of (m)able fcns.

(ex) ℓ^2 is all $\mathbb{C}$-valued sequences (z_i) where $\sum |z_i|^2 < \infty$
Can think of this as vectors in $\mathbb{R}^\infty$ or $\mathbb{C}^\infty$ with finite norm.
Rudin RA&CA p.86: any Hilbert space is iso to $\ell^2(A)$ for some set A.

Thm 2 Unique Closest Pt

$$\left.\begin{array}{l} X \text{ Hilb sp} \\ K \subseteq X \text{ non-empty,} \\ \quad\text{Convex} \\ \quad\text{clsd (topologically)} \end{array}\right\} \Rightarrow \begin{array}{l} \forall x \in X \ \exists \text{ unique } y_x \in K \ni \\ \|x - y_x\| = \text{dist}(x, K) = \inf_{y \in K} \{\|x-y\|\} \end{array}$$

An important special case is a subsp $\rightsquigarrow$ Convex

Pf $\quad \alpha := \inf_{\text{all } y \in K} \{\|x-y\|\}$

Why does this imply $\exists$ a seq $(y_n) \to y_x$?

warm up: Let J be a bdd set in $\mathbb{R}$ (J contained in some interval). We only care about the lower end.
If $\alpha = \inf(J)$ = greatest lower bnd (J), this means for any $n \in \mathbb{N}$, $\exists\, p \in J$ where $p \in (\alpha, \alpha + \frac{1}{n})$ or else p would be a better lower bnd and α not the g.l.b.
Then, for each n, we can choose $p_n \in (\alpha, \alpha + \frac{1}{n})$ and $(p_n) \to \alpha$.

Answer:
Now we can say $\exists\ y_n$ such that $\|x - y_n\| = p_n \in (\alpha, \alpha + 1/n)$

$\triangleright$ We want to show (y_n) is a Cauchy seq. [Then we have $y_x \in K$ since X complete and K clsd]

$$\|y_n - y_m\|^2 = \|\underbrace{(y_m - x)}_{a} - \underbrace{(y_n - x)}_{b}\|^2 \quad \text{add \& subtr } x$$

parallelogram law: $|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2$
$|a-b|^2 = 2|a|^2 + 2|b|^2 - |a+b|^2$

$$= 2\|y_n - x\| + 2\|y_m - x\| - \underbrace{\|y_m + y_n - 2x\|^2}$$

factor out 2: $\left\|2\left[\frac{1}{2}(y_n+y_m) - x\right]\right\|^2 = 4\left\|\underline{\tfrac{1}{2}(y_n+y_m)} - x\right\|^2$

This pt is in K by convexity for any m,n

Then since α is inf: $\quad \alpha \le \left\|\frac{1}{2}(y_n + y_m) - x\right\|$

$$\le 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\alpha^2$$ we are subtracting a smaller pos quantity.

Take lim:

$$\lim_{n,m\to\infty} \|y_n - y_m\|^2 \le \lim_{n,m} \Big[2\underbrace{\|y_n - x\|}_{2\alpha} + 2\underbrace{\|y_m - x\|}_{2\alpha} \underbrace{-4\alpha^2}_{-4\alpha}\Big] = 0$$

Thus (y_n) is Cauchy and its limit pt exists in K: $(y_n) \to y_x$

$$\|x - y_x\| = \underbrace{\|x - \lim y_n\|}_{\text{continuity of } \|\cdot\|} = \lim_{n\to\infty} |x - y_n| = \alpha \quad \text{by def of } y_n$$

$\triangleright$ Now we must show y_x is unique. [From Simmonds ITAMA p. 247]

$\$\ \exists$ another pt $z_x \ni \|x - z_x\| = \alpha$. Then $\frac{1}{2}y_x + \frac{1}{2}z_x =: g_x \in K$ by convexity.

$$|g_x - x|^2 = \left|\tfrac{1}{2}y_x + \tfrac{1}{2}z_x - x\right|^2 = \left|\tfrac{1}{2}(y_x - x) + \tfrac{1}{2}(z_x - x)\right|^2 = 2\left|\tfrac{1}{2}(y_x - x)\right|^2 + 2\left|\tfrac{1}{2}(z_x - x)\right|^2 - \underbrace{\left|\tfrac{1}{2}(y_x - x) - \tfrac{1}{2}(z_x - x)\right|^2}_{\left|\frac{1}{2}y_x - \frac{1}{2}z_x\right|^2}$$

$$= \tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha^2 + \tfrac{1}{4}|y_x - z_x|^2$$

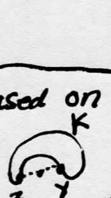
$$< \alpha^2 \text{ if } y_x \neq z_x$$

$\Rightarrow |g_x - x| < \alpha \quad \Rightarrow\Leftarrow$ since α is inf

$\Rightarrow z_x = y_x$

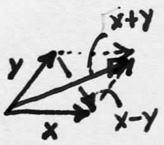
$\boxed{\text{QED}}$

Remark: A. Friedman FOMA p. 202, 205 tries to give a pf based on the Hilb sp norm being strictly convex: $\left|\frac{x+y}{2}\right| < \frac{1}{2}|x| + \frac{1}{2}|y|$
But it is NOT when $y = kx$ and I can't rule out this case



Thm 5 Parallelogram law holds in n.l.s. X $\iff$ X is inner prod sp

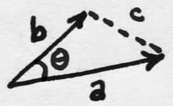
$$\boxed{\left(\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2\right)}$$



pf ($\Leftarrow$) follows from IP $\langle\cdot,\cdot\rangle$ defining a norm p.1

($\Rightarrow$) skipped here.

▷ Define the angle between 2 vectors via IP: $\|x-y\|^2 = \langle x-y, x-y\rangle$
$= \|x\|^2+\|y\|^2-2\mathrm{Re}\langle x,y\rangle$



Law of Cosine: $c^2 = a^2+b^2-2ab\cos\theta$

$$|x-y|^2 = |x|^2+|y|^2-2|x||y|\cos\theta$$

$$\Rightarrow |x||y|\cos\theta = \mathrm{Re}(\langle x,y\rangle)$$

$$\boxed{\theta = \mathrm{Cos}^{-1}\left(\frac{\mathrm{Re}(\langle x,y\rangle)}{|x||y|}\right)}$$

Here are some problems from the end of ch 3.1: To solve them, we need to refer to B. Friedman PATOAM p. 29-31 but avoid using his bad notation as much as possible.

(20) Find all sol'ns to $a\langle b,x\rangle = f$

(22) $x + a\langle b,x\rangle = f$

(23) specific example: $x(s) + s\int_0^1 x(t)t^2\,dt = \cos(s)$

Here is the presentation from B. Friedman:

(a) $a\langle b,x\rangle = f$ $\quad Lx = f$ where $L = a\langle b,\cdot\rangle$

a sol'n is only possible if $f = ya$

In that case, $\langle b,x\rangle \overset{!}{=} y$

One soln is $x = \frac{y}{\|b\|^2}b$

and, more generally, if c is any vector $\ni \langle b,c\rangle = 0$, ($c\in\ker(L)$), then $\boxed{x = \frac{y}{\|b\|^2}b + c}$ is a sol'n.

(b) $a_1\langle b_1,x\rangle + a_2\langle b_2,x\rangle + \dots + a_k\langle b_k,x\rangle = f$

$L := \sum a_i\langle b_i,\cdot\rangle$

We assume $f\in\mathrm{Span}\{a_1,\dots,a_k\}$ i.e. $f = \sum^k y_i a_i$

Then we seek $x \ni \langle b_i,x\rangle = y_i$

Now assume $x = \sum^k \xi_i b_i$

This leads us to solving

$$\begin{bmatrix}\langle b_1,b_1\rangle & \cdots & \langle b_1,b_k\rangle\\ \vdots & & \\ \langle b_k,b_1\rangle & \cdots & \langle b_k,b_k\rangle\end{bmatrix}\begin{bmatrix}\xi_1\\ \vdots\\ \xi_k\end{bmatrix} = \begin{bmatrix}y_1\\ \vdots\\ y_k\end{bmatrix}$$

and we can add on any $c\in\ker(L)$

Cast this in familiar terms as vectors in $\mathbb{R}^n$:

$ab^Tx = f$

Define $y = b^Tx \in\mathbb{R}$

$\vec{a}y = \vec{f}$

and $\vec{x} = \vec{b}\xi$

$\mathbb{R}^k$

$\sum^k \vec{a}_i b_i^T x = f \quad AB^Tx = f$

$$\begin{bmatrix}| & & |\\ a_1 & \cdots & a_k\\ | & & |\end{bmatrix}\begin{bmatrix}- & b_1 & -\\ & \vdots & \\ - & b_k & -\end{bmatrix}\begin{bmatrix}|\\ x\\ |\end{bmatrix} = \begin{bmatrix}|\\ f\\ |\end{bmatrix}$$

Let $y := B^Tx$

Then $Ay = f \iff f\in\mathrm{Col}(A)$

We now assume $x = B\xi$ and seek to solve

$$(B^TB)\xi = Y$$

$L = I(\cdot) + a\langle b, \cdot\rangle$

(c) $x + a\langle b, x\rangle = f$ $\qquad\qquad$ $x + ab^T x = f$

apply $\langle b, \cdot\rangle$ to both sides:

$\langle b, x\rangle + \langle b, a\rangle\langle b, x\rangle = \langle b, f\rangle$

Solve for $\langle b, x\rangle$:

$$\langle b, x\rangle\left[1 + \langle b, a\rangle\right] = \langle b, f\rangle$$

$$\langle b, x\rangle = \frac{\langle b, f\rangle}{\left[1 + \langle b, a\rangle\right]} \quad \text{provided } \left[1 + \langle b, a\rangle\right] \neq 0$$

[We can solve this for $x = \frac{1}{1+\langle b,a\rangle} f$, but that is NOT a soln of orig eq!]

Instead, plug it back in to orig eq:

$$x + a\frac{\langle b, f\rangle}{\left[1 + \langle b, a\rangle\right]} = f$$

$$\Rightarrow \boxed{x = f - \frac{\langle b, f\rangle}{\left[1 + \langle b, a\rangle\right]} a} \quad \text{is soln}$$

If, however, $1 + \langle b, a\rangle = 0$ then

$\langle b, x\rangle\left[1 + \langle b, a\rangle\right] = \langle b, f\rangle$ is $\langle b, x\rangle\, 0 = \langle b, f\rangle$

so there is no soln unless $\langle b, f\rangle = 0$. In this case, $x = f$ is a soln, but more generally $x = f + \alpha a$ is a soln (since $1 + \langle b, a\rangle = 0 \Rightarrow \langle b, a\rangle = -1$):

$$f + \alpha a + a\big[\underbrace{\langle b, f + \alpha a\rangle}_{\langle b,f\rangle + \alpha \underbrace{\langle b,a\rangle}_{-1}}\big] \stackrel{?}{=} f \quad \Rightarrow f + \alpha a + a\left[0 + -\alpha\right] = f \;\checkmark$$

(d) $x + \sum a_i\langle b_i, x\rangle = f$

Lets take $k=3$ and successively apply $\langle b_i, \cdot\rangle$ for $i = 1, 2, 3$

$$\langle b_1, x\rangle + \langle b_1, a_1\rangle\langle b_1, x\rangle + \langle b_1, a_2\rangle\langle b_2, x\rangle + \langle b_1, a_3\rangle\langle b_3, x\rangle = \langle b_1, f\rangle$$

$$\langle b_2, x\rangle + \langle b_2, a_1\rangle\langle b_1, x\rangle + \langle b_2, a_2\rangle\langle b_2, x\rangle + \langle b_2, a_3\rangle\langle b_3, x\rangle = \langle b_2, f\rangle$$

$$\langle b_3, x\rangle + \langle b_3, a_1\rangle\langle b_1, x\rangle + \langle b_3, a_2\rangle\langle b_2, x\rangle + \langle b_3, a_3\rangle\langle b_3, x\rangle = \langle b_3, f\rangle$$

$$(1 + \langle b_1, a_1\rangle)\overset{y_1}{\langle b_1, x\rangle} + \langle b_1, a_2\rangle\overset{y_2}{\langle b_2, x\rangle} + \langle b_1, a_3\rangle\overset{y_3}{\langle b_3, x\rangle} = \langle b_1, f\rangle$$

$$\langle b_2, a_1\rangle\langle b_1, x\rangle + (1 + \langle b_2, a_2\rangle)\langle b_2, x\rangle + \langle b_2, a_3\rangle\langle b_3, x\rangle = \langle b_2, f\rangle$$

$$\langle b_3, a_1\rangle\langle b_1, x\rangle + \langle b_3, a_2\rangle\langle b_2, x\rangle + (1 + \langle b_3, a_3\rangle)\langle b_3, x\rangle = \langle b_3, f\rangle$$

$$\begin{bmatrix} (1 + \langle b_1, a_1\rangle) & \langle b_1, a_2\rangle & \langle b_1, a_3\rangle \\ \langle b_2, a_1\rangle & (1 + \langle b_2, a_2\rangle) & \langle b_2, a_3\rangle \\ \langle b_3, a_1\rangle & \langle b_3, a_2\rangle & (1 + \langle b_3, a_3\rangle) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \langle b_1, f\rangle \\ \langle b_2, f\rangle \\ \langle b_3, f\rangle \end{bmatrix}$$