Chapter 2

Theory of Periodic Orbits

2.1 Persistence of Periodic Orbits and Lyapunov's Center Theorem

In doing numerical work, there is always the danger that the numbers the computer is producing are not related to the mathematical problem, but have somehow become corrupted. Extraneous solutions sometimes can be created by the numerical algorithm, for example. Therefore, whenever possible, we should back up the computations with theory. The halo orbits originate as a pair of families branching off from the planar Lyapunov family of L_2. In the next chapter, we will find the halo orbits by numerically continuing the planar family until we find the point of bifurcation. We do not have the methods to be able to *prove* mathematically[1] that the halo orbits exist, but there is a classical theory to prove the existence of the planar family, at least close to L_2.

Over a hundred years ago, A. M. Lyapunov proved that, assuming certain technical conditions, there exists a family of "small"[2] periodic orbits around a fixed point of a Hamiltonian vector field. The idea is that periodic orbits for the linearized vector field persist for the full nonlinear field, at least

[1] In principle, techniques exists that may enable one to do this. See the paper by Koch, Shenkel, and Wittwer "Computer Assisted Proofs in Analysis and Programming in Logic: A Case Study". UT Austin electronic Mathematical Physics Archives #94 – 394.

[2] It would be interesting, to determine bounds for how large the "small" orbits can be.

close to the fixed point. After we have established this result, we will apply it to the fixed point L_2.

First we give a general result, not restricted to Hamiltonian systems. The exposition follows Abraham & Marsden [1985 p. 496], who in turn followed the more general work of Duistermaat [1972], also [1983].

Theorem 1 (Persistence of Periodic Orbits) *Let $\epsilon \in [0, \epsilon_0]$, $\{X_\epsilon\}$ a smooth family of vector fields on a manifold M, with $\{F_t^\epsilon\}$ the associated flows.*

Let $\{H_\epsilon\}$ be a corresponding family of smooth first integrals.

Let γ be a periodic orbit for X_0 of period $T_0 > 0$, passing though the point $x_0 \in M$, and lying on the energy surface $\Sigma_{e_0} = H_0^{-1}(e_0)$, where $e_0 = H_0(x_0)$.

Let S_{x_0} be a local transverse Poincaré section to γ at x_0, which will be specified during the proof. Assume, furthermore, the following conditions:

1. *$ker[D(F_{T_0}^0)_{x_0} - Id]$ is 2 dimensional.*
2. *$X_0(x_0) \notin Image[D(F_{T_0}^0)_{x_0} - Id]$*

THEN *for $\epsilon, \mid e - e_0 \mid$ sufficiently small, there exists uniquely a periodic point $x_{e,\epsilon} \in S_{x_0}$ near x_0 , and a time value $T_{e,\epsilon}$ near T_0 such that the solution $F_{x_{e,\epsilon}}^\epsilon$ of X_ϵ is periodic with period $T_{e,\epsilon}$ and energy e.*

Proof. This is basically an exercise in using the implicit function theorem.

We know there exists a chart ($\alpha : V \to U, x_0 \in V, U \subset \mathbb{R}^n$), such that in the chart image $\Sigma_{e_0,0} := H_0^{-1}(e_0) = (\mathbb{R}^{n-1} \times \{0\}) \bigcap U$. (If M is a symplectic manifold, we can take α to be a symplectic chart). We are going to work in the image of this chart from here on.

Define the map

$$\begin{aligned} \Theta : U \times (0, \infty) \times [0, \epsilon_0) &\longrightarrow \mathbb{R}^n \\ (x, T, \epsilon) &\longmapsto F_T^\epsilon(x) - x \end{aligned}$$

A zero of the map Θ corresponds to a periodic orbit; we note that $\Theta(x_0, T_0, 0) = 0$. The first idea one might think of is to use the implicit function theorem to find more zeroes, but it turns out that we don't have strong enough hypotheses to make this work; we have to reduce the dimension of the domain, for one thing. So we are going to define a variant of Θ shortly. But first, it will be convenient for us to rename $\Sigma_{e_0,0}$ as Z and write $\bar{x}$ for $(x_1, \ldots, x_{n-1}, 0) \in Z = (\mathbb{R}^{n-1} \times \{0\}) \bigcap U$.

Claim 1: For ϵ near 0, e near e_0, and $\bar{x}$ near $\bar{x_0}$, we can express $H_\epsilon^{-1}(e)$ as the graph of a function $\tilde{\phi}(\bar{x}, e, \epsilon)$.

We use the implicit function theorem to prove this. Let us write the function $H_\epsilon(x)$ as $H((\bar{x}, \epsilon), x_n)$, so it looks like the form always used for the implicit function theorem. We know $H((\bar{x_0}, 0), 0) = e_0$. We also have $D_{x_n} H(\bar{x_0}, 0, 0) : \mathbb{R} \to \mathbb{R}$ is an isomorphism, because it is the last component of the gradient vector; since the gradient must be orthogonal to level surfaces, in particular the level surface Z, we must have its n-th component nonzero.

Then by the implicit function theorem, locally there is a real valued function $\tilde{\phi}$ such that $H[(\bar{x}, \epsilon), \tilde{\phi}((\bar{x}, \epsilon), e)] = e$

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$\diamond$

Then we shrink Z to the open set around $(\bar{x_0})$ proven to exist in the claim, and we define

$$\begin{aligned} \phi_{e,\epsilon} : Z &\longrightarrow \mathbb{R}^n \\ \bar{x} &\longmapsto (\bar{x}, \tilde{\phi}(\bar{x}, e, \epsilon)) \end{aligned}$$

Let $\pi : \mathbb{R}^n \longrightarrow Z$ be the standard projection. Let I_T denote an open interval around T_0, I_e an open interval around e_0, and $I_\epsilon = [0, \epsilon_0)$. Then define

$$\begin{aligned} \Psi : Z \times I_T \times I_e \times I_\epsilon &\longrightarrow Z \\ (\bar{x}, T, e, \epsilon) &\longmapsto \pi \circ \Theta(\phi(\bar{x}, e, \epsilon), T, e) = \pi(F_T^\epsilon(x) - x) \end{aligned}$$

Since $\Sigma_{e,\epsilon}$ is a graph, $\pi : \Sigma_{e,\epsilon} \longrightarrow Z$ is one to one, which means that Θ has a zero if and only if Ψ has a zero.

Let $p = (\bar{x_0}, T_0, e_0, 0)$ in the domain of Ψ. The plan now is to show that $D_{(\bar{x},T)}\Psi_p : L \times \mathbb{R} \longrightarrow Z$ is an isomorphism, where L is a hyperplane in Z of codimension 1, transverse to $X_0(x_0)$. Then we can appeal to the implicit function theorem again, and the proof is completed. Note that

$$D_{(\bar{x},T)}\Psi_p(h,t) = D_{\bar{x}}\Psi_p(h) + D_T\Psi_p(t)$$

We will show that $D_{\bar{x}}\Psi_p$ is one to one on the subspace L of Z, and that the image of $D_T\Psi_p$ is a 1-dim subspace that is not contained in the image of $D_{\bar{x}}\Psi_p$.

By direct computation we find,

$$D_{\bar{x}}\Psi_p = (D(F_{T_0}^0)_{x_0} - Id)\mid_Z$$

Now we show $\ker[D_{\bar{x}}\Psi_p] = \text{Span}\{X_0(x_0)\}$.

Observe $\ker[D_{\bar{x}}\Psi_p] = \ker[(D(F^0_{T_0})_{x_0} - Id)\mid_Z] = \ker[D(F^0_{T_0})_{x_0} - Id] \bigcap Z$. Also note that $\ker[DH_{x_0}] = T_{x_0}\Sigma_{e_0,0} = Z$ where we use the representation of the tangent space of Σ as the hyperplane in $\mathbb{R}^n$ tangent to $\Sigma = Z$ at the point x_0.

Claim 2: We can use conditions (1) and (2) to establish

$$\ker[D(F^0_{T_0})_{x_0} - Id] \bigcap \ker[DH_{x_0}] = \text{Span } \{X_0(x_0)\}.$$

Claim 2 will be proven at the end. From the above argument, we have established $\ker[D_{\bar{x}}\Psi_p] = \text{Span}\{X_0(x_0)\}$. Hence we can choose an arbitrary subspace L transverse to $X_0(x_0)$ and write $Z = L \oplus \text{Span } \{X_0(x_0)\}$. $D_{\bar{x}}\Psi_p$ is one to one on L and its image has dimension $n-2$.

By another direct computation we find, $D_T\Psi_p = X_0(x_0)$. Then from condition (2), it is apparent that $D_T\Psi_p$ spans a subspace that is not contained in the image of $D_{\bar{x}}\Psi_p$.

At last we are ready to apply the implicit function theorem, and we apply it to the map $\hat{\Psi}$ defined as Ψ restricted to $L \times I_T \times I_e \times I_\epsilon$. We have $\hat{\Psi}(p) = 0$ and $D_{\hat{\Psi}_p}$ is an isomorphism. The implicit function theorem gives us that in a neighborhood of $(e_0, \epsilon = 0)$ there exist smooth functions $\bar{x}(e,\epsilon), T(e,\epsilon)$ such that $\hat{\Psi}(\bar{x}(e,\epsilon), T(e,\epsilon), e, \epsilon) = 0$. Then $x(e,\epsilon) = \phi(\bar{x}(e,\epsilon), e, \epsilon)$ is the periodic point we seek in the local Poincaré section L.

This completes the proof, except for establishing Claim 2, which we now do. First, it is easy to see that $D(F^0_{T_0})_{x_0}(X_0(x_0)) = X_0(x_0)$. Therefore

$X_0(x_0) \in \ker[D(F^0_{T_0})_{x_0} - Id]$. Second, by conservation of energy, trajectories that start in $\Sigma_{e,\epsilon}$ must stay in it, hence $X_0(x_0) \in T_{x_0}\Sigma_{e,\epsilon} = Z$, and so $DH_{x_0}(X_0(x_0)) = 0$. Observe that these two statements imply that

$$X_0(x_0) \in \ker[D(F^0_{T_0})_{x_0} - Id] \bigcap \ker[DH_{x_0}]$$

and thus the dimension of the intersection is greater than or equal to 1. Now we show its dimension can be at most 1.

Since H is constant along trajectories, we know $(H \circ F_{T_0})(x) = H(x)$. Then $DH_{F_{T_o}} \cdot D(F_{To})_x = DH_x$. We take $x = x_0$, and because $F_{T_0}(x_0) = x_0$ we get

$$DH_{x_0}[D(F^0_{T_0})_{x_0} - Id] = 0$$

and thus the image of $D(F^0_{T_0})_{x_0} - Id$ is also contained in $\ker[DH_{x_0}]$. By the rank + nullity theorem, Image[$D(F^0_{T_0})_{x_0} - Id$] has dimension $n-2$. Combining this with Span $\{X_0(x_0)\}$, we have used up all of $\ker[DH_{x_0}]$, and therefore we have established

$$\ker[D(F^0_{T_0})_{x_0} - Id] \bigcap \ker[DH_{x_0}] = \text{Span } \{X_0(x_0)\}.$$

□

Now we are ready to use this theorem to prove Lyapunov's Center theorem,[3] so named because of a two dimensional eigenspace where the lin-

[3]The reader should be away that there are many versions of this theorem, and also strengthenings, such as Al Kelley's paper "On the Liapunov Sub-Center Manifold" in the first edition of Abraham & Marsden, 1967.

earized vector field has a center-type fixed point; we therefore get a center manifold, which contains the family of periodic orbits.

Theorem 2 (Lyapunov Center Theorem) *Let (M, ω) be a symplectic manifold, with a Hamiltonian function $H \in C^l(M \to \mathbb{R})$, $l \geq 2$. Let m_0 be a critical point of H.*

Assume the linearized vector field $(X'_H)_{m_0}$ has as eigenvalues

$$\{\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n\}^4$$

satisfying the conditions

1. *$\lambda_1 = i\alpha_1$ for some $\alpha_1 > 0$.*
2. *$\lambda_j \neq k\lambda_1$ for any integer k, and $j = 2, \ldots n$. (This is called the* **nonresonance condition***).*

THEN *there exists a 1-parameter family $\{\gamma_\epsilon\}, \epsilon \in [0, \epsilon_0]$ of closed orbits of X_H.*

The orbit γ_ϵ shrinks to the point m_0 as $\epsilon \longrightarrow 0$, and Period$(\gamma_\epsilon) \longrightarrow \frac{2\pi}{\alpha_1}$.

Proof. Throughout this proof we work exclusively in the image of a symplectic chart mapping into $\mathbb{R}^{2n}$. We can take $m_0 = 0$ and, by adding an inessential constant, we can have $H(m_0) = 0$.

Since the first two terms vanish, the Taylor expansion of H about the point 0 is just:

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[4] The eigenvalues come in positive and negative pairs because $(X'_H)_{m_0}$ is infinitesimally symplectic.

$$H(x) = \tfrac{1}{2!}D^2H_0(x,x) + R_3(0;x)$$

We now introduce a "blowing up", or rescaling parameter ϵ:

$$\begin{aligned}\mathcal{H}(x,\epsilon) &:= \frac{1}{\epsilon^2}H(\epsilon x)\\ &= \frac{1}{2}D^2H_0(x,x) + \frac{1}{\epsilon^2}R_3(0;\epsilon x)\\ &= H_2(x) + h(x,\epsilon)\end{aligned}$$

The Hamiltonian vector field of $\mathcal{H}_\epsilon$, which we will denote X_ϵ, corresponds to what we had in the Persistence of Periodic Orbits theorem. As will be seen shortly, we can find a specific periodic orbit for the linearized vector field corresponding to $\epsilon = 0$.

It is easy to see that when $\epsilon = 0$, the Hamiltonian vector field X_0 of $\mathcal{H}$ equals the linearized vector field $(X'_H)_{m_0}$ of H. Let us define $A := X_0$; it is linear and has the eigenvalues specified in the theorem statement. We now show there exists a linear, symplectic change of variables so that H_2 takes the form

$$H_2(q,p) \;=\; \frac{1}{2}\alpha_1[q_1^2 + p_1^2] + \tilde{H}_2(q_2,\ldots,q_n,p_2,\ldots,p_n)$$

For the eigenvalue $\lambda_1 = i\alpha_1$, we have an eigenvector $\mathbf{v_1} \in \mathbb{C}^{2n}$; let $\mathbf{v_1} = \mathbf{e} + i\mathbf{f}$. Note $A\bar{\mathbf{v}}_1 = -i\alpha_1\bar{\mathbf{v}}_1$ since the entries of A are real. By comparing real and imaginary parts, we obtain the eigenvector relations

$$\begin{aligned}A\mathbf{e} &= -\alpha_1\mathbf{f}\\ A\mathbf{f} &= \alpha_1\mathbf{e}\end{aligned}$$

The goal now is to find an ordered basis for $\mathbb{R}^{2n}$,

$$\mathcal{B} = [\mathbf{e}, \mathbf{b_2}, \ldots, \mathbf{b_n}, \mathbf{f}, \ldots, \mathbf{b_{2n}}]$$

such that

$$\mathbf{A}[\mathbf{e}, \mathbf{b_2}, \ldots, \mathbf{b_n}, \mathbf{f}, \ldots, \mathbf{b_{2n}}] =$$

$$[\mathbf{e}, \mathbf{b_2}, \ldots, \mathbf{b_n}, \mathbf{f}, \ldots, \mathbf{b_{2n}}] \begin{bmatrix} 0 & \cdots & 0 & \alpha_1 & 0 & \cdots \\ \vdots & * & * & 0 & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_1 & 0 & \cdots & 0 & \cdots & \cdots \\ 0 & * & * & 0 & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

while preserving $[\omega]_{\mathcal{B}} = J$. Denote the matrix on the right hand side $[A]_{\mathcal{B}}$.

Since $\mathbf{v_1}$ and $\bar{\mathbf{v_1}}$ come from distinct eigenvalues, they are linearly independent. From this it can be shown $\mathbf{e}$ and $\mathbf{f}$ are linearly independent. Define $E := \text{Span}\{\mathbf{e}, \mathbf{f}\}$. Obviously, $A\mid_E\colon E \longrightarrow E$. Let W be the ω-orthogonal complement in $\mathbb{R}^{2n}$. We will now show that $\omega(\mathbf{e}, \mathbf{f}) \neq 0$. By construction of the W, $\omega(\mathbf{e}, \mathbf{v}) \neq 0$ for all $\mathbf{v} \in W$. Therefore, the nondegeneracy of ω forces the existence of $\mathbf{v} \in E$ such that $\omega(\mathbf{e}, \mathbf{v}) \neq 0$. Since $\mathbf{e}, \mathbf{f}$ are a basis, $\mathbf{v} = \theta\mathbf{e} + \phi\mathbf{f}$. Thus since $\omega(\mathbf{e}, \mathbf{e}) = 0$ we must have $\omega(\mathbf{e}, \mathbf{f}) \neq 0$. By rescaling and relabeling, if necessary, we can arrange it so that $\omega(\mathbf{e}, \mathbf{f}) = 1$, while preserving the form of the eigenvector relations.

We can then follow the standard linear algebra procedure for inductively proving the existence of a symplectic basis and determine the basis $\mathcal{B}$ such that $[\omega]_{\mathcal{B}} = J$. The crucial zero rows and columns in $[A]_{\mathcal{B}}$ result from A being invariant on E and W. Since the form ω has the matrix representation

J in both the old and the new bases, the co-ordinate map to the new basis is symplectic.

We compute the form of the Hamiltonian associated to R by the formula

$$\begin{aligned} H_2(x) &= \frac{1}{2}\omega(Ax, x) \\ &= \frac{1}{2}([A]_{\mathcal{B}}x)^T Jx \\ &= \frac{1}{2}\alpha_1[q_1^2 + p_1^2] + \tilde{H}_2(q_2, \ldots, q_n, p_2, \ldots, p_n) \end{aligned}$$

Now that we have a nice form for our Hamiltonian, it is easy to find periodic orbits. Hamilton's canonical equations, for the q_1 and p_1 components only, lead us to two harmonic oscillator equations, which can be easily solved to yield

$$\begin{aligned} q_1(t) &= q_1(0)\cos(\alpha_1 t) - p_1(0)\sin(\alpha_1 t) \\ p_1(t) &= p_1(0)\cos(\alpha_1 t) - q_1(0)\sin(\alpha_1 t) \end{aligned}$$

By choosing initial conditions $q_1(0) = 1$ and $p_1(0) = 0$ we get a nice circular orbit γ_0 that is the circle $q_1^2 + p_1^2 = 1$, having period $T_0 = \frac{2\pi}{\alpha_1}$. Choose an arbitrary point $x_0 \in \gamma_0$.

Let us verify conditions (1) and (2) of the Persistence of Periodic Orbits theorem. Recall F_t^ϵ is the flow generated by X_ϵ.

Condition 1: $\ker[D(F_{T_0}^0)_{x_0} - Id]$ is 2 dimensional.

To establish this we will need

Claim 1 *The eigenspace E corresponding to the eigenvalues $i\alpha, -i\alpha$ is precisely $ker[e^{T_0\mathbf{A}} - I]$.*

Since we have complex eigenvalues, we must allow the domain of A to be $\mathbb{C}^{2n}$. Let A_C denote $A : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$. Define $N_C :=$ker$[e^{T_0\mathbf{A_C}} - I]$. The general solution to $\dot{x} = A_C x$ is $\phi_{x_0}(t) = e^{t\mathbf{A}}x_0$. Then we see $x_0 \in N_C \Longleftrightarrow x_0$ is an initial condition that produces a periodic orbit ϕ_{x_0} of period kT_0 for some positive integer k.

We know that all periodic solutions of $\dot{x} = A_C x$ have initial conditions in the (complexified) eigenspace associated with the purely imaginary eigenvalues of A_C. If we now require the solutions to be T_0 - periodic, any x_0 that is a linear combination of $\mathbf{v_1}$ and $\bar{\mathbf{v_1}}$ qualifies, because since $T_0 = \frac{2\pi}{\alpha_1}$

$$e^{T_0\mathbf{A}}\mathbf{v_1} = e^{i2\pi}\mathbf{v_1} = \mathbf{v_1}.$$

The same idea hold for $\bar{\mathbf{v_1}}$. We now have, in fact, *all* possible T_0-periodic solutions, because of the nonresonance condition: suppose y were an element of the eigenspace associated with some eigenvalue $i\beta$ and $e^{TA}y = y$. Then we must have $Ay = i\beta y$ i.e. $TAy = Ti\beta y = \frac{2\pi}{\alpha_1}i\beta y$. Thus

$$e^{TA}y = e^{i2\pi\frac{\beta}{\alpha_1}} \Rightarrow e^{i2\pi\frac{\beta}{\alpha_1}} = 1 \Rightarrow i\beta = ki\alpha_1$$

for some integer k. But this violates the nonresonance condition. Thus we have shown $N_C =$Span$\{\mathbf{v_1}, \bar{\mathbf{v_1}}\}$.

But we are interested in only real initial conditions. Let N_R denote the real part of N_C. We must show the dimension of N_R is 2. Choose any

$w \in N_C$.

$$w = \gamma v_1 + \delta \bar{v}_1$$

$$w_1 + iw_2 = (\gamma + i\hat{\gamma})(e + if) + (\delta + i\hat{\delta})(e - if)$$

The real part of this is $w_1 = (\gamma+\delta)e + (\hat{\delta}-\hat{\gamma})f$. Since w_1 is arbitrary, this shows e and f span N_R. But e and f are also linearly independent, which means N_R =Span$\{e, f\} = E$, thus the dimension is 2 and the claim, as well as condition (1) of the Persistence of Periodic Orbits theorem, is verified.

End Claim $\diamond$

Condition 2. Now we must establish that $Ax_0 \notin \text{Image}[e^{T_0\mathbf{A}} - I]$. Since x_0 is a periodic point, $x_0 \in E$. Let $Ax_0 \neq 0$. Since $A\mid_E: E \longrightarrow E$, $A_{x_0} \in E$. We also know $A\mid_W: W \longrightarrow W$. For a contradiction, suppose there is a $u \in \mathbb{R}^{2n} = E \oplus W$ such that $(e^{T_0\mathbf{A}} - I)u = Ax_0$. It is easy to see that $(e^{T_0\mathbf{A}} - I)$ is invariant on both of the subspaces E and W, thus for $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_W$,

$$[e^{T_0\mathbf{A}} - I](u_E + u_W) = [e^{T_0\mathbf{A}} - I]u_W = Ax_0$$

But this is a contradiction because the left side of the last equality is in W, while the right side is in E, and we specified $Ax_0 \neq 0$. Therefore, condition (2) is established, and we can apply the theorem.

We now know: For $\epsilon, \mid e - H(x_0) \mid$ sufficiently small, there exists a unique periodic point $x_{e,\epsilon}$ near x_0 with period T near T_0 such that the solution $F_{x_{e,\epsilon}}$ of X_ϵ is $T-$periodic with energy e. The proof will be finished after we show if $F_{x_{e,\epsilon}}$ is a periodic orbit for X_ϵ, then $\epsilon \cdot F_{x_{e,\epsilon}}$ is a periodic orbit for X_H.

For convenience, let us drop the subscripts on $x_{e,\epsilon}$. Recall $\mathcal{H}_\epsilon = \frac{1}{\epsilon^2}H(\epsilon x)$. Then

$$\begin{aligned}
\frac{d}{dt}(F^\epsilon(t,x)) &= X_{\mathcal{H}_\epsilon}(F^\epsilon(t,x)) \\
&= J\nabla\mathcal{H}_\epsilon(F^\epsilon(t,x)) \\
&= J\nabla(\frac{1}{\epsilon^2}H\circ E)(F^\epsilon(t,x)) \\
&= \frac{1}{\epsilon^2}J\nabla H(\epsilon F^\epsilon(t,x))\epsilon \\
&= \frac{1}{\epsilon}J\nabla H(F^\epsilon(t,x)) \\
\frac{d}{dt}(\epsilon F^\epsilon(t,x)) &= J\nabla H(\epsilon F^\epsilon(t,x))
\end{aligned}$$

and thus $\epsilon F^\epsilon(t,x)$ is a periodic orbit for X_H.

Q.E.D.

2.2 Application to L_2

Theorem 3 *The collinear Lagrange points are unstable. Each has a one parameter family of closed orbits around it, together with a stable and unstable manifold.*

To prove this theorem, we must compute the eigenvalues of the linearized vector field. We can work with either X_H or X; since they are related by the Legendre transformation the eigenvalues are the same. To be consistent with the two dimensional version in Abraham & Marsden [1985 pp.683 - 685], we choose to work with X.

Let us recall

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Then we compute the linearized vector field[5], where we abbreviate $D_{ij}\Omega = \Omega_{ij}$

$$DX_{(x,y,z,\dot{x},\dot{y},\dot{z})} = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ \Omega_{11} & \Omega_{12} & \Omega_{13} & & -2 & \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & 2 & & \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & & & \end{bmatrix}$$

For convenience computing the eigenvalues, we would like to express this matrix as a block diagonal matrix, where one block corresponds to the planar R3BP. To accomplish this, we form the new matrix $P \circ D(X) \circ P^{-1}$, where P is the permutation matrix

$$P = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & 1 & & & \\ & & & & & 1 \end{bmatrix}$$

and we obtain

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \Omega_{11} & \Omega_{21} & 0 & -2 & \Omega_{13} & 0 \\ \Omega_{21} & \Omega_{22} & 2 & 0 & \Omega_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \Omega_{31} & \Omega_{32} & 0 & 0 & \Omega_{33} & 0 \end{bmatrix} \qquad (2.1)$$

At L_2, $\Omega_{ij} = 0$ if $i \neq j$

So we lose annoying terms.

[5]We should remark that this also defines the first order variational equations, which we work with extensively in the numerical computations later on.

It is straightforward to compute the partial derivatives of Ω, and many of them turn out to be zero when evaluated at L_2.

$$
\begin{aligned}
D_1\Omega_{(x,y,z)} &= x - \frac{(1-\mu)(x-x_e)}{r_1^3} - \frac{(\mu)(x-x_m)}{r_2^3} \\
D_2\Omega_{(x,y,z)} &= y[1 - \frac{(1-\mu)}{r_1^3} - \frac{(\mu)}{r_2^3}] \\
D_3\Omega_{(x,y,z)} &= -z[\frac{(1-\mu)}{r_1^3} + \frac{(\mu)}{r_2^3}] \\
D_{11}\Omega_{(x,y,z)} &= 1 - \frac{(1-\mu)}{r_1^3} - \frac{(\mu)}{r_2^3} + 3[\frac{(1-\mu)(x-x_e)^2}{r_1^5} + \frac{(\mu)(x-x_m)^2}{r_2^5}] \\
D_{12}\Omega_{(x,y,z)} &= 3y[\frac{(1-\mu)(x-x_e)}{r_1^5} + \frac{(\mu)(x-x_m)^2}{r_2^5}] \\
D_{13}\Omega_{(x,y,z)} &= 3z[\frac{(1-\mu)(x-x_e)}{r_1^5} + \frac{(\mu)(x-x_m)}{r_2^5}] \\
D_{23}\Omega_{(x,y,z)} &= -3yz[\frac{(1-\mu)}{r_1^5} \frac{(\mu)(x-x_m)}{r_2^5}] \\
D_{22}\Omega_{(x,y,z)} &= 1 - \frac{(1-\mu)}{r_1^3} - \frac{(\mu)}{r_2^3} - 3y^2[\frac{(1-\mu)}{r_1^5} + \frac{\mu}{r_2^5}] \\
D_{33}\Omega_{(x,y,z)} &= -\frac{(1-\mu)}{r_1^3} - \frac{(\mu)}{r_2^3} - 3z^2[\frac{(1-\mu)}{r_1^5} + \frac{\mu}{r_2^5}]
\end{aligned}
$$

The ones not shown are found by the symmetry of the mixed partials: $D_{ij}\Omega = D_{ji}\Omega$. Now when we evaluate these at L_2, with co-ordinates $(\xi, 0, 0)$, we find all $D_{ij}\Omega(\xi, 0, 0) = 0$ if $i \neq j$. To compute the characteristic polynomial, we use the block structure we now have

$$
\begin{bmatrix} B_4 & 0 \\ 0 & C_2 \end{bmatrix}
$$

Then $0 = det(DX - \lambda I) = det(B_4 - \lambda I) \cdot det(C_2 - \lambda I)$. The B_4 block is just what is found for the *planar* R3BP, and explicitly computed in Abraham & Marsden [1985 pp. 683-685].

The characteristic polynomial for B is

$$P(\lambda) := \lambda^4 + (4 - \Omega_{22} - \Omega_{11})\lambda^2 + \Omega_{11}\Omega_{22} = 0$$

We will show that this has two real roots of opposite sign, and two conjugate pure imaginary roots. To do this, we define $\eta := \lambda^2$. This yields the polynomial

$$\hat{P}(\eta) := \eta^2 + (4 - \Omega_{22} - \Omega_{11})\eta + \Omega_{11}\Omega_{22} = 0$$

If $\hat{P}$ has 2 real roots of opposite sign, then we are done. We apply the quadratic formula, and get

$$\eta = -\frac{1}{2}(4 - \Omega_{11} - \Omega_{22}) \pm \frac{1}{2}\sqrt{(4 - \Omega_{11} - \Omega_{22})^2 - 4\Omega_{11}\Omega_{22})}$$

Define $\Delta := \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}$. Then at the point L_2, $\Omega_{11} = 1 + 2\Delta$ (note that this is strictly positive), and $\Omega_{22} = 1 - \Delta$. It can be shown that the argument of the above square root in the quadratic formula is $\Delta(9\Delta - 8)$. The roots are therefore real if $\Delta > \frac{8}{9}$. We are actually going to show $\Delta > 1$, and this follows immediately if we show $\Omega_{22} < 0$ at L_2.

If we establish $\Omega_{22}(\xi, 0, 0) < 0$, this also shows that we have two real roots of opposite sign. Let $\delta := -4\Omega_{11}(\xi, 0, 0)\Omega_{22}(\xi, 0, 0)$; this is strictly positive. Then the roots are are $-p + \sqrt{p^2 + \delta}$ and $-p - \sqrt{p^2 + \delta}$; these are of opposite sign because $|p| < \sqrt{p^2 + \delta}$.

To show that $\Omega_{22}(\xi, 0, 0) < 0$, it is useful to define a function U such that

$$\Omega(x, y, 0) = -U(r_1(x, y, 0), r_2(x, y, 0))$$

This is accomplished by

$$U(r_1, r_2) := -\frac{1}{2}[\mu r_2^2 + (1 - \mu)r_1^2 - \mu(1 - \mu)] - \frac{(1 - \mu)}{r_1} - \frac{\mu}{r_2}$$

We then find

$$D_2\Omega(x,y,0) = -y[D_1U\frac{1}{r_1} + D_2U\frac{1}{r_2}]$$
$$D_{22}\Omega(x,0,0) = -D_1U\frac{1}{r_1} - D_2U\frac{1}{r_2}$$

Note that the last equation is evaluated at L_2. We also are going to need

$$D_1\Omega = -D_1U\cdot\frac{\partial r_1}{\partial x} - D_2U\frac{\partial r_2}{\partial x}$$
$$D_1U = -(1-\mu)(r_1 - \frac{1}{r_1^2})$$
$$D_2U = -\mu(r_2 - \frac{1}{r_2^2})$$

Since the vector field X has a zero at L_2, $D_1\Omega$ is zero there. From the expression for $D_1\Omega$, we find

$$0 = -D_1U\cdot\frac{\partial r_1}{\partial x} - D_2U\cdot\frac{\partial r_2}{\partial x}$$

Since along the x axis $r_1 = r_2 + 1$,we have $\frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x}$. Thus $-D_1U = D_2U$ or

$$(1-\mu)(r_1 - \frac{1}{r_1^2} = -\mu(r_2 - \frac{1}{r_2^2}) \qquad (2.2)$$

Then, plugging this into the expression for Ω_{22}, we find

$$\Omega_{22}(\xi,0,0) = [\frac{1}{r_2} - \frac{1}{r_1}]\mu(r_2 - \frac{1}{r_2^2})$$

The last detail is to show $r_2 < 1 < r_1$ at L_2. Obviously, we must have $r_2 > 0$ and $r_1 > 1$. Hence $r_1^2 > 1$ and then $\frac{1}{r_1^2} < 1 < r_1$, which implies that the left hand side of equation (2.2) is positive, and hence the right hand side must be positive. This means that $[r_2 - \frac{1}{r_2^2}] < 0$, which in turn implies that $r_2 < 1$.

We have now established that B_4 has 2 real eigenvalues of opposite sign and 2 pure imaginary eigenvalues of opposite sign; they are given explicitly by

$$\lambda = \pm\sqrt{\eta} = \pm\sqrt{\frac{\delta-2}{2} \pm \frac{1}{2}\sqrt{9\delta^2 - 8\delta}}$$

Let us call these eigenvalues $\rho, -\rho, i\alpha, -i\alpha$. Now it is a trivial calculation to verify that the eigenvalues of C_2 are

$$\pm\sqrt{\Omega_{33}(\xi, 0, 0)} = \pm\sqrt{-\delta}$$

And thus we get two more pure imaginary eigenvalues, $i\beta, -i\beta$.

By appealing to a well known result of dynamical systems theory, there is an 1-dimensional unstable manifold tangent to the eigenspace spanned by the eigenvector associated with the positive, real eigenvalue; there is a stable manifold for the negative, real eigenvalue. There is also a center manifold associated to the 4 imaginary eigenvalues. Now we have come to a problem: to satisfy the nonresonance condition of Lyapunov's theorem, we must have $\beta \neq k\alpha$ for any integer k. It can be shown easily that $\beta > 1$ and $\alpha > 0$. Therefore, a priori, we know that $\beta \leq k\alpha$ for only finitely many integers k, all positive. By making precise numerical calculations with error bounds, we should be able to show that the nonresonance condition is satisfied and we get two families of periodic orbits. In Chapter 4 we compute numerical values for α, β (without error bounds) and we find $\alpha \doteq 1.863$ and $\beta \doteq 1.786$, which look quite nonresonant, provided our errors are less than 0.04!

$\square$

The family associated to the imaginary eigenvalues $i\alpha, -i\alpha$ of B_4 lies in the x, y plane in the configuration space, and this is the family we follow to locate the halo orbits later in this paper.

2.3 Poincaré Maps

Theorem 4 (Existence and Degree of Uniqueness of Poincaré Maps) *Let X be a smooth vector field on a manifold M, with a closed orbit γ of period τ. Let $m_0 \in \gamma(\mathbb{R})$.* **Then** *there exists a Poincaré map of γ,*

$$\begin{aligned} \Theta : W_0 &\longrightarrow W_1 \\ m &\longmapsto F(m, \tau - \delta(m)) \end{aligned}$$

where W_0 and W_1 are open sets in a codimension 1 submanifold S which is transverse to $\gamma(\mathbb{R})$ at m_0. S is called a **Poincaré section.**

If Θ' is another Poincaré map of γ in another Poincaré section S', then $\Theta' = h^{-1} \circ \Theta \circ h$, where h is a diffeomorphism.

Proof. This follows Abraham & Marsden [1985 p.521 - 523].

Step 1. First we define the domains we are going to need. Choose any $m_0 \in \gamma(\mathbb{R})$. We know $X(m_0) \neq 0$, so we can look at a neighborhood of m_0 in a chart (ϕ, U) which straightens the flowlines by making the chart image vector field $X_\phi = \hat{e}_n$; we take it as well known that such charts exist. We can arrange it so that $\phi(m_0) = 0$ and $\phi(U) = \mathcal{V} \times \mathcal{I} \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$. The notation for a general point in that $\phi(m) = x$.

The trajectory through m_0 is defined for all $t \in \mathbb{R}$ since it is a periodic orbit, of period τ. Since the flow is smooth, we can cut down the size of U to a smaller open set and guarantee that for each point in the new U, the flow is defined on the time interval $[-\tau, \tau]$.[6] Define $S := \phi^{-1}(\mathcal{V} \times \{0\})$.

[6] For more details, see Abraham & Marsden [1985 p.70].

Since $X_\phi(0) = \hat{e}_n$, the basis for $T_0(\mathcal{V} \times \{0\})$ is $\hat{e}_1, \ldots, \hat{e}_{n-1}$, and $T\phi_0^{-1}$ is an isomorphism, S is transverse to the flow at m_0 (and since transversality is an open condition, S is transverse to the flow in a neighborhood of m_0).

$F_\tau : M \longrightarrow M$ is a diffeomorphism, and $F_\tau(m_0) = m_0$. Therefore $U_0 := F_\tau^{-1}(U) \bigcap U$ is open and nonempty. Let us also define

$$\begin{aligned} W_0 &:= S \bigcap U_0 \\ U_2 &:= F_\tau(U_0) \\ W_2 &:= F_\tau(W_0) \end{aligned}$$

W_2 is a local transverse section to the flowline γ at m_0 because (1) $T_{m_0}(W_0) \subseteq T_{m_0}(S)$ and we know $X(m_0) \notin T_{m_0}(S)$, (2) $T(F_\tau)_{m_0} : T_{m_0}(M) \longrightarrow T_{m_0}(M)$ is an isomorphism, and (3) $T(F_\tau)_{m_0}(X(m_0)) = X(m_0)$.

Step 2. Define the Poincaré map Θ. We can do all the work down in the image of the straightening chart ϕ. Define

$$\tilde{\Theta}_\phi := \pi \circ (F_\tau)_\phi : \mathcal{W}_0 \longrightarrow \mathcal{V} \times \{0\}$$

where π is the canonical projection that kills the n^{th} component. *Presumably, the problem is that $\mathcal{W}_2$ might not be a nice graph over $\mathcal{V}$ like we've drawn it; we know that, even outside the straightening chart, the flowlines can't cross, since our vector field is autonomous, but maybe $\mathcal{W}_2$ can still get twisted or kinked in some weird way so that $\pi \mid_{\pi(\mathcal{W}_2)}$ isn't a diffeomorphism.* To guard against this, we appeal to the inverse function theorem. We need that, dropping the ϕ subscripts on the maps,

$$D(\tilde{\Theta})_0 = D(F_\tau)_0 \circ \pi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$$

is an isomorphism. Since clearly $D(F_\tau)_0$ is, we only need to establish that $\pi : T_0\mathcal{W}_2 \longrightarrow T_0\mathcal{W}_2$ is an isomorphism. This follows immediately because we can characterize $T_0\mathcal{W}_2$ as the perpendicular hyperplane to a vector $\alpha \in \mathbb{R}^n$, that is

$$\{(x_1, \ldots, x_n) \mid \alpha_1 x_1 + \ldots + \alpha_n x_n = 0\}$$

$\alpha_n \neq 0$, because if it did, $T_0\mathcal{W}_2$ would not be transverse to the straightened flow. This allows us to solve for x_n in terms of the other variables; thus the set of points defining $T_0\mathcal{W}_2$ can be written

$$\{(x_1, \ldots, x_{n-1}, -\frac{1}{\alpha_n}[\alpha_1 x_1 + \ldots + \alpha_{n-1}x_{n-1}]) \mid x_i \in \mathbb{R}\}$$

Then obviously, π restricted to $T_0\mathcal{W}_2$ is an isomorphism.

We apply the inverse function theorem and conclude that π is a local diffeomorphism in a neighborhood of 0.

$$\pi : \mathcal{W}_2' \longrightarrow \mathcal{W}_0'$$

Choose $\mathcal{W}_0''$ to be an open set containing 0 such that $F_\tau(\mathcal{W}_0'') \subseteq \mathcal{W}_2'$. Then

$$\Theta_\phi := \pi \circ (F_\tau)_\phi \mid_{\mathcal{W}_0''}$$

is a diffeomorphism onto its image; call its image W_1. Rename $\mathcal{W}_0''$ as $\mathcal{W}_0$.

Step 3. Now we show that the map we have defined $\Theta : W_0 \longrightarrow W_1$ meets the criteria to be a Poincaré map.

1. By construction, we have that W_0, W_1 are open neighborhoods of m_0 in S, and that Θ is a diffeomorphism.

2. There exists a smooth time difference function $\delta : W_0 \longrightarrow (0, \infty)$ such that

$$\Theta(m) = F(m, \tau - \delta(m)) \quad \forall m \in W_0$$

Choose an arbitrary $x \in \mathcal{W}_0$. Since $\mathcal{W}_2$ is contained in the chart image $\mathcal{U}$, we can define

$$\begin{aligned} \delta : W_0 &\longrightarrow \mathbb{R} \\ x &\longmapsto \pi_n \circ F_\tau(x) \end{aligned}$$

where π_n is the standard projection onto the n^{th} co-ordinate, which is the time difference in our straightening chart. Thus

$$F_{\tau-\delta(x)}(x) = F_{-\delta(x)} \circ F_\tau(x) = F_{-\delta(x)}(\pi \circ F_\tau(x), \pi_n \circ F_\tau(x))$$

$$= F_{-\delta(x)}(\pi \circ F_\tau(x), \delta(x)) = (\pi \circ F_\tau(x), 0) = \Theta_\phi(x)$$

3. It is obvious from the straightening chart that $F_\tau(x)$ does not meet S for $t \in (0, \tau - \delta(x))$.

Step 4. We now show that any two Poincaré sections at m_0 are conjugate in a small enough neighborhood of m_0. By transitivity, it is enough to show this for an arbitrary such section S' and our standard section S. As usual, let us work in the image of the straightening chart map. The result follows by exactly the same ideas that we used in step 2 to show that $\pi \mid T_0\mathcal{W}_2$ is a diffeomorphism in a neighborhood of 0.

Step 5. The last ingredient that we need to establish the conjugacy of two arbitrary Poincaré sections in a small enough neighborhood of the orbit is the conjugacy of our standard section S at m_0 and an arbitrary section T at m_0'.

We can assume that $F_a(m_0) = m'_0$ for some $0 < a < \tau$. Then $F_a(U)$ is an open set containing m'_0. Let $S' := F_a^{-1}(F_a(U) \cap T)$. This is transverse to γ because F_a is a diffeomorphism. Then the Poincaré map Θ' for S' is conjugate to Θ for S by step 4. Θ'' for T is conjugate to Θ' by F_a and thus to Θ.

$\square$

Important Remark This shows that the eigenvalues of any Poincaré map depend only upon γ and not on the base point m_0 or the particular Poincaré section.

2.4 Orbit Cylinders

In this section, we prove some theorems about periodic orbits in Hamiltonian dynamical systems. A knowledge of this theory is essential to be able to make sensible numerical computations, and especially to know when the results the computer is producing are bogus. A topic of crucial importance is 1-parameter families of periodic orbits. In the older literature of celestial mechanics, they are called *natural* families of periodic orbits, and a member of such a family is called a *singular* orbit; an isolated orbit is called *ordinary*.[7] Being more geometrically minded, Abraham & Marsden refer to such a family of periodic orbits as an *orbit cylinder*, and the following discussion is based on Abraham & Marsden [1985 p. 576].

[7] See Deprit & Henrard [1967], [1968] for more details.

Definition 1 *An* **orbit cylinder** *of X_H is an embedding $\Gamma : S^1 \times (a,b) \longrightarrow M$ such that for all $e \in (a,b)$, $\gamma_e := \Gamma[S^1 \times \{e\}]$ is a closed orbit of X_H.*

An orbit cylinder is **regular** *if $H[\gamma_e] = e$ and Γ is transversal to every energy surface Σ_e. This is equivalent to $H \circ \Gamma$ having no critical points.*

Theorem 5 (Regular Orbit Cylinder Theorem) *Let γ be a closed orbit of X_H. Then γ is contained in a regular orbit cylinder if and only if the Poincaré map Θ of γ has $+1$ as an eigenvalue of multiplicity exactly one.*

Proof Again this will be an exercise in using the implicit function theorem.

Fix any point $x_0 \in \gamma$. We can work in a special "straightening " symplectic chart ϕ such that, in the image in $\mathbb{R}^n$, the q_1 axis represents the difference in period from the period of the point x_0, namely T_0. The p_1 axis represents the change in energy from $H(x_0) = e_0$. For a picture of this, see Fig. (2.1) in the proof of the Eigenvalue Inheritance theorem, which follows. Then we would write $\phi(x) = (t, q_2, \ldots, q_n, e, p_2, \ldots, p_n)$. The "horizontal" hyperplane segments $\{p_1 = e\}$ are the constant energy submanifolds $\Sigma_e = H^{-1}(e)$. In the chart image, the local Poincaré section is $S := \{q_1 = 0\}$; up on the manifold, the inverse image, $\tilde{S}$, is a local submanifold, transverse to the orbit γ. Therefore, there is a well defined Poincaré map $\Theta : W_0 \longrightarrow W_1$, where W_0 is an open set in the transverse section $\tilde{S}$ containing x_0, and sufficiently small so that $\Theta\,|_{W_0}$ maps diffeomorphically onto W_1.

As usual, we want to reduce the problem of finding periodic orbits to finding fixed points, or more precisely, to finding zeroes of the function ψ, to be defined shortly. Let $W := \phi(W_0 \bigcap W_1)$; this is an open set in the $\{q_1 = 0\}$

hyperplane, that we called S. Define $q := (q_2, \ldots, q_n)$ and $p := (p_2, \ldots, p_n)$. Define $\Theta_\phi := \phi \circ \Theta \circ (\phi \mid_W)^{-1}$. Then

$$\begin{aligned} \psi : W &\longrightarrow \mathbb{R}^{2n-2} \\ (q, e, p) &\longmapsto \pi \circ \Theta_\phi(q, e, p) - \pi(q, e, p) \end{aligned}$$

where $\pi : S \longrightarrow \Sigma_e$ (projection). Actually, for convenience with the implicit function theorem, we reorder the domain variables as $((q, p), e)$ and therefore $W =: W_{qp} \times I_e$.

Claim: $D_{(q,p)}\psi_{((0,0),0)} : \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^{2n-1}$ is an isomorphism.

This is where we use the eigenvalue information. $D_{(q,p)}\psi_{((0,0),0)} = D\psi \mid_{T(0,0,0)\Sigma_0} = D(\Theta_\phi)_{(0,0,0)} \mid_{T\Sigma_0} - I$. We know that the Poincaré map restricted to a constant energy surface Σ_e is symplectic.[8] Therefore, if $+1$ occurs as an eigenvalue for $D(\Theta_\phi)_{(0,0,0)} \mid_{T\Sigma_0}$, it must occur with even multiplicity. But this can't happen, because by hypothesis, $D(\Theta_\phi)_{(0,0,0)}$ only has one $+1$ eigenvalue. This proves that $D_{(q,p)}\psi_{((0,0),0)}$ is an isomorphism. We also know $\psi((0,0),0) = 0$ since $\phi^{-1}(0) = m_0 \in \gamma$. The implicit function theorem then gives us (q, p) as a function of e, *i. e.* there exists a smooth function $g : (-\epsilon, \epsilon) \longrightarrow W_{qp}$ such that $\psi(g(e), e) = 0$. Let us rewrite g as a graph:

$$\begin{aligned} G : (-\epsilon, \epsilon) &\longrightarrow S \\ e &\longmapsto (g(e), e) \end{aligned}$$

Now we must show how a curve of fixed points in S gives us a regular embedded cylinder Γ of periodic orbits. For F the flow of our Hamiltonian vector field, define

[8] We will prove this statement as the Symplectic Inheritance theorem later in this chapter.

$$\begin{aligned} \Gamma : S^1 \times (-\epsilon, \epsilon) &\longrightarrow M \\ (s, e) &\longmapsto F(s, \phi^{-1}(G(e)) \end{aligned}$$

To be an embedding, Γ must be a one to one, proper, immersion. Since Γ is clearly a homeomorphism, the first two are immediate. Now we show $T\Gamma_{(s,e)}$ is one to one for all (s, e). This follows because the image is 2-dimensional. We can regard the tangent space to $S^1 \times (-\epsilon, \epsilon)$ as $\mathbb{R}^2$. Then can compute without too much difficulty (except for notation!) that

$$T\Gamma_{(s,e)}(1, 0) = X_H(F(s, \Gamma(s, e)))$$

which is obviously contained in $T_{\Gamma(s,e)}\tilde{\Sigma}_e$, the tilde symbolizing the energy surface is in M, not down in the chart image we were working with previously.

For the other standard basis vector for $\mathbb{R}^2$, we have

$$T\Gamma_{(s,e)}(0, 1) = T(F_s)_{\phi_{-1} \circ G(e)} \circ T\phi^{-1}_{G(e)} \circ DG_e(1)$$

The first two terms on the right side are isomorphisms, and $DG_e(1) \notin T_{\Gamma(s,e)}\Sigma_e$ because that would correspond to a "vertical tangent" and hence g would not be a smooth function. Therefore, $T\Gamma_{(s,e)}(0, 1) \notin T_{\Gamma(s,e)}\tilde{\Sigma}_e$ and the image of $T\Gamma_{(s,e)}$ must have dimension 2, and thus $T\Gamma_{(s,e)}$ must be one to one.

Now we have an embedded orbit cylinder. To show it is regular, we must establish the transversality condition at an arbitrary point (s, e):

$$T\Gamma_{(s,e)}(\mathbb{R}^2) + T_{\Gamma(s,e)}\tilde{\Sigma}_e = T_{\Gamma(s,e)}M$$

but this follows immediately from our earlier discussion, and the fact that the dimension of $T_{\Gamma(s,e)}\tilde{\Sigma}_e$ is $2n - 1$.

□

The following result is given as an exercise in Abraham & Marsden [1985]. We will now establish

Theorem 6 *The family of closed orbits proved to exist in Lyapunov's theorem is a regular orbit cylinder.*

Proof. In the proof of the Liapunov theorem, we exhibited a periodic orbit γ for the vector field $X_{\mathcal{H}_\epsilon}(x) = Ax$ and $\gamma(t) = F^0(t, x_0)$. This corresponded to the periodic orbit $\tilde{\gamma}(t) = \epsilon \cdot F^0(t, x_0)$ for X_H.

For γ, the flow was $F^0(t, x_0) = e^{tA}x_0$, and then for the monodromy map we have $D(F^0_{T_0})_{x_0} = e^{T_0 A}$. $T_0 A$ as eigenvalues $i2\pi, -i2\pi$ and *no* others that are integer multiples of $i2\pi$, because of the nonresonance condition. This implies $e^{T_0 A}$ has two and only two $+1$ eigenvalues. Hence we can apply the orbit cylinder theorem to conclude that γ is embedded in an orbit cylinder.

One might ask, how do we know this orbit cylinder coincides with the Liapunov family found in that theorem? The reason is that in the Poincaré section, the periodic point $x_{e,\epsilon}$ near x_0 with period $T_{e,\epsilon}$ was unique. Since any orbit cylinder containing γ must intersect this Poincaré section, we see that it must be the Liapunov family.

Then we get a cylinder for X_H because if $\{F_{x_{e,\epsilon}}\}_\epsilon$ is the cylinder for $X_{\mathcal{H}_\epsilon}$, then $\{\epsilon \cdot F_{x_{e,\epsilon}}\}_\epsilon$ is the cylinder for X_H.

□

2.5 Inherited Properties of Poincaré Maps

The following is a theorem that caused the author much trouble, since the numerical results persistently contradicted it. For a long time, it was not clear whether the numerical work was in error, or whether the theorem was false. The following proof establishes the theorem, it was taken from Meyer & Hall [1992 p.135], where many details were omitted.[9]

Theorem 7 (Eigenvalue Inheritance) *Let F_t be a Hamiltonian flow with a periodic orbit γ, of period T. If the eigenvalues of the monodromy map $T(F_T)_x$ are $1, 1, \lambda_3, \ldots, \lambda_{2n}$, and the Hamiltonian H is nondegenerate along γ ($TH_x \neq 0$), then the eigenvalues of the Poincaré map restricted to a constant energy surface through x are $\lambda_3, \ldots, \lambda_{2n}$.*

Proof. Here, just as in the proof of the Regular Orbit Cylinder theorem, we use the special chart guaranteed to exist by the well known Hamiltonian Flow Box theorem, or the Symplectic Straightening theorem. See Figure (2.1). What we did not mention before is that the vector field $X_H(x) = \frac{\partial}{\partial q_1}$ for all x in the chart domain; the flow is therefore straightened.

We work entirely in the image of this chart. The Poincaré section S is the hyperplane $\{q_1 = 0\}$. Let the origin 0 be the point where γ meets S; the period of γ is T. Then we can compute the differential of the flow map F and

[9] Eventually, the problem was found to be a faulty expression for the Poincaré map in the numerical algorithm.

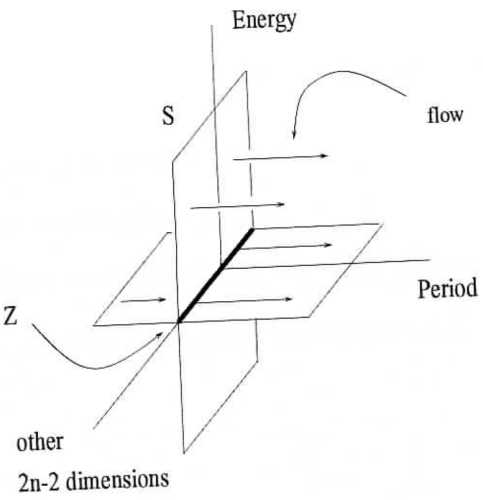

Figure 2.1: Image of Straightening Chart

we get

$$D(F_T)_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & D\Theta_0 & \\ 0 & & & \end{bmatrix}$$

Let us show why there must be a $+1$ in the upper left corner. Recalling the standard notational convention for flow maps $F : (-a, a) \times M \longrightarrow M$:

$$F_t(x) = F(t, x) = F_x(t)$$

where $F_t : M \longrightarrow M$ is a diffeomorphism and $F_x : (-a, a) \longrightarrow M$ is a curve map such that $F_x(0) = x$. And, of course, we know $F_{s+t}(x) = F_s \circ F_t(x)$. So, in the case here we observe that

$$F_T \circ F_x(t) = F_x(T + t) \Rightarrow D(F_T)_{F_x(t)} \circ D(F_x)_t = D(F_x)_{T+t}$$

If we take $t = 0$ and $x = \mathbf{0}$ we see that $D(F_T)_0 \cdot X_H(0) = +1 \cdot X_H(0)$. But in our chart, $X_H(0)$ is just the standard basis vector e_1. Thus we see that the monodromy map $D(F_T)$ of a periodic orbit of a time independent vector field has a $+1$ eigenvalue associated with the direction of the flow.

Let us denote the monodromy matrix $D(F_T)_0$ by A, and let Σ_0 be the constant energy submanifold passing through the origin, i.e. $\Sigma_0 = \{p_1 = 0\}$. We know that the Poincaré map Θ leaves energy surfaces invariant: $Q : Z \longrightarrow Z$, where $Q := \Theta \mid Z$. Now we are going to permute our basis vector such that A will take the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & & & \\ 0 & 0 & & DQ_0 & \\ \vdots & \vdots & & & \end{bmatrix}$$

The new basis will not be symplectic, but that will not pose a problem for us. Let P be the matrix that permutes the co-ordinate vectors as follows:

$$(q_1, q_2, \ldots, q_n, p_1, \ldots p_n) \longrightarrow (q_1, p_1, q_2, q_3, \ldots, q_n, p_2, p_3, \ldots p_n)$$

The matrix A is then changed to $B = P \cdot A \cdot P^{-1}$. Note that the first row and column are not changed. In the new basis, Z consists of all vectors whose first 2 components are zero. Therefore, DQ_0 must belong in the lower right $(2n-2) \times (2n-2)$ block in the matrix B. Now we must show that the 2nd row and the 2nd column of B are both the vector $\hat{e}_2$.

To show that the 2nd row is $\hat{e}_2$, recall that $H(F(t,x)) = H(x)$ for any t and x. Then $DH_{F(t,x)} \cdot D(F_t)_x = DH_x$. If we evaluate this at $t = T$ and $x = 0$ we get $\nabla H(0)^T \cdot A = \nabla H(0)^T$. In the straightening chart, $\nabla H(x) = \hat{e}_{n+1}$. Therefore, in the new basis $\nabla H(0) = \hat{e}_2$. Thus $\hat{e}_2^T \cdot B = \hat{e}_2^T$ and this forces the 2nd row of B to be $\hat{e}_2^T$. We must have the 2nd column of B be $\hat{e}_2$ because all the entries of the first 2 columns of B below the 2nd row must be zero. If they were not, vectors not originally in TZ_0 could get mapped into TZ_0 by $D(F_T)_0$, which would violate the fact that $D(F_T)_0 : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is an isomorphism and $DQ_0 : TZ_0 \longrightarrow TZ_0$ is an isomorphism.

From the form of matrix B it is obvious that the eigenvalues are $\{1,1\} \bigcup \{\text{eigenvalues of } DQ_0\}$. These are also the eigenvalues of $D(F_T)_0$ since eigenvalues are preserved under similarity transforms.

We know that these eigenvalues do not depend on the special straightening chart from our theorem on the Poincaré map.

□

Theorem 8 (Symplectic Inheritance) *The Poincaré map $\Theta : W_0 \longrightarrow W_1$, restricted to a constant energy submanifold $H^{-1}(e) \bigcap W_0$, is symplectic.*[10]

Proof (Abridged version of Abraham & Marsden [1985 p. 579]). We work exclusively in the image of our familiar symplectic straightening chart, see Fig (2.1). Here

$$\omega = dH \wedge dt + \sum_{i=2}^{n} dq^i \wedge dp^i$$

If i is the inclusion map of the Poincaré section $S = \{t = 0\} \times \mathbb{R}^{2n-1}$, then $i^*\omega = \sum_{i=2}^{n} dq^i \wedge dp^i$. We define ω^0 and ω^1 to be the restrictions of this form to to $\mathcal{W}_0$ and $\mathcal{W}_1$, respectively. Recall that $\Theta(x) = F(x, \tau - \delta(x))$ for all $x \in \mathcal{W}_0$, and let $\lambda(x) := \tau - \delta(x)$. The proof will be accomplished by showing

$$(\Theta^*\omega)_x(\xi, \eta) = \omega_x(\xi, \eta) - d\delta_x \wedge dH_x(\xi, \eta)$$

[10]With respect to the standard symplectic form ω restricted to $H^{-1}(e) \bigcap S =: S_e$.

where we extend the domains of Θ and δ to a slight thickening of $\mathcal{W}_0$ into $\mathbb{R}^{2n}$, and $\xi, \eta \in T_x\mathbb{R}^{2n}$. The desired result

$$(\Theta^*\omega^1)_x(\xi, \eta) = \omega_x^0(\xi, \eta)$$

will follow by restricting x to $\mathcal{W}_0 \bigcap S_e$ (for then $dH_x = 0$) and $\xi, \eta \in T_x(S_e)$.

Note that

$$\begin{aligned} D\Theta_x(\xi) &= D_1 F_{(x,\lambda(x))}(\xi) + D_2 F_{(x,\lambda(x))} D\lambda_x(\xi) \\ &= D_1(F_{\lambda(x)})_x(\xi) - d\delta_x(\xi) X_H(\Theta(x)) \end{aligned}$$

Now

$$\begin{aligned} (\Theta^*\omega)_x(\xi, \eta) &= \omega_{\Theta(x)}(D\Theta_x(\xi), D\Theta_x(\eta)) \\ &= \omega_{\Theta(x)}(\ D_1(F_{\lambda(x)})_x(\xi) - d\delta_x(\xi) X_H(\Theta(x)), \\ &\qquad D_1(F_{\lambda(x)})_x(\eta) - d\delta_x(\eta) X_H(\Theta(x))) \end{aligned}$$

Using the bilinearity of ω to expand this out, and noting that $\omega(X_H, X_H) = 0$, we obtain

$$\begin{aligned} &\omega_{F_\lambda(x)}(D_1(F_{\lambda(x)})_x(\xi), D_1(F_{\lambda(x)})_x(\eta)) - \omega_{F_\lambda(x)}(D_1(F_{\lambda(x)})_x(\xi), d\delta_x(\eta) X_H(\Theta(x))) - \\ &\qquad \omega_{F_\lambda(x)}(d\delta_x(\xi) X_H(\Theta(x)), D_1(F_{\lambda(x)})_x(\eta)) \end{aligned}$$

Observe in the first term that the time value $\lambda(x)$ is fixed; also $F_{\lambda(x)}$ is a symplectic map. Thus

$$\omega_{F_\lambda(x)}(D_1(F_{\lambda(x)})_x(\xi), D_1(F_{\lambda(x)})_x(\eta)) = (F^*_{\lambda(x)}\omega)_x(\xi, \eta) = \omega_x(\xi, \eta)$$

We will also use the facts that $\omega(X_H, \cdot) = dH(\cdot)$ and $dH_{F_\lambda(x)} \circ D_1(F_\lambda(x))_x(\cdot) = dH_x(\cdot)$ to obtain

$$\begin{aligned}
(\Theta^*\omega)_x(\xi, \eta) &= \omega_x(\xi, \eta) + d\delta_x(\eta)\omega_{F_\lambda(x)}(X_H(F_\lambda(x)), D_1(F_{\lambda(x)})_x(\xi)) - \\
&\quad d\delta_x(\xi)\omega_{F_\lambda(x)}(X_H(F_\lambda(x)), D_1(F_{\lambda(x)})_x(\eta)) \\
&= \omega_x(\xi, \eta) + d\delta_x(\eta)dH_x(\xi) - d\delta_x(\xi)dH_x(\eta) \\
&= \omega_x(\xi, \eta) - d\delta_x \wedge dH_x(\xi, \eta)
\end{aligned}$$

$\square$