Waves

Thus an idealized string transmits waves at a certain speed v; $f(x-(v+\delta)t)$ is NOT a soln to D'lambert's wave eq and does NOT make the wave go faster!

5/3/2002

The wave equation is $\boxed{y_{tt} = v^2 y_{xx}}$ where $v^2 = \frac{T}{\rho}$ ← Tension in String; ρ ← linear mass density

v is the speed of propagation in non-dispersive media.

Towne WP p.4 Sheets → I derived this assuming $\begin{cases}(1) \text{ purely transverse motion} \\ (2) \text{ homogeneous string}\end{cases}$

Then I regarded a small "string elt" as a rigid body and wrote expressions for $\sum$ Horizontal Forces $= 0$, $\sum$ Vertical forces $= m\ddot{y}$

To have Dispersion we need a different eq, like Klein-Gordon: $u_{tt} = u_{xx} - u$ or $u_t = -u_{xxx}$ Strang ITAM p.556

Schrödinger $iu_t = u_{xx}$

The 1-D wave eq has the general soln $y(x,t) = f(x-\check{v}t) + g(x+vt)$
$= f\circ\tau_v(x,t) + g\circ\tau_{-v}(x,t)$

Towne WP p.7-8

Why is this the general soln?

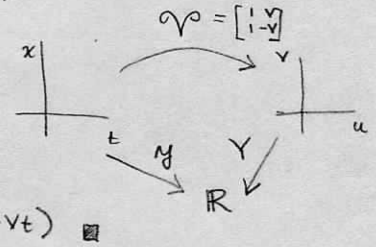
Let $y(x,t)$ be an arb soln.

Since φ is a diffeo, $\exists$ fcn $Y \ni y = Y\circ\varphi$

Then subs the expression for Y into $y_{tt} = v^2 y_{xx}$

and get $Y_{uv} = 0$ which implies $Y(u,v) = f(u) + g(v)$

$\varphi = \begin{bmatrix}1 & v \\ 1 & -v\end{bmatrix}$



More generally the wave eq is $\boxed{u_{tt} = v^2\nabla^2 u}$ or $\boxed{\Box^2 u = 0}$ $= f(x-vt) + g(x+vt)$ ■

For a stiff string, the eq becomes $D_t^2 y = v^2[D_x^2 y - \alpha D_x^4 y]$ Main YAWIP p. 214

<u>Wavelength</u> λ

<u>Period</u> T

<u>Freq</u> $\nu = \frac{1}{T}$

<u>Speed</u> $v = \lambda\nu$

<u>Wave Number</u> $k = 2\pi/\lambda$

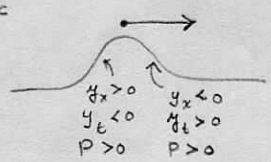
<u>angular freq</u> $\omega = kv$ thus $\omega = 2\pi\nu$ radians/sec

$\sin(\omega t)$ has period $T = \frac{2\pi}{\omega}$

Power $P(x,t) = -T\cdot \underbrace{y_x}_{F_{\text{transverse}}}\cdot \underbrace{y_t}_{\text{Vel.}}$ (T: tension) ← The neg sign is there to make P pos for waves traveling left → right

$P = F\cdot v$

H&R p.300



If we specialize to a sin wave $y(x,t) = A\sin(kx - \omega t)$

$\Rightarrow P(x,t) = A^2 k\omega T\cos^2(kx-\omega t)$

average over 1 period $\bar{P} = 2\pi^2 A^2 \nu^2 \mu v$ — μ: linear mass density. $\bar{P} \sim A^2$

[for a 3-D wave, Intensity $I :=$ Power/Unit Area Normal to wave propagation.]

<u>Standing waves</u>: nodes are fixed pts – energy can't flow past nodes. [a priori, nodes are spaced $\frac{\lambda}{2}$ apart]

a <u>Fixed</u> end pt causes a pulse to reflect with inversion

a <u>Free</u> end pt " " " " " " no inversion.

a string (with fixed end pts) of length l has nodes at pts where $\frac{\lambda}{2} = \frac{l}{n}$ $n \in \mathbb{Z}$.

ie natural freqs $\nu_n = \frac{n}{2l}v$

Observe $k(x - vt) = kx - \omega t$

That is how we go from $f(x-vt)$ to $\sin(kx - \omega t)$

"f" $= \sin \circ k\cdot(x,t)$ ← compose

Complex form of wave (sinusoidal)

$$\Psi(x,t) = Ae^{i(\omega t - kx)}$$ $\mathbb{R}$e part: $A\cos(\omega t - kx)$

<u>Dispersion</u> – each freq component of the wave moves thru the medium at a different speed and thus the wave changes shape as it moves.

<u>Attenuation</u> – amplitude decreases with time.

We must have a different wave eq to model a dispersive medium — Strang ITAM

Marion CDOPAS ch 13

13.3 For the wave eq $\Psi_{tt} - v^2 \Psi_{xx} = 0$ we assume a soln of the form $\Psi(x,t) = \psi(x) e^{i\omega t}$

Then substituting in, we obtain the ODE $\boxed{\psi''(x) + \overbrace{\frac{\omega^2}{v^2}}^{k^2} \psi(x) = 0}$ Helmholtz Eq p. 477

we can express a general $\Psi(x,t) = \sum_n \psi_n(x) e^{i\omega_n t}$

13.4 Consider the Complex form of a sin wave: $\Psi(x,t) = A e^{i(\omega t - kx)} = A e^{-ikx} e^{i\omega t}$.

Phase velocity: How must x change with t for the arg $(\omega t - kx) = \text{const}$?

$\Rightarrow \omega - k\dot{x} = 0 \Rightarrow \dot{x}(t) = \boxed{\frac{\omega}{k}}$ phase velocity V

If $V = V(k)$ the wave exhibits <u>Dispersion</u>

If we add an imaginary term onto wave number k: $k - i\beta$ $\beta > 0$ we can model <u>Attenuation</u>

$$\Psi(x,t) = A e^{-\beta x} e^{i(\omega t - kx)}$$

No attenuation for $\omega \le \underbrace{2\sqrt{\tau/md}}_{\omega_c \text{ cutoff freq}}$ p. 481–

13.5 Group velocity and Wave Packets

Consider 2 waves whose freqs and wave numbers are very close:

$$\Psi_1(x,t) = A e^{i(\omega t - kx)}$$

$$\Psi_2(x,t) = A e^{i([\omega + \Delta\omega]t - [k + \Delta k]x)}$$

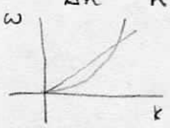
Then let $\Psi = \Psi_1 + \Psi_2$ and take the Real part:

since Δk and $\Delta\omega$ are small, this term varies slowly with a long period $\Rightarrow$ it defines the amplitude envelope.

$$\Psi(x,t) = \underbrace{2A \cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right]} \cos\left[\left(\omega + \tfrac{1}{2}\Delta\omega\right)t - \left(k + \tfrac{\Delta k}{2}\right)x\right]$$

The "phase velocity" of the <u>envelope</u> is the $\boxed{\text{Group velocity } \frac{\Delta\omega}{\Delta k}} = \omega'(k)$

In non-dispersive media $\frac{\Delta\omega}{\Delta k} = \frac{\omega}{k}$



$$\Psi(x,t) = \sum_{n=1}^{N} A_n e^{i(\omega_n t - k_n x)}$$

For continuous frequency distribution:

$$\Psi(x,t) = \int_{\mathbb{R}} A(k) e^{i(\omega t - kx)} dk = \int_{\mathbb{R}} \left(A(k) e^{i\omega t}\right) e^{-ikx} dk = \tilde{\mathcal{F}}\left(A(k) e^{i\omega t}\right)$$

If $A(k)$ is non-zero only on $(k_0 - \Delta k, k_0 + \Delta k)$ then we have

$$\Psi(x,t) = \int_{k_0 - \Delta k}^{k_0 + \Delta k} A(k) e^{i(\omega t - kx)} dk$$

<u>Wave Packet</u>

Group velocity only applies to wave packets.

Marion p. 485

For a wave packet around k_0, lets expand ω in a Taylor series

$$\omega(k) = \omega(k_0) + \omega'(k_0)(k-k_0) + \dots \; O(k^2)$$

$$\omega t - kx = (\omega_0 t - k_0 x) + \omega_0' \cdot (k-k_0)t - (k-k_0)x$$
$$= (\omega_0 t - k_0 x) + (k-k_0)(\omega_0' t - x)$$

wave packet becomes

$$\psi = \int_{k_0-\Delta k}^{k_0+\Delta k} \underbrace{A(k)\, e^{i(k-k_0)(\omega_0' t - x)}}_{\text{Amplitude}} e^{i(\omega_0 t - k_0 x)}\, dk$$

Then if we set for const phase in amplitude: $\frac{d}{dt}\left((k-k_0)(\omega_0' t - x)\right) \overset{!}{=} 0$

$$(k-k_0)(\omega_0' - \dot{x}) = 0$$

$k = \frac{\omega}{V}$

$$U = \frac{V_0}{}$$

$$\Rightarrow \quad \dot{x} \overset{!}{=} \omega_0' = \omega'(k_0) = U$$

Energy Propagation in LOADED string

KE of j^{th} particle: $T_j = \frac{1}{2} m \dot{q}_j^2$

PE " " $W_j = \frac{1}{2}\frac{\tau}{d}(q_j - q_{j+1})^2$

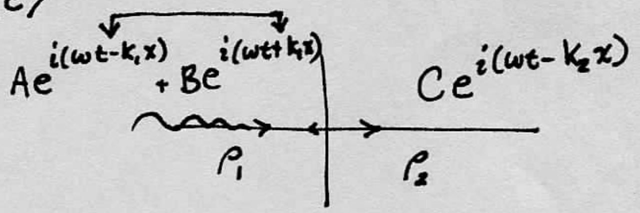
Avg KE per unit length $\langle \mathcal{T} \rangle = \frac{1}{2}\frac{m}{d}\langle \dot{q}_j^2 \rangle$

PE " " $\langle \mathcal{W} \rangle = \frac{1}{2}\frac{\tau}{d^2}\langle (q_j - q_{j+1})^2 \rangle$

$$\langle \mathcal{W} \rangle = \frac{\tau A^2}{d^2}\sin^2\frac{kd}{2} = \langle \mathcal{T} \rangle \qquad \langle \mathcal{E} \rangle = \langle \mathcal{T} \rangle + \langle \mathcal{W} \rangle$$

Group velocity $\omega'(k) = \frac{\langle P \rangle}{\langle \mathcal{E} \rangle}$ if $\omega < \omega_c$.

Reflected and Transmitted Waves



$$\psi_1(0) \overset{!}{=} \psi_2(0)$$
$$(\psi_1)_x(0) \overset{!}{=} (\psi_2)_x(0)$$

Reflection coeff $R := \frac{|B|^2}{|A|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \qquad T := \frac{k_2}{k_1}\frac{|C|^2}{|A|^2}$

$R + T \overset{!}{=} 1$

reflected energy ↑ ↑ transmitted energy

13.9 Fourier Integral Representation of Wave Packets

Here we are taking a duality $x \leftrightarrow k$

$$A(k) = \tilde{\mathcal{F}}(f)_{(k)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)\, e^{iku}\, du$$

$A : k \longmapsto A(k)$ spectral distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} A(k)\, e^{-ikx}\, dk = \tilde{\mathcal{F}}^{-1}(\tilde{\mathcal{F}}(f))(x)$$

we can also take the duality $t \leftrightarrow \omega$

$$A(\omega) = \tilde{\mathcal{F}}(f)_{(\omega)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, e^{i\omega t}\, dt$$

$$f(t) = \tilde{\mathcal{F}}^{-1}(A)$$

<u>example 13.1</u> <u>finite sin wave train</u>

freq $\omega_0 = k_0 v$

$$\Psi_{(x,t)} = \begin{cases} \mathrm{Re}\left[e^{i(k_0 x - \omega_0 t)}\right] \\ 0 \end{cases} = \begin{cases} \cos k_0(x-vt) & |x-vt| < L \\ 0 & |x-vt| > L \end{cases}$$

then
$$\tilde{\mathcal{F}}(\Psi)_{(k)} = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \cos k_0 u \cdot \cos ku\, du$$

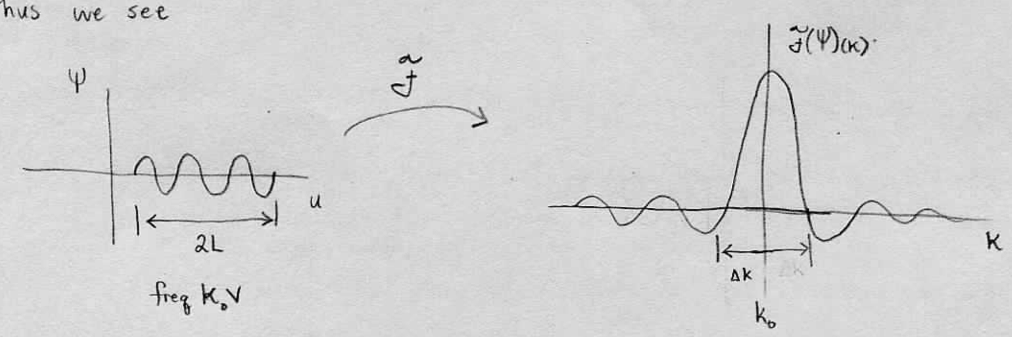
we transform wrt the combined variable $u = x - vt$

$$\overset{\text{evenness}}{=} \frac{2}{\sqrt{2\pi}} \int_0^L \cos k_0 u \cos ku\, du$$

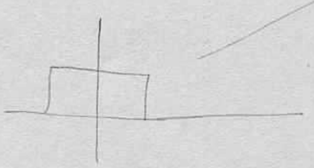
$$= \sqrt{\tfrac{2}{\pi}} \left[\frac{\sin(k_0+k)L}{(k_0+k)} + \frac{\sin(k_0-k)L}{(k_0-k)} \right]$$

This term dominates for k near k_0

Thus we see



Supposedly, same result for



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