Do Sym Prbs have Sym solns? waterhouse

Smart Modern Geos p. 131

Heron Area of Triang: $A = \sqrt{s(s-a)(s-b)(s-c)}$ 8/3/2024

Brahmagupta cyclic quad $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ 1

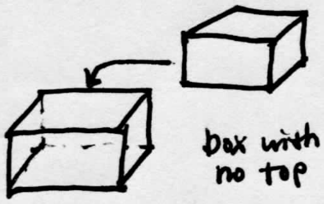
Purkiss Sym Princ

Examples

1. rectangle perim vs area $2(x+y) = P_0$ fixed $A = xy$ is max when $x = y$

2. Triangle inscribed in fixed circle  min area when Δ equilateral

3. $x, y, z, w > 0$ $x \cdot y \cdot z \cdot w = 16 \Rightarrow x+y+z+w \geq 8$ extrema when $x = y = z = w = 2$

4.  box with no top

5. Find min $f = (r-1)^2 + \left(\frac{s}{r}-1\right)^2 + \left(\frac{t}{s}-1\right)^2 + \left(\frac{4}{t}-1\right)^2$ [1981 Putnam]
subj to $1 \leq r \leq s \leq t \leq 4$ ans. $\langle r, s, t\rangle = \langle \sqrt{2}, 2, 2\sqrt{2}\rangle$
Problem becomes sym with new vars $R = r,\ S = \frac{s}{r},\ T = \frac{t}{s},\ W = \frac{4}{t}$

6. Fixed $\bar{x} = \frac{1}{n}\sum^n x_i$ $\rho = \sum_i x_i^2$ is min when all x_i are equal

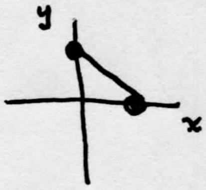
7. Arithmetic vs Geo Means: $x_i \geq 0\ \forall\ i = 1 \ldots n$
$$\sum^n \frac{x_i}{n} \geq \left(\prod x_i\right)^{1/n}$$ (generalizes 3 above)

Buniakowski counterexample: $f(x,y) = [x^2 + (y-1)^2]\,[(x-1)^2 + y^2] \geq 0$

Buniakowski proved the principle holds for all polys of degree ≤ 3, so we need degree 4

This obviously takes its absolute (global) min value at $\langle 0,1\rangle$ and $\langle 1,0\rangle$ rather than $x = y$

or for a problem with a constraint:
$$f(x,y) = \left(x^2 + y^2 - \tfrac{5}{8}\right)^2$$
consider $x, y > 0$ $x + y = 1$



See Hazewinkel 'Sym Problemen' dutch

at $x = \frac{1}{2} = y$ $f(\frac{1}{2}, \frac{1}{2}) = \left(\frac{1}{8}\right)^2$ but there are larger values $f(\frac{1}{8}, \frac{7}{8}) = \left(\frac{5}{32}\right)^2$
and smaller $f(\frac{1}{4}, \frac{3}{4}) = 0$

So Purkiss principle must only mean a local extrema.

we want Lag Mult cond $\nabla f = \sum \lambda_i Dg^i$ or for 1 constraint $g: \mathbb{R}^n \to \mathbb{R}$
$$\nabla f = \lambda \nabla g$$
This is what he deals with until thread

example $\quad f(x,y,z) = xy + yz + zx \qquad$ elementary sym polys

$g(x,y,z) = (xyz)^2$

$$\nabla f = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix} \qquad \nabla g = 2xyz \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

Plug in $\langle r, r, r \rangle = \vec{r}$

$$\nabla f_r = \begin{bmatrix} 2r \\ 2r \\ 2r \end{bmatrix} \qquad \nabla g_r = 2r^5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

All entries are equal in ∇ so we always will have $\nabla f_r = \lambda \nabla g_r$

Lemma 1 $\quad f(Sx) = f(x) \;\; \forall S \in S_n \Rightarrow$ When all x_i are equal, all $D_i f$ are equal:

$x_1 = r,\ x_2 = r \ldots x_n = r$

so $x = \langle r, r, \ldots r \rangle = \vec{r}$

Then $D_i f_r = \alpha \;\; \forall i$

Maybe should name the fcn "h" since we want to apply these lemmas to $h=f$ and $h=g$

$$\left[\text{Thus if } \begin{matrix} D_i f_r = \alpha \;\; \forall i \\ D_i g_r = \beta \;\; \forall i \end{matrix} \Rightarrow \nabla f_r = \left(\frac{\alpha}{\beta}\right) \nabla g_r \right]$$

Pf. $\quad f(Sx) = f(x)$

$\Rightarrow Df_{Sx} DS_x = Df_x \qquad$ but S is a perm, so linear $\quad DS_x = S$

Say $S = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$

$Df_{Sx} S = Df_x$

let $q = \langle y, x, z \rangle = S(x)$

$$\begin{bmatrix} D_1 f_q & D_2 f_q & D_3 f_q \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} D_1 f_x & D_2 f_x & D_3 f_x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & D_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & D_2 \end{bmatrix} + \begin{bmatrix} D_3 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} D_3 f_q & D_1 f_q & D_2 f_q \end{bmatrix} = \begin{bmatrix} D_1 f_x & D_2 f_x & D_3 f_x \end{bmatrix}$$

Fixed pt.

Now if $x = r = \langle r, r, r \rangle$ then $\underbrace{Sx}_{q} = x$ that is to say $Sr = r$

$$\begin{bmatrix} D_3 f_r & D_1 f_r & D_2 f_r \end{bmatrix} = \begin{bmatrix} D_1 f_r & D_2 f_r & D_3 f_r \end{bmatrix}$$

$D_3 f_r = D_1 f_r$

$D_1 f_r = D_2 f_r$

$D_2 f_r = D_3 f_r$

so all must be equal

So for any

any perm that is a big cycle of all the elts.

But in general, I think we'd need all perms $S \in S_n$

no cycles, no fixed pts

$\square$

For the 2^{nd} deriv, we need Robinson

Robinson Lemma 1 $\nabla f_{x^*} = \lambda \nabla g_{x^*}$ on $S = g^{-1}(0)$ $\Rightarrow$ $\frac{d^2}{dt^2} f(\sigma(t))\Big|_{t=0} = v^T [D^2 \mathcal{L}_x] v$

σ is a curve $\sigma(0) = x^*$, $\sigma'(0) = v$ $\uparrow$ $T_x M$

so $v \in T_x S$

$\mathcal{L} = f - \lambda g$

$v^T [D^2 f_{x^*} - \lambda D^2 g_{x^*}] v$

This gives what Waterhouse has

quadratic $Q(v) := v^T [D^2 f_p - \lambda D^2 g_p] v$ this should be pos or neg def

ex Let's see it for our example:

$f = xy + yz + zx$ $\qquad g = (xyz)^2$

$Df_x = [(y+z) \ (x+z) \ (x+y)]$ $\qquad Dg_x = 2xyz [yz \ \ xz \ \ xy]$

$= [2xy^2z^2, \ 2x^2yz^2 \ \ 2x^2y^2z]$

then $D^2 f_x = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\qquad D^2 g_x = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

but $D^2 g_r = r^4 \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$

Lemma 2 $f(Sx) = f(x) \ \forall S \in \mathcal{S}_n$ (or at least S is one big cycle)

f C^2 smooth $\Longrightarrow$ $D^2 f_r = \begin{bmatrix} \alpha & \beta & \beta \\ & \alpha \ \cdot & \beta \\ \beta & & \cdot \\ \beta & \beta & \alpha \end{bmatrix}$

- main diag elts equal
- off diag elts equal

Pf.

From Lemma 1, $(f \circ S)(x) = f(x)$

$Df_{Sx} S = Df_x$ $\qquad Sr = r$ $\qquad [D_1 f_r \ D_2 f_r \ D_3 f_r][S] = [D_1 f_r \ D_2 f_r \ D_3 f_r]$

operating from the back, S perms the cols of Df

Now do 2^{nd} deriv: $D^2(f \circ S)_x (h,k) = D(Df_{Sx}(Sh))(k)$

$= D^2 f_{Sx}(Sh, Sk) = [Sk]^T [D^2 f_{Sx}] [Sh]$

or $D^2(f \circ S)_x = S^T [D^2 f_{Sx}] S$

Thus $S^T [D^2 f_{Sx}] S = D^2 f_x$ $\qquad$ plug in $x = r = \langle r, \ldots r \rangle^{1 \ldots n}$ $\quad S^T D^2 f_r S = D^2 f_r$

cont'd →

Lemma 2 pf cont'd

Lets look at a 3x3 example. we know $D^2 f_x$ is symmetric matrix

$$D^2 f_x = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

$S^T = S^{-1}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} e & a & d \\ f & d & b \\ c & e & f \end{bmatrix}} \stackrel{!}{=} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

$$\begin{bmatrix} c & e & f \\ e & a & d \\ f & d & b \end{bmatrix}$$

main diag is sent to itself and permed. Same for non-diag elts.

How can we prove this holds for any perm and a $n \times n$ matrix?

To presene the main diag consider elt (i,i) if row $i \longrightarrow$ row j

Then we must also map col $i \longrightarrow$ col j

$$\begin{bmatrix} (i,i) & & \\ \downarrow & \ddots & \\ (j,i) & \longrightarrow & (j,j) \end{bmatrix}$$

when S acts from the back, it perms the rows
when S^T acts from the front, it has exactly the same perm of the cols.

Consider this index vector $y^T = [1\ 2\ 3]$ then $\overset{y^T}{[1\ 2\ 3]} \overset{S}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = [3\ 1\ 2]$

But $S^T y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

same perm

And, in general, we see $(S^T y)^T = y^T S$ So what S in the back does to the cols, S^T in front does to rows

Thus main diag is preserved as a set, but permed.

Thus $D_{ii} f_r = D_{jj} f_r$ for any i, j if we consider all perms $S \in \mathcal{S}_n$

By a similiar arg, if $(i+p, i)$ off diag

This is going to yield $D^2 f_r = \begin{bmatrix} b & c & c \\ c & b & c \\ c & c & b \end{bmatrix}$ (no relation to previous use of these letters)

$\square$

Lemma 3 Consider quadratic form

$$Q(v) = v^T \begin{bmatrix} b & c & c \\ c & b & c \\ c & c & b \end{bmatrix} v = v^T \begin{bmatrix} b & & \\ & b & \\ & & b \end{bmatrix} v + v^T \begin{bmatrix} 0 & c & c \\ c & 0 & c \\ c & c & 0 \end{bmatrix} v$$

If $\sum v_i = 0 \implies Q(v) = (b-c)\sum v_i^2 = (b-c)\|v\|_2^2$

pos def if $b>c$

neg def $b<c$

Pf. Observe $\sum v_i = [1\ 1\ 1]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Then $\nabla g_r \cdot v = 0$ is $g(r)[1\ 1\ 1]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$ so we recover this cond

$v \perp \nabla g_r$ i.e. $v \in T_x S$

should call it M

$$v^T \begin{bmatrix} b & & \\ & b & \\ & & b \end{bmatrix} v = b\,[v_1\ v_2\ v_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = b\|v\|_2^2$$ so that one is easy

$$v^T \begin{bmatrix} 0 & c & c \\ c & 0 & c \\ c & c & 0 \end{bmatrix} v = [v_1\ v_2\ v_3]\begin{bmatrix} 0 & c & c \\ c & 0 & c \\ c & c & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = [v_1\ v_2\ v_3]\begin{bmatrix} c(v_2+v_3) \\ c(v_1+v_3) \\ c(v_1+v_2) \end{bmatrix}$$

$-v_1$, $-v_2$, $-v_3$

$\sum v_i = 0 \quad v_1 + v_2 + v_3 = 0$

$v_2 + v_3 = -v_1$

$$= -c v_1^2 - c v_2^2 - c v_3^2$$

$$= -c\|v\|_2^2$$

$$\implies Q(v) = (b-c)\|v\|_2^2$$

$\square$

Purkiss Principle

we only need subgroup of full size cycle $(1\ 2 \dots n) \subset S_n$ see example

$S: \mathbb{R}^n \to \mathbb{R}^n$ linear perm matrix O.N.

$f: \mathbb{R}^n \to \mathbb{R}$ are $f(Sx) = f(x)\ \forall S \in S_n$

$g: \mathbb{R}^n \to \mathbb{R}$ $\quad g(Sx) = g(x)$ " " " "

Let $M := g^{-1}(0)$

$S(M) = M$ invariant.

distances between pts unchanged

$\implies f|_M$ has a local extrema where $M \cap \mathcal{E}$

any pt m where $Sm = m$ $\forall S \in \mathcal{G}$. Fixed pt

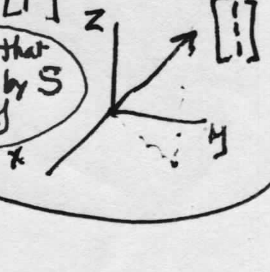
$\mathcal{E}$ is the line where all components are equal

$$\mathcal{E} = \left\{ r \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \;\middle|\; r \in \mathbb{R} \right\}$$ all

example $\quad f(x,y,z) = \dfrac{x}{y^2+1} + \dfrac{y}{z^2+1} + \dfrac{z}{x^2+1}$ $\quad$ Michael Penn youtube

$g(x,y,z) = x+y+z-3$ $\quad$ this is a plane in $\mathbb{R}^3$ invariant under S

$\mathcal{E}$ is pts that are fixed by S $\forall S \in \mathcal{G}$



The inequality is $\dfrac{x}{y^2+1} + \dfrac{y}{z^2+1} + \dfrac{z}{x^2+1} \le \dfrac{3}{2}$

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Pf of Purkiss $p = \langle r, r, \dots r \rangle$

assume $\nabla g(p) \neq \vec{0}$ Lemma 1 $\Rightarrow$ $\nabla f_p = \lambda \nabla g_p$

$D^2 f_p - \lambda D^2 g_p = D^2 \mathcal{L}_p$ where $\mathcal{L} := f - \lambda g$ as in Robinson

$$D^2 \mathcal{L}_p = \begin{bmatrix} b & c & c \\ c & b & c \\ c & c & b \end{bmatrix}$$

$$\nabla g_p = \varphi(p) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Thus $v \perp \nabla g_p \Rightarrow \nabla g_p \cdot v = 0 \Rightarrow \varphi(p)\left(\sum v_i\right) = 0 \Rightarrow \sum^n v_i = 0$

Then by Lemma 3 $v^T \left[D^2 \mathcal{L}_p\right] v = (b - c)\|v\|_2^2$ pos or neg def

$b = D_{ii}\mathcal{L}_p$ $c = D_{ij}\mathcal{L}_p$ for any $i \neq j$

take $i = 1$
since always same value.

$\Rightarrow$ p is an extrema of $f|_M$ $\square$

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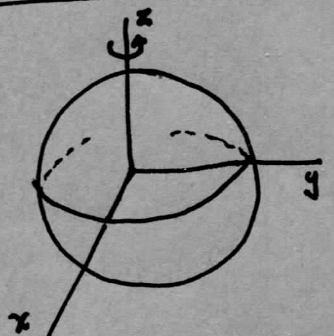
examples of degeneracy



Rotation $Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ take $\theta = \pi/2$

$g(x,y,z)$ $x^2 + y^2 + z^2 - \rho^2$ is invariant

$g(Qx) = g(x)$ $M = S^2 = g^{-1}(\rho^2)$ is invariant

$f = z^2$ is invariant (coriolis deflection as fcn of φ could be a better example? ↳ more non-trivial)

North pole N, S would be extrema of f, not saddle pts.

Instead of Sym wrt perms of the variables, consider more general linear maps $T \ni f(Tx) = f(x)$ → we will call f an invariant fcn

We don't need to consider continuous Sym groups (Lie Groups)

$T \in G$ $\mathcal{G}$ some finite group

$T: \mathbb{R}^n \to \mathbb{R}^n$ (he says $T: V \to V$)

$Tp = p$ means p fixed pt. ($Sr = r$ was FP before)

Lemma 4 I don't know what he is doing

But we want $\langle Tx, Ty \rangle = \langle x, y \rangle$ $\forall T \in \mathcal{G}$. Say $T = Q$ O.N. matrix.

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$$Tp = p \qquad v^T A v$$

Consider Taylor expansion of f about $\overset{FP}{p}$

$$f(p+v) = f(p) + \underbrace{Df_p(v)}_{\nabla f_p \cdot v} + \underbrace{\frac{1}{2!} D^2 f_p(v,v)}_{q(v) \text{ quadratic form}} + \mathcal{O}(v^3)$$

$$= f(p) + \langle \nabla f_p, v \rangle + q_p^f(v) + \mathcal{O}(v^3)$$

Now since $f(Tx) = f(x)$

let $x = p + v$

$$f(p+v) = f(T(p+v)) = f(p + Tv) \overset{!}{=} f(p) + \langle \nabla f_p, Tv \rangle + q_p^f(Tv, Tv) + \mathcal{O}$$

By uniqueness (of terms in Taylor expansion) we see $q_p^f(Tv) \overset{!}{=} q_p^f(v)$

and $\langle \nabla f_p, v \rangle \overset{!}{=} \langle \nabla f_p, Tv \rangle$

Invariance of IP: (or just def of adjoint) $\langle \nabla f_p, Tv \rangle = \langle T^{-1} \nabla f_p, v \rangle$ (This follows easily if $T = Q$ O.N. matrix)

Thus $\langle \nabla f_p, v \rangle = \langle T^{-1} \nabla f_p, v \rangle \; \forall v$

$$\Rightarrow \nabla f_p \overset{!}{=} T^{-1} \nabla f_p$$

or $T(\nabla f_p) = \nabla f_p$ and ∇f_p is fixed by T.

This proves:

Lemma 5: f is invariant under T $(f(Tx) = f(x) \; \forall x)$, $Tp = p \quad \forall T \in \mathcal{G}$ $\Rightarrow$ $T(\nabla f_p) = \nabla f_p$, $q_p^f(Tv) = q_p^f(v) \; \forall T \in \mathcal{G}$

To get the analogue of Purkiss, we need more hypotheses:

Lemma 1 had $\nabla f_r = \underset{\text{scalar}}{\alpha(r)} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $\nabla g_{\bar r} = \underset{\text{scalar}}{\beta(r)} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ Thus $\nabla f_{\bar r} \overset{!}{=} \lambda \nabla g_{\bar r}$ since both in span$\left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$.

Here we require $Tx = x$ only if $x \in U_0$, a 1-dim subsp.

We need an analogue of $\sum N^{(i)} = 0$ i.e $\nabla g_{\bar r} \cdot v = \beta(r) [1 \; 1 \cdots 1] \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix} = 0$

It will be an invariant subsp

Lemma 6 $\left.\begin{array}{l} U \text{ invariant under } T \\ T|_U : U \to U \\ \text{Let } W := U^\perp \text{ wrt } \langle \cdot, \cdot \rangle \end{array}\right\} \Rightarrow$ W is also invariant under T $\quad T|_W : W \to W$

Pf see ~~Sch~~ Schuams LA in particular, this holds for Q O.N. matrix

In particular $U_0^\perp =: W_0$

In Lemma 2 we needed the perms to move all the matrix elts, no elt mapped to itself.
In Lemma 2 we needed the perms to move all the elts, no elt mapped to itself.
Here, the analogue is we require W_0 to have no invariant subspaces (only $\{0\}$)
They call this irreducibility. Lemma 3 generalized as:

Lemma 7: $\left.\begin{array}{l} \mathcal{G} \text{ fin group linear maps } T: \mathbb{R}^n \to \mathbb{R}^n \\ \text{Each } T \text{ acts irreducibly on } W \end{array}\right\} \Rightarrow$ quadratic form $q(w) = \alpha \langle w, w \rangle$ for some $\alpha \in \mathbb{R}$
$= \alpha \|w\|_2^2$
Pos or neg definite

pf. $q(w) = \langle Bw, w \rangle = w^T B w$ for uniq self-adj map $B: W \to W$
we know $q_p^f(Tv) = q(v)$
in particular $q(w) = q(Tw)$

$T^* = T^{-1}$ for T ON and other cases ($T^T = T^T$)

$$\langle Bw, w \rangle = q(w) = q(Tw) = \langle BTw, Tw \rangle = \langle T^* BTw, w \rangle \ \forall w$$

(adjoint)

$$\Rightarrow \quad B = T^{-1} BT \quad \text{or} \quad TB = BT$$

The $\mathbb{R}$ self-adj map B has a $\mathbb{R}$ EW λ_0
~~Let~~ Let X be the eigenspace $\{\text{all } w \in W \mid Bw = \lambda_0 w\}$

~~Thus X is invariant under~~
For $w \in X$, $T \in \mathcal{G}$ we have $B(Tw) = TBw = T(\lambda_0 w) = \lambda_0 (Tw)$
Thus $Tw \in X$

Thus X is an invariant subsp of W, so by hypoth $X = W$

Thus $B = \lambda_0 (Id)$ and so $q(w) = \lambda_0 \langle w, w \rangle$

$\square$

extended to linear maps that are not perms.

Thm Extended Purkiss Prince

$\mathcal{G}$ fin dim group of linear maps $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\exists$ 1-dim subsp U_0 where $Tu = u$ $\forall\, T \in \mathcal{G}$, $u \in U_0$

There is no other non-trivial invariant subsp in $U_0^{\perp}$

$f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad f(Tx) = f(x)$ $\forall\, T \in \mathcal{G}$

$g(Tx) = g(x)$ $\forall\, x \in$ nbhd of p where $Tp = p$

$\Longrightarrow$

let $c := g(p)$

$M := g^{-1}(c)$

$f|_M$ has local extrema at p (except in degenerate cases)

Pf. is just like previous Purkiss.

Modern Purkiss

M smooth mfd $\quad \tau: M \longrightarrow M$

$\mathcal{G}$ finite group of smooth maps $\tau: M \longrightarrow M$

$\exists\, p \in M$ s.t. $\tau(p) = p$ $\forall\, \tau \in \mathcal{G}$

$f: M \longrightarrow \mathbb{R}$ smooth

$f(\tau(x)) = f(x)$ $\forall\, x \in M$, all $\tau \in \mathcal{G}$

$d\tau_p : T_pM \longrightarrow T_pM$ has no nontrivial invariant subspaces [irreducibility]

$\Longrightarrow$

$df_p = 0$ (critical pt)

If non-degen, p is a local extrema of f on M

pf

f gives us $df_p : T_pM \longrightarrow \mathbb{R}$

$\ker(df_p)$ is invariant under all $\tau \in \mathcal{G}$

Irreducibility $\Longrightarrow df_p = 0$

Hence p is critical pt of f

2^{nd} order term for "$d^2 f$" induce quadratic form q on tangent sp T_pM

"clearly" $q(\tau v) = q(v)$ $\forall\, v \in T_pM$, $\tau \in \mathcal{G}$