Marion CDOPAS
Ch 9
prob # 9-4

shape of water in a rotating bucket

12/13/2000

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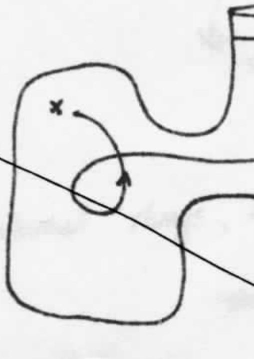
Preludes:
- How do we define ~~pressure~~ pressure in a static fluid.
- why is the surface flat for a stationary bucket?
- ~~Descr~~ Demonstrate why the surface slants for a uniformly accelerated bucket on a railway car.

Definition of Pressure in a static fluid:

Fix an arbitrary reference pt x_0 (However, for convenience we will take it to be at the free surface where, a priori, the fluid pressure is 0)

We can compute the pressure difference between any 2 pts. by connecting them by a tube (of const cross sectional area) wholly contained in the fluid.



Fix a pt x

Pressure at x. $P(x) = \dfrac{F}{A} = \dfrac{\text{"weight of fluid in tube"}}{\text{Area of tube}}$

Let γ be any curve connecting A to x;

Then thicken γ to a tube having const cross sectional area A

Fluid has const density ρ.

$$\text{Weight} = \text{mass}\cdot g \approx \sum_{\substack{\text{all segments}\\ \text{of curve}}} \overbrace{\rho A \cdot \Delta \vec{s}_i}^{\text{mass}} \cdot \vec{g}$$

$$= \rho A \int_\gamma \vec{g}\cdot d\vec{s}$$

specifically, the ~~line integral~~ "signed weight"

If there could be no flow thru the walls of the tube, and gravity had to pull the fluid thru the tube.

Then

$$P(x) = \frac{\rho A \int_\gamma \vec{g}\cdot d\vec{s}}{A} = \rho \int_0^1 [0, g]\cdot[\dot{\gamma}_1(t), \dot{\gamma}_2(t)]\, dt$$

$$= \rho g \int_0^1 \gamma_2'(t)\, dt$$

$$= \rho g\left(\gamma_2(1) - \gamma_2(0)\right)$$

so all it depends upon is the vertical height difference, no matter how wierd the curve γ.

$$P(x) = \rho g (y - y_0)$$

This shows clearly that the free surface of a container of water is flat, and that the level surfaces of const pressure are all planes.

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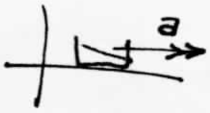
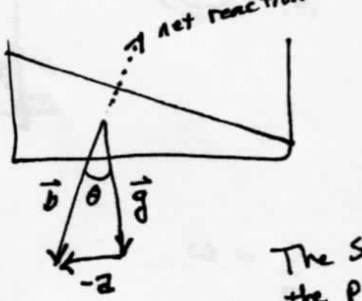


Now consider an uniformly accelerating cart w/ a tank of water.

Transform to a co-ord system moving with the cart so that the water is at rest.



Consider all the accelerations on a "particle" of water.

$$\ddot{\beta} = \ddot{\alpha} - \ddot{\sigma}$$

$$\ddot{\alpha} \to \vec{g} + \frac{1}{m}\vec{F}_R \qquad \ddot{\sigma} \to a$$

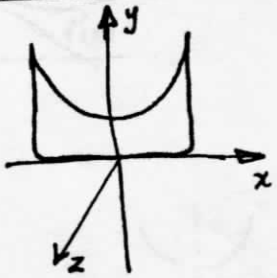
The surface is $\perp$ to the pressure gradient.

Distinguish between accelerations due to external things, like gravity, and the reaction force of the water. ~~The surface~~

$$\ddot{\beta} = (\vec{g} - \vec{a}) - \frac{1}{m}F_R$$

Then we have reduced the problem to the previous case. (Pressure is measured along the line $\vec{b}$ (If you like we could further rotate the axes by θ).

Now consider a uniformly rotating bucket of water.



Transform to rotating co-ord system so water is at rest.

again we will find the non-reaction forces and the surface (level surface of 0 pressure) will be O.G. to the net external force.

$$\cancel{m}\ddot{\alpha} = \cancel{m} Q_t(\ddot{\beta} + \underbrace{\dot{\omega}\times\beta}_{0} + \underbrace{2\omega\times\dot{\beta}}_{0} + \omega\times(\omega\times\beta)) + \underbrace{\cancel{m}\ddot{\sigma}}_{0}$$

$$\ddot{\alpha} = Q_t(\ddot{\beta} + \omega\times(\omega\times\beta)) \qquad \vec{\omega} = \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix}$$

$$\ddot{\beta} = Q_t^T\ddot{\alpha} - \omega\times(\omega\times\beta)$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{\beta}) = -\omega^2\begin{bmatrix} \beta_1 \\ 0 \\ \beta_3 \end{bmatrix}$$

$$Q_t = \begin{bmatrix} \cos(\omega t) & 0 & -\sin\omega t \\ 0 & 1 & 0 \\ \sin(\omega t) & 0 & \cos(\omega t) \end{bmatrix} \qquad \ddot{\alpha} = \vec{g} + \frac{1}{m}F_R$$

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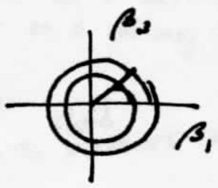
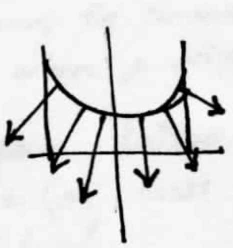
$$\ddot{\beta} = -g\hat{e}_2' + Q_t^T(F_w) + \omega^2 \begin{bmatrix}\beta_1\\0\\\beta_3\end{bmatrix}$$





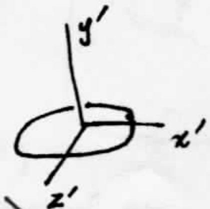
convert to polar co-ords (cylindrical co-ords)

$$\ddot{\beta} = \underbrace{\left(-\vec{g} + \omega^2\vec{r}\right)} + \underbrace{Q_t^T(F_w)}_{\text{reaction force of water.}}$$

Then the surface is $\perp$ to this vector at all pts.

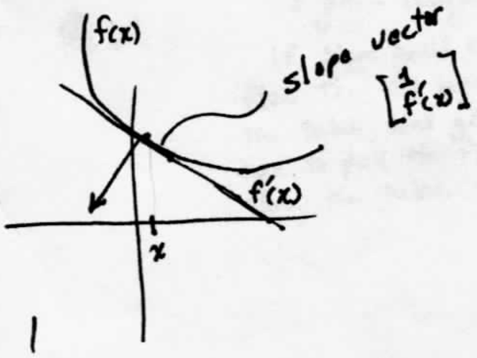
$\omega = \frac{v}{r}$

$a_N = \frac{v^2}{r} = \frac{\omega^2 r^{\cancel{2}}}{\cancel{r}}$



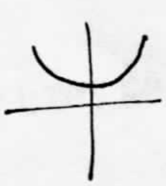
Fix a particular pt on the circle say the x' axis. (aka the $\theta = 0$ axis in ~~polar~~ cylindrical co-ords)

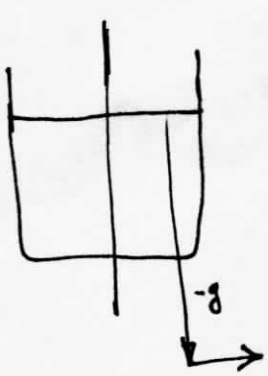
then $\beta(t) = \begin{bmatrix}x\\0\\0\end{bmatrix}$



$$\ddot{\beta} = \begin{bmatrix}0\\-g\\0\end{bmatrix} + \omega^2\begin{bmatrix}x\\0\\0\end{bmatrix} = \begin{bmatrix}\omega^2 x\\-g\\0\end{bmatrix}$$

want $[1 \;\; f'(x)] \cdot [\omega^2 x \;\; -g] \overset{!}{=} 0$





$$\omega^2 x + g f'(x) = 0$$

$$f'(x) = +\frac{\omega^2}{g}x$$

$$\Rightarrow \boxed{f(x) = +\frac{\omega^2}{2g}x^2 + c}$$

parabolic ~~but concave down~~?