9/21/2022

Vectors ('polar' vectors) vs. Axial vectors (psuedo vectors)

axial vectors are a derived "arrow object" that comes from the cross-prod of 2 vectors

First we show how axial vectors behave differently under reflections $R: \mathbb{R}^3 \to \mathbb{R}^3$

Result 1 Cross-prod in different bases

usually we have $\{f\} = \{e\}P$ so $v_f = P^{-1} v_e$

Here I will write $\{f\} = \{e\} A^{-1}$, $v_f = A v_e$

$$\cancel{\{e\}}(a_e \times b_e)_e = \{f\} \frac{1}{\det A} (a_f \times b_f)_f$$

Reflection changes the orientation of the basis.

$$= \cancel{\{e\}} \frac{A^{-1}}{\det A} (a_f \times b_f)_e$$

$$\Rightarrow A(a_e \times b_e) = \frac{1}{\det A}(A a_e \times A b_e) \Rightarrow \boxed{A(a \times b) = \frac{1}{\det A}(Aa \times Ab)}$$

Now if we plug in a reflection $R: \mathbb{R}^3 \to \mathbb{R}^3$ (we know $\det(R) = -1$)

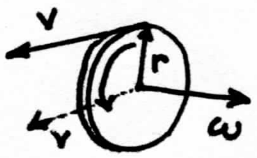
$$R(a \times b) = -1 (Ra \times Rb) \quad \text{or} \quad \boxed{Ra \times Rb = -R(a \times b)}$$

reflection thru mirror plane

reflection wrt origin

Thus if we reflected the vectors a, b to Ra and Rb and took the cross prod (right handed cross prod) we do not get $R(a \times b)$ but instead $-R(a \times b)$.

▷ Lets see how this works in a physical setting: Angular velocity $\vec{\omega}$ is the prototype of axial vectors $\vec{v} = \vec{\omega} \times \vec{r}$



we can solve for $\vec{\omega}$: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(a \cdot c) - \vec{c}(a \cdot b)$

Thus

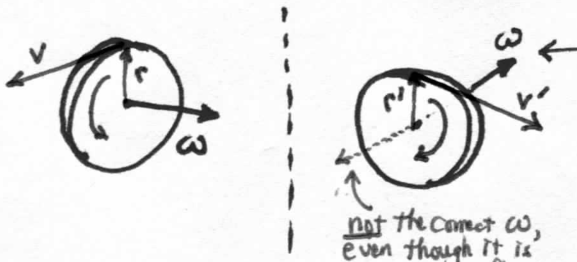
$$r \times v = r \times (\omega \times r)$$
$$= \vec{\omega}(r \cdot r) - \vec{r}(\cancel{r \cdot \omega})$$

0 since ⊥

$$\boxed{\vec{\omega} = \frac{1}{r^2} \vec{r} \times \vec{v}}$$

Key Idea when we do a reflection (a math operation on $\mathbb{R}^3$) in a physics problem, we want the reflected object to have "artifacts" of the reflection — it should be just like the real object looked at from a different vantage pt.

For example, the spinning wheel

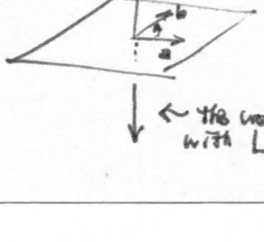


not the correct ω, even though it is the vector reflection

We want $\vec{\omega} = \vec{r} \times \vec{v}$ in the orig and the reflection, otherwise we can tell it is a reflection

$r' \times v'$ points as shown, and it is $-R(r \times v)$

Aside observe



← this would be a×b with Left hand rule.

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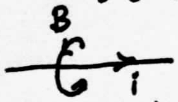
The magnetic field $\vec{B}$ is also an axial vector.

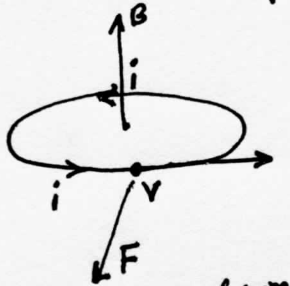
Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$

Note: This is the force on a particle with charge q moving with velocity $\vec{v}$ in a pre existing mag field $\vec{B}$ — not the $\vec{B}$ generated by the moving particle.

we could solve $\vec{B} = \frac{-1}{v^2}(\vec{v} \times \vec{F})$

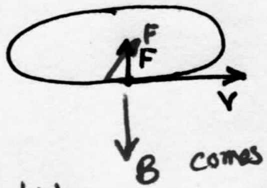


Point thumb in dir of i; fingers curl around wire as shown for Right hand.



Circular loop of wire

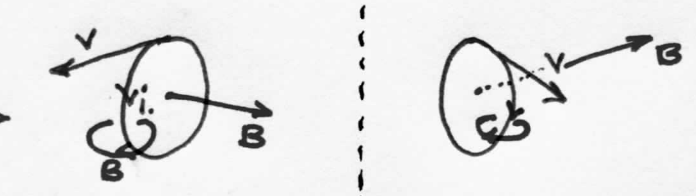
By contrast, to make the particle move in a circle, there must be an accel (Force) pointing to the center. But if I try



$\vec{B} = \frac{-1}{v^2}(v \times F)$

comes out pointing down

But the main idea is that this is just like the rotating wheel

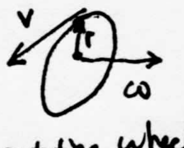


Important Result: For axial vector $\vec{\omega}$ under reflection

$\omega = \vec{\omega}_{\parallel} + \vec{\omega}_{\perp}$ Then $\omega' = -\vec{\omega}_{\parallel} + \vec{\omega}_{\perp}$ That is $\omega_{\parallel} \longrightarrow -\omega_{\parallel}$, $\omega_{\perp} \longrightarrow \omega_{\perp}$

exactly opposite a real vectors reflection!

Pf. I couldn't show this analytically, but I can show it with pictures:



rotating wheel

edge view →

tilt