Done better on prev page

Then for $Ax = b$ $\quad x = A^{-1}b = \frac{1}{\det A} A_{cof} b$

▷ **Cramer's Rule** $\quad x_j = \frac{\det B_j}{\det A}$ where $B_j = \begin{bmatrix} | & & | & & | \\ a_1 & \cdots & b & \cdots & a_n \\ | & & | & & | \end{bmatrix}$ replace j^{th} col of A with vector b

Pf. Expand $\det(B_j)$ in cofactors down col j (i.e. b)

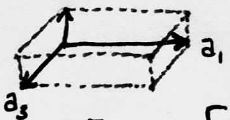
$$\det B_j = b_1 |A|_{1j} + b_2 |A|_{2j} + \dots + b_n |A|_{nj} = \underbrace{\begin{bmatrix} & \cdots & \\ |A|_{1j} & |A|_{2j} \cdots & |A|_{nj} \\ & \cdots & \end{bmatrix}}_{A_{cof}} \begin{bmatrix} \\ b \\ \\ \end{bmatrix} \quad \square$$

p.234 Thm

▷ $\boxed{\det A = \text{volume of } n\text{-dim parallelipiped with edges from the rows of } A \text{ (we could also take cols since } \det A = \det A^T)}$

Pf. Assume A nsing. If A sing, $\det A = 0$ — and the rows would only determine a lower dim parallelipiped, which would have n-dim vol $= 0$.

Easy case: Rows of A are O.G.



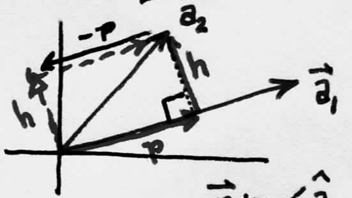
vol $= \|a_1\| \cdot \|a_2\| \cdot \|a_3\|$

$$A = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{bmatrix} \qquad AA^T = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{bmatrix} \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \|a_1\|^2 & & \\ & \|a_2\|^2 & \\ & & \|a_3\|^2 \end{bmatrix}$$

$$\underbrace{\det(AA^T)}_{\substack{\det A \det A^T \\ = (\det A)^2}} = \|a_1\|^2 \cdot \|a_2\|^2 \cdot \|a_3\|^2 = (\text{vol})^2$$

$$\Rightarrow \det A = \pm \text{Vol}$$ [sign depends on O^R of rows of A as a basis for $\mathbb{R}^n$]

General case we would have something like:



We can reduce this to the easy case by doing the Gram-Schmidt process, and we can do this by elem row ops that preserve the det.

$\vec{p} := \langle \hat{a}_1, a_2 \rangle \hat{a}_1$ proj $\vec{a}_2$ onto $\hat{a}_1$.

$h := \vec{a}_2 - \vec{p}$

Given LI set $\{\vec{a}_1 \dots \vec{a}_n\}$

$a_1' := a_1 \qquad \hat{q}_1 := \frac{1}{\|a_1\|} a_1'$

$a_2' := a_2 - \langle \hat{q}_1, a_2 \rangle \hat{q}_1$

$$= a_2 - \underbrace{\left(\frac{a_2^T a_1}{a_1^T a_1}\right)}_{\text{Just a scalar}} a_1$$

$$a_3' := a_3 - \frac{a_1^T a_3}{a_1^T a_1} a_1 - \frac{a_2^T a_3}{a_2^T a_2} a_2$$

We can express this in the form of elem matrices (I'll combine all of them into one matrix)

$$\underset{\sim}{A} = \begin{bmatrix} — a_1' — \\ — a_2' — \\ — a_3' — \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & \\ \frac{-a_1^T a_2}{a_1^T a_1} & 1 & \\ \frac{-a_1^T a_3}{a_1^T a_1} & \frac{-a_2^T a_3}{a_2^T a_2} & 1 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} — a_1 — \\ — a_2 — \\ — a_3 — \end{bmatrix}}_{A}$$

$$\det(\underset{\sim}{A}) = \underbrace{\det G}_{1} \det A = \det A$$

Then $\underset{\sim}{A}\underset{\sim}{A}^T = \begin{bmatrix} \|a_1'\|^2 & & \\ & \|a_2'\|^2 & \\ & & \|a_3'\|^2 \end{bmatrix}$ $\quad (\det \underset{\sim}{A})^2 = \text{vol}^2$

$\overset{\shortparallel}{(\det A)^2}$ so again $\det A = \pm \text{vol}$

(cf Munkres ADM ch 5 §21 p.183) □

▷ Before <u>Formula for Pivots during Gaussian Elimination</u>, let's do a digression on block matrices:

In ch 5.6 sheet ⑧ we use

$$\begin{array}{c} 1 \\ (n-1) \end{array}\left[\begin{array}{c|c} \alpha & b^T \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} \beta & e^T \\ \hline f & G \end{array}\right] = \left[\begin{array}{c|c} \alpha\beta + b^T f & \alpha e^T + b^T G \\ \hline \beta\vec{c} + D\vec{f} & ce^T + DG \end{array}\right] \overset{\substack{\text{if } \vec{f}=\vec{0} \\ \vec{c}=\vec{0}}}{=\!=} \left[\begin{array}{c|c} \alpha\beta & \alpha e^T + b^T G \\ \hline 0 & DG \end{array}\right]$$

(columns of the first factor: 1, $(n-1)$)

(we started with block upper triang)

still upper triang

Here we shall work with bigger blocks in $A = LDU$ and at the end we shall compute dets of block matrices, like $\mathbb{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ $\quad \det \mathbb{A} = \det A \det D$

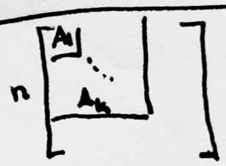
<u>Thm 4E</u> If the factorization exists $A = LDU$ $\Rightarrow$ In fact, for every $k \le n$, the principal submats from the upper left corner satisfy

$$A_k = L_k D_k U_k$$

(Each A_k is going thru Gaussian elim on its own)

Pf. We have $A = LDU$

For any given $k < n$, partition a size k block in the corner of each L, D, U. We will do block mult and show $A_k = L_k D_k U_k$



$$A = \begin{bmatrix} L_k & 0 \\ B & C \end{bmatrix} \underbrace{\begin{bmatrix} D_k & 0 \\ 0 & E \end{bmatrix}\begin{bmatrix} U_k & F \\ 0 & G \end{bmatrix}}$$

$$\left[\begin{array}{c|c} D_k U_k + 00 & D_k F + 0G \\ \hline 0U_k + E0 & 0F + EG \end{array}\right] = \left[\begin{array}{c|c} D_k U_k & D_k F \\ \hline 0 & EG \end{array}\right]$$

cont'd →

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$$\begin{bmatrix} L_k & O \\ B & C \end{bmatrix} \begin{bmatrix} D_kU_k & D_kF \\ O & EG \end{bmatrix} = \left[\begin{array}{c|c} L_kD_kU_k + OO & L_kD_kF + OEG \\ \hline BD_kU_k + CO & BD_kF + CEG \end{array}\right]$$

$$= \left[\begin{array}{c|c} \boxed{L_kD_kU_k} & L_kD_kF \\ \hline BD_kU_k & BD_kF + CEG \end{array}\right] = \left[\begin{array}{c|c} A_k & \\ \hline & \end{array}\right] = A$$

□

Strang also has the convention that writing $A = LDU$ means no row exchanges during the elimination process. If there were row exchanges we have $PA = LDU$.

Thm 4F Gaussian elim on A requires no row exchanges $\Leftrightarrow$ For every principal submat A_k, $\det(A_k) \neq 0$

($\Rightarrow$) show $A = LDU \Rightarrow \det A_k \neq 0 \ \forall k$

GE succeeded with no row exch so $A = LDU$

By Thm 4E, the submats factor $A_k = L_kD_kU_k$

we know $\det A_k = \det L_k \det D_k \det U_k = 1 \cdot \det D_k \cdot 1 = d_1 \cdots d_k$

Because GE completed with no row exch, $d_i \neq 0$ for all $i \leq k < n$ (possibly $d_n = 0$? or $d_{n-1}, d_n = 0$ if A sing?)

Anyway $d_1, \ldots, d_k \neq 0 \Rightarrow \det A_k \neq 0$

($\Leftarrow$) show: $A = LDU \Leftarrow \det A_k \neq 0 \ \forall k \leq n$

pf by induction

($k=1$) $\det A_1 \overset{\det}{=} [a_{11}] = a_{11} \neq 0$ so $d_1 = a_{11}$ and $\underset{A}{[a_{11}]} = \underset{L}{[1]}\underset{D}{[a_{11}]}\underset{U}{[1]}$ no row exch

Assume true for k-1, show k

GE has given us $A_{k-1} = L_{k-1}D_{k-1}U_{k-1}$ where all $d_i \neq 0$ up to $k-1$ and no row exch

Form the larger block $A_k = L_kD_kU_k$

Since $\det A_k \neq 0$ and since $\det A_k = \underbrace{d_1 \ldots d_{k-1}}_{\neq 0} d_k \Rightarrow d_k \neq 0$

pivot is nonzero $\Rightarrow$ no row exch

Thus it continues up to $k = n$

□

▷ Strang addresses one further question: How do we know that a perm is either even or odd, and that this is well defined across all ways to make the perm?

I do a detailed discussion in my Abstract Algebra sheets, sheet (S3) on Sym Group.

Strang LAAIA ch 4

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⑧

Block Matrices and Dets

(out-of-seq with book)

$\boxed{4.3.9}$ Consider block matrix $\mathbb{A} = \begin{array}{c|cc} & k & \ell \\ \hline k & A & B \\ \ell & 0 & D \end{array}$ Block Triang

p.239

$\boxed{\text{Show } \det \mathbb{A} = \det A \det D}$

$\left[\text{NOT true } \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det D - \det B \det C \text{ see further on}\right]$

We can apply elementary row ops to make this matrix upper triangular:

$$E\mathbb{A} = \begin{bmatrix} T_A & \tilde{B} \\ 0 & T_D \end{bmatrix} \text{ where } T_A = \begin{bmatrix} d_1 & * & * \\ & \ddots & * \\ 0 & & d_k \end{bmatrix} \quad T_D = \begin{bmatrix} d_{k+1} & & * \\ & \ddots & \\ 0 & & d_{k+\ell} \end{bmatrix}$$

possibly some $d_i = 0$ if $\mathbb{A}$ sing

Let p be the number of row exchanges nec to make A into T_A and q those for $D \rightsquigarrow T_D$

$$\det \mathbb{A} = \underset{1}{\det E} \cdot \det \mathbb{A} = \underbrace{(-1)^p \prod_{i=1}^{k} d_i}_{\det A} \underbrace{(-1)^q \prod_{j=k+1}^{\ell+k} d_j}_{\det D} = (-1)^{p+q} \prod_{i=1}^{k+\ell} d_i$$

← This should have come before I split it into 2 pieces

$\boxed{4.3.13}$ Show $\det \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det(AB) = \det A \det B$

Let us post-mult by $\underset{\det = 1}{\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}}$: $\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \overset{\mathbb{H}}{\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix}}$

Then by prob $\boxed{4.3.9}$ $\det \mathbb{H} = \det(AB) \det I = \det AB$

p.239

$\boxed{4.4.10}$ $\boxed{\text{Schur Complement}}$

Let $\mathbb{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and let's say A^{-1} exists

(i) We can do block elimination, in the same manner as elim for single entries

$$\begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix} \quad r_2 \rightsquigarrow r_2 - \tfrac{2}{3} r_1 \quad \begin{bmatrix} 1 & 0 \\ -2 \cdot \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 6 - 2 \cdot \frac{1}{3} \cdot 1 \end{bmatrix}$$

$$\underset{\mathbb{E}}{\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}} \underset{\mathbb{A}}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} = \underset{\mathbb{T}}{\begin{bmatrix} A & B \\ 0 & \boxed{D - CA^{-1}B} \end{bmatrix}}$$

— This is called the Schur complement

(ii) $$\det \mathbb{A} = \underset{1}{\det \mathbb{E}} \det \mathbb{A} = \det \mathbb{T} = \det A \det(D - CA^{-1}B)$$ by $\boxed{4.3.9}$

$$= \det(AD - ACA^{-1}B)$$

if we have $AC = CA \longrightarrow$ $$= \det(AD - CB)$$

cont'd ⟶

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⑨

(iii) Note that we cannot say $\det\begin{bmatrix} A & B \\ C & D\end{bmatrix} = \det A \det D - \det B \det C$ even if all the blocks commute - as was erroneously stated in Schuams LA

Counter-ex $\left[\begin{array}{cc|cc} 2 & & 1 & \\ & 2 & & 1 \\ \hline 1 & & 2 & \\ & 1 & & 2 \end{array}\right] = \begin{bmatrix} 2I & I \\ I & 2I \end{bmatrix}$

Then $\det A \det D - \det B \det C = 4 \cdot 4 - 1 \cdot 1 = 15$ ⊛

But from the formula we just proved $\det \mathbb{A} = \det(AD - CB) = \det(4I - I) = \det(3I) = 3^2$
$= 9$

So ⊛ is incorrect

(iv) Here is another, closer form to $\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad - bc$
if we instead assume D^{-1} exists and $CD = DC$

$$\underset{\mathbb{A}}{\begin{bmatrix} A & B \\ C & D\end{bmatrix}} \begin{bmatrix} D & 0 \\ -C & I\end{bmatrix} = \begin{bmatrix} AD-BC & B \\ \underbrace{CD-DC}_{0} & D\end{bmatrix}$$

det both sides

$\det \mathbb{A} \det D \cdot 1 = \det(AD-BC)\det(D)$

if $\det D \neq 0$ $\quad \det \mathbb{A} = \det(AD - BC)$

Gemini (iv) We can actually get the forms $\det \mathbb{A} = \det(AD - BC)$ without requiring D^{-1}
(or the prev one without needing A^{-1})

We use the facts that (1) polys are cont (det is a poly)
(2) Invertible matrices are dense in $\mathbb{R}^n \times \mathbb{R}^n$ well $\mathbb{R}^{n^2}$

Let ~~A~~ $D_\varepsilon := D + \varepsilon I$

D is sing, but for arb small $\varepsilon > 0$ D_ε is nsing

D_ε commutes with C because D does and I commutes with everything

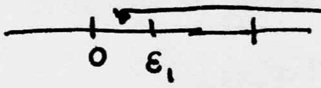
Then we get to $\det\begin{bmatrix} A & B \\ C & D_\varepsilon\end{bmatrix} = \det(AD_\varepsilon - BC)$

Take $\lim_{\varepsilon \to 0}$ and we can take the lim operation inside det, since it is cont □

Notes D_ε is nsing $\forall$ suff small values of $\varepsilon > 0$ because $p(\varepsilon) = \det(D + \varepsilon I)$ is characteristic poly of D for EWs
(have -1 times EWs)

D sing $\Rightarrow$ $\varepsilon = 0$ is EW and there can be at most n

They are discrete points on $\mathbb{R}$



none would be inside $(0, \varepsilon_1)$
thus $\forall\, \varepsilon \in (0, \varepsilon_1)$ $\det(D + \varepsilon I) \neq 0$

Why are invertible matrices (the set Ω of invertible matrices) dense in $\mathbb{R}^{n^2}$?

I wanted to say: let $f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$
$A \longmapsto \det(A)$

Singular matrices $S = f^{-1}(0)$ is a n^2-1 dim submfd and since it is locally diffeo to $\mathbb{R}^{n^2-1}$ any open ball in $\mathbb{R}^{n^2}$ would meet $S^c = \Omega$

But 0 is NOT a Reg value of $f = \det$

Jacobi formula
Anz DC ch1 sheets

$df_A(X) = \mathrm{Tr}(A_{cof} X)$ if rank $A \leq n-2$, this map df_A is not onto $\mathbb{R}$

Gemini says the base idea holds, but S is an algebraic variety because of the singularities

□

(4.17) $\det A > 0$ Show $\exists$ htpy $A \rightsquigarrow I \ni \det A_t > 0 \ \forall_t$ 2/2/2026

LAMA ch 4 ⑩

Gemini Pro 3 : group of Real mats w/ pos det $GL^+(n, \mathbb{R})$ is path conn

(1) SVD $A = U \Sigma V^T$ U, V OG det ± 1

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$ diag

$\det U \cdot \det \Sigma \cdot \det V > 0 \Rightarrow \det U \det V > 0 \overset{\text{O.N.}}{\Rightarrow} \det U = \det V$

(2) Deform sing vals $\Sigma_{(t)}$ has entries $\sigma_i(t) = (1-t)\sigma_i + t \cdot 1$

since $\sigma_i > 0$, intermediate vals are pos $\det U \Sigma_t V^\dagger > 0$

This results in $A \rightsquigarrow U I V^\dagger = U V^\dagger$

(3) Deform ON mats

$Q = U V^\dagger$ $\det Q = \det U \det V = 1$ $Q \in SO(n)$

spectral decomp ON mat Q is collection of blocks $\begin{bmatrix} c(\theta) & -s \\ s & c \end{bmatrix}$

define path Q_t by $\theta(t) = (1-t)\theta$

when $t = 1$ $\theta = 0$ $\begin{matrix} \cos = 1 \\ \sin = 0 \end{matrix}$ $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

Conclusion $A(t) : Q \rightsquigarrow I$