Determinants

Define det by the properties it possesses. Start with 3 axioms:

Axioms

① det depends linearly on the 1^{st} row of matrix
② det changes sign when 2 rows are transposed

} using these together, det is linear in each row independently, not together $\det(A+B) \neq \det A + \det B$

③ $\det(I) = 1$

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④ **Thm**: If 2 rows of A are equal then $\det(A) = 0$

Let row $\vec{r}_i = \vec{r}_j$ Let $\det\begin{bmatrix} - r_i - \\ - r_j - \end{bmatrix} = \beta$ Then by (2) $\det\begin{bmatrix} - r_j - \\ - r_i - \end{bmatrix} = -\beta$

But $r_i = r_j$ so A is unchanged $\Rightarrow \beta = -\beta$ so $\beta = 0$ ie $\det(A) = 0$

⑤ **Thm**: Certain elementary row operations leave det unchanged.
[Those row ops given by an elementary matrix which is triangular with only 1s on main diag]

OK: $r_1 \rightsquigarrow r_1 \pm \alpha r_2$

NOT OK: $r_1 \rightsquigarrow \alpha r_1 \pm r_2$

The row that is being replaced must always have its own coeff equal to 1.

For the OK case, show it for row 1, row 2 in a tiny matrix. It then follows for any other 2 rows in a larger matrix by transposing them into r1, r2 position (Axiom 2), Do the elem operation, transpose them back to orig positions, (nullifying sign changes).

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow \det\begin{bmatrix} a-\alpha c & b-\alpha d \\ c & d \end{bmatrix} \overset{①}{=} \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \alpha \underbrace{\cancel{\det\begin{bmatrix} c & d \\ c & d \end{bmatrix}}}_{0 \text{ by } ④} = \det A$$

For NOT OK case:

$$\det\begin{bmatrix} \alpha a+c & \alpha b+d \\ c & d \end{bmatrix} \overset{①}{=} \alpha \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \underbrace{\cancel{\det\begin{bmatrix} c & d \\ c & d \end{bmatrix}}}_{0} = \alpha \det(A)$$

So $\det A \neq \alpha \det A$ in general.

▷ There is an easy way to see this, if we anticipate Thm ⑨ and Thm ⑦ we can write row ops in terms of elementary matrices:

OK $A \rightsquigarrow EA$

$$\underset{E}{\begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix}} \qquad \det(EA) \overset{⑨}{=} \underbrace{\det E}_{1 \text{ by } ⑦} \cdot \det A$$

NOT OK

$$\underset{E}{\begin{bmatrix} \alpha & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix}} \qquad \text{here } \det E = \alpha \neq 1$$

□

⑥ **Thm** A has $\vec{0}$ row $\Rightarrow \det A = 0$

Let A′ be A with $\vec{0}$ row transposed to r_1. Then $\det(A') = -\det A$ by ②
$\det(A')$ is invariant under $r_1 \rightsquigarrow r_1 + r_2 = \vec{0} + r_2 = r_2$. Now A′ has 2 rows equal to r_2
so $\det A' = 0$ by ④. Then $\det A = -0 = 0$ □

7 **Thm** If A is triangular then $\det(A) = a_{11} a_{22} \cdots a_{nn}$ (product along main diag)

$$\begin{bmatrix} a_{11} & \times & \times & \cdot & \times \\ & a_{22} & \times & \times & \\ & & \ddots & & \times \\ 0 & & & & a_{nn} \end{bmatrix}$$

Idea:
pf. The det is invariant under back subs, so we can reduce it to a pure diag matrix, then use linearity to factor out each diag value: $\det A = (\prod a_{ii}) \det I$
Finally we use axiom 3 that $\det I = 1$. More later.

▷ Let's step back and recap Gaussian elimination and Back substitution from ch 1
[This will also be relevant when we get to Thm 10: $\det A = \det A^T$]

Following ch 1 p.12 and 31 we do an easy case where no 0 pivots occur, no row exchanges nec.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \Rightarrow \begin{matrix} E_1: r_2 \to r_2 - 2r_1 \\ E_2: r_3 \to r_3 - -r_1 \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \; E_3: r_3 \mapsto r_3 + r_2 \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

we can represent this as

$$\underbrace{\underset{E_3}{\begin{bmatrix} 1 & & \\ & 1 & \\ 0 & 1 & 1 \end{bmatrix}} \underset{E_2}{\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & 0 & 1 \end{bmatrix}} \underset{E_1}{\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & & 1 \end{bmatrix}}}_{L^{-1}} \underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

Then $A = LU$ and

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \underset{E_1^{-1}}{\begin{bmatrix} 1 & & \\ +2 & 1 & \\ & & 1 \end{bmatrix}} \underset{E_2^{-1}}{\begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix}} \underset{E_3^{-1}}{\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix}}$$

Just flip the signs - that is easy trick that works only for elementary matrices!

So the triangular "A" that we have in Thm 7 is a "U". Only for the purpose of the pf of Thm 7, we continue with Elementary matrices to do Back subs to make it diagonal.

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} r_2 \to r_2 + 2r_3 \\ r_1 \to r_1 - r_3 \end{matrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I'm going to stop here because you get the idea - we can reduce the matrix to a diagonal one by elementary ops that do not change det.

▷ what about the case where we start with triangular A with a 0 on main diag? The back subs produces a 0 row, so $\det A = 0$ but this again follows from the product of main diag entries: $a_{11} \cdot a_{22} \cdots 0 \cdots a_{nn} = 0$

8 **Thm** A singular $\Rightarrow \det A = 0$
A nsing $\Rightarrow \det A \neq 0$

pf. If A sing, elimination will produce a $\vec{0}$ row (even if we did row interchanges) so $\det A = 0$ by 6 Thm.
If A nsing, we can reduce to triangular U with all main diag entries (the pivots d_i) not equal 0. Thus $\det A = \pm d_1 d_2 \dots d_n \neq 0$ (+/- because maybe we needed to do row exchanges)

▷ To actually compute $\det(A)$ (for A bigger than 3×3), we reduce to U by Gaussian elim, keeping track of number of row exchanges. Then $\boxed{\det A = \pm \det U = \pm \prod d_i}$ □

9 $\boxed{\text{Thm} \quad \det(AB) = \det A \cdot \det B}$ In particular $\det(AA^{-1}) = \det I = 1$

$\det(A^{-1}) = \frac{1}{\det A}$

pf. If A or B is sing, then AB is sing and $\det(AB) = 0$

For A, B nsing, define $d(A) := \frac{\det AB}{\det B}$ show this satisfies Axioms 1, 2, 3.

observe $\begin{bmatrix} -a_1- \\ -a_2- \\ -a_3- \end{bmatrix}\begin{bmatrix} | & | & | \\ B_1 & B_2 & B_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} a_1B_1 & a_1B_2 & a_1B_3 \\ a_2B_1 & a_2B_2 & a_2B_3 \\ a_3B_1 & a_3B_2 & a_3B_3 \end{bmatrix}$ R ★

$$d\left(\begin{bmatrix} \alpha a_1 + c \\ -a_2- \\ -a_3- \end{bmatrix}\right) = \frac{1}{\det B}\left(\det\left(\begin{bmatrix} \alpha a_1 + c \\ -a_2- \\ -a_3- \end{bmatrix} B\right)\right) = \frac{1}{\det B}\det\begin{bmatrix} (\alpha a_1 + c)B_1 & (\alpha a_1 + c)B_2 & (\alpha a_1 + c)B_3 \\ & R & \end{bmatrix}$$

Axiom 1

$$= \frac{1}{\det B}\left(\alpha \det\begin{bmatrix} a_1B_1 & a_1B_2 & a_1B_3 \\ & R & \end{bmatrix} + \det\begin{bmatrix} cB_1 & cB_2 & cB_3 \\ & R & \end{bmatrix}\right)$$

$$= \alpha\, d\left(\begin{bmatrix} -a_1- \\ -a_2- \\ -a_3- \end{bmatrix}\right) + d\left(\begin{bmatrix} -c- \\ -a_2- \\ -a_3- \end{bmatrix}\right)$$ linear in 1st row ✓

▷ Transpose 2 rows:

$$d\begin{pmatrix} -a_2- \\ -a_1- \\ -a_3- \end{pmatrix} = \frac{1}{\det B}\left(\det\begin{bmatrix} -a_2- \\ -a_1- \\ -a_3- \end{bmatrix}\begin{bmatrix} | & | & | \\ B_1 & B_2 & B_3 \\ | & | & | \end{bmatrix}\right) = \frac{1}{\det B}\det\begin{bmatrix} a_2B_1 & a_2B_2 & a_2B_3 \\ a_1B_1 & a_1B_2 & a_1B_3 \\ a_3B_1 & a_3B_2 & a_3B_3 \end{bmatrix}$$

compare with ★ and see this is 2 rows transposed in the prod

$$= -\frac{\det(AB)}{\det B} = -d(A)$$

▷ $d(I) = \frac{1}{\det B}\det(IB) = \frac{\det B}{\det B} = 1$

Thus $d(A)$ is a det, the unique det, so $d(A) = \det(A)$ Thus $\underline{\det(AB) = \det A \det B}$ □

10 $\boxed{\text{Thm} \quad \det(A^T) = \det A}$

If A is sing, then A^T is sing so $\det A = 0 = \det A^T$.

So assume now A nsing. In the case no row exchanges needed, In 7 we showed $A = LU$. Here we want to make that LDU where the pivots are alone in a diag matrix, by factoring

$$U = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} \\ & 1 & \frac{u_{23}}{d_2} \\ & & 1 \end{bmatrix}$$

If A sing, $\exists\, x_0 \neq 0 \ni Ax_0 = 0$

$\langle Ax_0, y\rangle = 0 \quad \forall y$

$\parallel$

$\langle x_0, A^Ty\rangle = 0 \quad \forall y$

$\{x_0\}^\perp$ has dim $(n-1)$ but y is arb in n dim space

$\Rightarrow \exists\, y_0 \neq 0 \ni A^Ty_0 = 0$

▷ Now for the general case, we can account for row exchanges

$$PA = LDU \xRightarrow[\text{9}]{\text{take dets}} \det P \det A = \underbrace{\det L}_{1} \det D \underbrace{\det U}_{1}$$

also for transpose $(PA)^T = (LDU)^T \Rightarrow A^TP^T = U^TD^TL^T$

$$\det A^T \det P^T = \underbrace{\det U^T}_{1} \underbrace{\det D^T}_{\det D} \underbrace{\det L^T}_{1}$$

Thm 7

$$\Rightarrow \det A^T \cancel{\det P^T} = \det D = \cancel{\det P} \det A \Rightarrow \boxed{\det A^T = \det A}$$

since diag $D = D^T$

→ why is $\det P = \det P^T$?

Perm matrix P is clearly O.N. so $P^TP = I$

$\det P^T \det P = 1$ ★

Since any P is attained by a series of row transpositions from I, $\det P = +1$ or -1

If $\det P = +1$, $\det P^T \stackrel{!}{=} +1$ by ★

Likewise for -1 $\Rightarrow \det P = \det P^T$ □

11 $\boxed{\text{Thm} \quad A \text{ has EWs } \lambda_1 \ldots \lambda_n \Rightarrow \det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n}$

If we can diagonalize $A = S^{-1}\Lambda S$ $\det A = \det S^{-1} \det \Lambda \det S = \det(SS^{-1})\det\Lambda = \det \Lambda$

If A has non-diag Jordan form, use Schur decomp to triangular matrix

□

LAAIA
Ch 4.3

Formulas for det

6/6/2026

We specified 3 axioms for the det fcn, and we developed consequences of those axioms. But we did not prove such a fcn actually exists. We don't know if doing row ops in a different order would produce a different value of det.

We shall derive the Leibniz sum-over-all-perms explicit expression for det. This will show it does exist and is well-defined.

By using the axioms we can derive a formula — then we can just take the formula and show it implies the axioms.

<u>Consider 2×2 case</u> use linearity of det. a row can be broken up $[a\ b] = [a\ 0] + [0\ b]$

$$\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}} = ad\begin{vmatrix} 1 & \\ & 1 \end{vmatrix} + bc\begin{vmatrix} & 1 \\ 1 & \end{vmatrix}$$

0 These dets are 0 because 0 col 0

So the only survivors have 1 non-zero elt in each row & col
any zero col eliminates that term

Generally this expansion would have n^n terms: n in 1^{st} row, paired with n in 2^{nd} row, etc...

How many terms survive?

$$n! = \underline{n}\ \underline{(n-1)}\ \underline{(n-2)} \ldots \underline{1}$$

← choices for elt from 2^{nd} row
← choices for elt from 1^{st} row

This is the number of ways to perm n elts

<u>Consider 3×3</u>

Let's discuss the 3×3 expansion in more detail

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = a_{11}a_{22}a_{33}\underset{+}{\overset{\sigma_e}{\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1\end{vmatrix}}} + a_{12}a_{23}a_{31}\underset{+}{\overset{\sigma_a}{\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix}}} + a_{13}a_{21}a_{32}\underset{+}{\overset{\sigma_b}{\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}}}$$

$$+ a_{11}a_{23}a_{32}\underset{-}{\overset{\sigma_c}{\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}}} + a_{12}a_{21}a_{33}\underset{-}{\overset{\sigma_d}{\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1\end{vmatrix}}} + a_{13}a_{22}a_{31}\underset{-}{\overset{\sigma_f}{\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}}}$$

$$= \sum_{\text{all } \sigma\in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} \det[P_\sigma]$$

σ is a perm of the <u>cols</u>. We can list them out $\sigma_e = \text{id}$

$\text{sgn}(\sigma) = \begin{cases} +1 & \sigma \text{ is even \# of transp} \\ -1 & \text{odd}\end{cases}$

$\sigma_a(1) = 2$, $\sigma_a(2) = 3$, $\sigma_a(3) = 1$. This is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1\end{pmatrix}$ and is achived by 2 transposes $\{1\,2\,3\} \xrightarrow{(13)} \{3\,2\,1\} \xrightarrow{(12)} \{2\,3\,1\}$

elts circled when in final position

Thus this is the product of 2 cycles $(12)(13)$

$\text{sgn}(\sigma_a) = (-1)^2 = +1$

$\sigma_b(1) = 3$, $\sigma_b(2) = 1$, $\sigma_b(3) = 2$. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2\end{pmatrix}$ $\{1\,2\,3\} \xrightarrow{(12)} \{2\,1\,3\} \xrightarrow{(13)} \{3\,1\,2\}$

$(13)(12) \Rightarrow \text{sgn}(\sigma_b) = +1$

$\sigma_c = (2\,3)$ transpose cols 2, 3 $\quad \text{sgn}(\sigma_c) = -1$

$\sigma_d = (1\,2)$ cols 1, 2 $\quad \text{sgn}(\sigma_d) = -1$

$\sigma_f = (1\,3)$ $\quad \text{sgn}(\sigma_f) = -1$

So we would like to say $d(A) := \displaystyle\sum_{\text{all } \sigma\in S_n} a_{1\sigma(1)}\cdots a_{n\sigma(n)} \det P_\sigma$

But we can't <u>define</u> det by referencing det.

We need to understand the relationship between matrix P_σ and the perm σ (it is NOT the matrix rep of the perm, since P_σ has the <u>colums</u> permed, not the rows.

<u>Here it is</u>: If we have a perm σ, we can represent it my a matrix Q_σ by applying σ to the <u>rows</u> of I. But here, we applied σ to the cols, thus $P_\sigma = (Q_\sigma)^T$

But this is no problem. The 3 axioms imply $\det(P) = \det P^T$

We must show $\det(P_\sigma) = \text{sgn}(\sigma)$

$\boxed{\text{we must show } \det(P_\sigma) = \mathrm{sgn}(\sigma)}$

By the axioms, we perform row transposes on P_σ unil it becomes I (this is the sam as applying σ^{-1} to it) If it is accomplished in p transposes, then $\det(P_\sigma) = (-1)^p \det(I) = (-1)^p$

If n is large, different people might use different sets of transpositions to turn P_σ into I. But we have a theorem that perms consist of either an even or odd number of transposes, and no even perm can be achived by odd # of transp and vice versa.

See my sheets The Sym Group 10/9/2022

Therefore $\mathrm{sgn}(\sigma)$ is well-defined and we can write

$$\boxed{d(A) := \sum_{\text{all } \sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \,\mathrm{sgn}(\sigma)} \quad ★$$

$\triangleright$ Now we must show d satisfies Axioms 1,2,3 so $d \equiv \det$ and thus det exists

(3) is the easiest $I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ then all terms are zero except

$a_{11} a_{22} a_{33} \cdots a_{nn}$ and $\sigma(1) = 1$, $\sigma(2) = 2$, $\vdots$, $\sigma(n) = n$ Id perm

$= 1 \cdot 1 \cdot 1 \cdots 1$

$= 1 \cdot \mathrm{sgn}(id)$

$= 1$

(2) Let τ be a transposition, say $\tau = (2,4) = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \text{id} \\ 1 & 4 & 3 & 2 & \cdots \end{pmatrix}$

We want to show $d(\tau A) = -d(A)$

$$d(\tau A) = \sum_{\text{all } \sigma \in S_n} a_{1\sigma\tau(1)} a_{2\sigma\tau(2)} \cdots a_{n\sigma\tau(n)} \,\mathrm{sgn}(\sigma \circ \tau)$$

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- $\sigma \circ \tau$ is just another elt of S_n so the sum $\sum_{\sigma \in S_n}$ stays the same
- $\mathrm{sgn}(\sigma \circ \tau) = (-1)^1 \mathrm{sgn}(\sigma)$ since τ is a transposition

  We can write each σ as a product of transp — not uniq, but the parity is invariant

  $\sigma = \underbrace{(\;)(\;)\cdots(\;)}$

  then $\sigma \circ \tau = \underbrace{(\;)(\;)\cdots(\;)}\underset{\tau}{(2\,4)}$ This adds 1 to the total and changes even $\to$ odd, odd $\to$ even

  $\Rightarrow$ mult by -1

Cont'd →

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③

(1) To establish that d is linear in the 1^{st} row, let's return to the expanded 3×3 case

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \overset{+}{a_{11} a_{22} a_{33}} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + \overset{+}{a_{12} a_{23} a_{31}} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + \overset{+}{a_{13} a_{21} a_{32}} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ \underset{-}{a_{11} a_{23} a_{32}} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + \underset{-}{a_{12} a_{21} a_{33}} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + \underset{-}{a_{13} a_{23} a_{31}} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

This shows how we can factor out the terms from the first row

$$[redacted]\, d(A) = a_{11}\Omega_{11} + a_{12}\Omega_{12} + \dots + a_{1n}\Omega_{1n} \quad \text{LC of elts of } 1^{st} \text{ row}$$

If $\vec{a}_1 = \vec{b}_1 + \lambda \vec{c}_1$ row vectors

$$d\begin{bmatrix} - b+\lambda c - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix} = (b_{11} + \lambda c_{11})\Omega_{11} + \dots + (b_{1n} + \lambda c_{1n})\Omega_{1n}$$

$$= b_{11}\Omega_{11} + \lambda c_{11}\Omega_{11} + \dots + b_{1n}\Omega_{1n} + \lambda c_{1n}\Omega_{1n}$$

$$= d\begin{bmatrix} - b - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix} + \lambda\, d\begin{bmatrix} - c_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix}$$

$\boxed{\text{QED}}$

▷ Now we need to establish Laplace's Cofactor Expansion

Lets show it for a 3×3, let's expand along the 1^{st} [redacted] col by linearity

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}}_{M_{11}} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Transpose the nonzero elts to the top

$$(-1)\underbrace{\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix}}_{M_{21}} + (-1)^2 \underbrace{\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix}}_{M_{31}}$$

We shall show later that
If matrix A is block upper triang

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \qquad \det A = \det B \det D$$

Block M_{ij} is A missing row i and col j

Since $\det A = \det(A^T)$ we can expand down any row or column.

$$\Rightarrow \det(A) = a_{11}\det M_{11} + a_{21}(-1)\det M_{21} + a_{31}(-1)^2 \det M_{31}$$

Fix row i

$$\boxed{\det(A) = \sum_{j=1}^{n} a_{ij} \underbrace{(-1)^{i+j}\det(M_{ij})}_{\text{cofactor } C_{ij}}}$$

The signs of the cofactors are given by a checkboard pattern:

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix} \quad (-1)^{i+j}$$

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LAAIA ch 4.4 **Applications of det** 6/6/2026

Computation of A^{-1}

Expand $\det A$ in cofactors along row i

$$\text{(★)} \qquad \det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$$

If this 2nd index would be const, we are expanding down col j of A

If the 1st index is const (row i), we are expanding down that row, and that is where the as are expected to come from

We can write this as matrix mult if we make a Cofactor matrix by putting the above cofactors in the columns (so I call it C^T)

Let's show it in 3x3 case:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{C^T} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

(★) gives us the main diag terms. But why should off diag terms be 0? [Illustrate for this term]

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} \overset{?}{=} 0$$

Recall cofactor expansion for 1st row:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)\,a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^2 a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Now explicitly show cofactors C_{21}, C_{22}, C_{23} so transpose row 2 and row 1 and expand:

$$\det A = (-1)\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{(-1)\overset{a_{11}}{a_{21}}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}}_{C_{21}} + \underbrace{(-1)^2\overset{a_{12}}{a_{22}}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}}_{C_{22}} + \underbrace{(-1)^3\overset{a_{13}}{a_{23}}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}}_{C_{23}}$$

If we now plugged in the row 1 elts as I have written them in blue That would mean the RHS came from expansion of a matrix with a duplicate row

$$0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} \quad ✓$$

Thus $A\,C^T = (\det A)\,I$ Strang calls C^T 'A_{cof}'

$$A\left[\frac{1}{\det A}C^T\right] = I$$

or $A\left[\dfrac{1}{\det A}A_{cof}\right] = I$

Thus $\boxed{A^{-1} = \dfrac{1}{\det A}A_{cof}}$

So for 2x2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

□

Thm **Cramer's Rule** x is sol'n to $Ax=b$ $\Rightarrow$ $x^{(j)} = \dfrac{\det B_j}{\det A}$ where $B_j = \begin{bmatrix} | & | & & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{b} & \cdots & \vec{a}_n \\ | & | & & | & & | \end{bmatrix}$

replace col $\vec{a}_j$ by $\vec{b}$

Pf. $Ax=b \Rightarrow x = A^{-1}b = \dfrac{1}{\det A} A_{cof}\, b$

1 ▷ $x^{(j)}$ is the j^{th} component in $A_{cof}\, b$

Let's write it out in the 3x3 case:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let's just take $j=2$

$$(\det A)\, x_2 = b_1 C_{12} + b_2 C_{22} + b_3 C_{32} \quad ⊛$$

2 ▷ On the other hand, expanding det A down col 2 $\begin{vmatrix} a & a_{12} & a \\ a & a_{22} & a \\ a & a_{32} & a \end{vmatrix} = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

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If we had replaced col 2 by $\vec{b}$

$$b_1 C_{12} + b_2 C_{22} + b_3 C_{32} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = \det B_2$$

3 ▷ Thus ⊛ becomes $(\det A)\, x_2 = \det B_2$

$$x_2 = \frac{1}{\det A} \det B_2$$

$\square$