Normal Matrices

Real OG (i.e. ON) matrices, Hermitian, skew-Herm, Unitary, Perms, etc... all belong to the class of Normal Matrices

Def A is Normal if $A^H A = A A^H$ (commutes with its herm)

We shall follow the discussion in Horn & Johnson MA p. 101 which encompasses all Strang's examples. Let A be $n \times n$ matrix with EWs $\lambda_1 \dots \lambda_n$ (not nec distinct).

Thm TFAE
- (a) A is normal: $A^H A = A A^H$
- (b) A can be diagonalized by unitary $U^H A U = \Lambda$
- (c) $\sum_i^n \sum_j^n a_{ij}^2 = \sum^n |\lambda_i|^2$
- (d) $\exists$ ON set of n EVs of A

pf. **Step 1** Given A, from Schur's Lemma $\exists$ triangular T and unitary $U \ni U^H A U = T$

A normal $\Rightarrow$ T normal

$$\begin{aligned} T T^H = U^H A U (U^H A U)^H = U^H A \cancel{U U^H} A^H U &\overset{A \text{ Norm}}{=} U^H A^H A U \\ &= U^H A^H U U^H A U \\ &= (U^H A U)^H U^H A U = T^H T \end{aligned}$$

The plan is to show (a) $\Leftrightarrow$ (b)
and (b) $\Leftrightarrow$ (c)
(b) $\Leftrightarrow$ (d)

Step 2 (a) $\Rightarrow$ (b)
We established A normal $\Rightarrow$ T normal i.e. $T^H T = T T^H$
Now show this cond forces T to actually be diag. [A normal triang matrix is, in fact, diag.]

Writing out an example makes this clear:

$$\overset{T^H}{\begin{bmatrix} \bar t_{11} & 0 & 0 & 0 \\ \bar t_{12} & \bar t_{22} & 0 & 0 \\ \bar t_{13} & \bar t_{23} & \bar t_{33} & 0 \\ \bar t_{14} & \bar t_{24} & \bar t_{34} & \bar t_{44} \end{bmatrix}} \overset{T}{\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ & t_{22} & t_{23} & t_{24} \\ & & t_{33} & t_{34} \\ & & & t_{44} \end{bmatrix}} = \overset{T}{\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ & t_{22} & t_{23} & t_{24} \\ & & t_{33} & t_{34} \\ & & & t_{44} \end{bmatrix}} \overset{T^H}{\begin{bmatrix} \bar t_{11} & & & \\ \bar t_{12} & \bar t_{22} & & \\ \bar t_{13} & \bar t_{23} & \bar t_{33} & \\ \bar t_{14} & \bar t_{24} & \bar t_{34} & \bar t_{44} \end{bmatrix}}$$

Comparing the (1,1) elt on both sides, we see all terms in 1st row above main diag must be 0

$$\bar t_{11} t_{11} \overset{!}{=} t_{11} \bar t_{11} + \sum_{j=2}^n t_{1j} \bar t_{1j} = |t_{11}|^2 + \underbrace{\sum_{j=2}^n |t_{1j}|^2}$$

This must be 0 to equal LHS
But since each term is non-neg, each term must be 0 $\Rightarrow$ only t_{11} is nonzero in 1st row
[In fact, we know t_{11} = EW λ_1]

Cont'd →

Now look at the (2,2) term:

$$\overset{T^H}{\begin{bmatrix} \bar{t}_{11} & 0 & 0 & 0 \\ 0 & \bar{t}_{22} & & \\ 0 & \bar{t}_{23} & \bar{t}_{33} & \\ 0 & \bar{t}_{24} & \bar{t}_{34} & \bar{t}_{44} \end{bmatrix}} \overset{T}{\begin{bmatrix} t_{11} & 0 & 0 & 0 \\ & t_{22} & t_{23} & t_{24} \\ & & t_{33} & t_{34} \\ & & & t_{44} \end{bmatrix}} = \overset{T}{\begin{bmatrix} t_{11} & 0 & 0 & 0 \\ & t_{22} & t_{23} & t_{24} \\ & & t_{33} & t_{34} \\ & & & t_{44} \end{bmatrix}} \overset{T^H}{\begin{bmatrix} \bar{t}_{11} & & & \\ 0 & \bar{t}_{22} & & \\ 0 & \bar{t}_{23} & \bar{t}_{33} & \\ 0 & \bar{t}_{24} & \bar{t}_{34} & \bar{t}_{44} \end{bmatrix}}$$

$$\bar{t}_{22} t_{22} \stackrel{!}{=} t_{22}\bar{t}_{22} + \underbrace{\sum_{j=3}^{n} t_{2j}\bar{t}_{2j}}_{\text{by same arg, each term in this sum is 0}}$$

By working down the diag elts $(3,3), \dots, (n-1,n-1)$ we see each row is 0 off main diag and thus $T = \Lambda$ [we already knew from Schur diag elts are EWs].

<u>Step 3</u> (a) $\Leftarrow$ (b)

Obviously diag Λ is normal: $\Lambda^H\Lambda = \Lambda\Lambda^H$

Thus $U^H A U = \Lambda \Rightarrow A$ is normal by similiar arg as Step 1.

<u>Step 4</u> (b) $\Leftrightarrow$ (c) I am skipping this. The calculation would make more sense in the context of matrix norms. cf Atkinson AITNA p. 415-421 and prob 22 p. 431

<u>Step 5</u> (b) $\Leftrightarrow$ (d)

$U^H A U = \Lambda \Leftrightarrow AU = U\Lambda$ the cols of U are the full set of O.N. EWs. $\square$

▷ Jordan Form of matrix (Jordan Normal Form – no relation to prev word 'normal')

If A has full set of EVs, we can diagonalize $AS = S\Lambda$

$$\underbrace{\Lambda}_{J} = S^{-1}AS$$

What if A does not have a full set?
How close to diagonal can we make it?

<u>Jordan form thm</u> If A has $p \leq n$ indep EWs

$$\Rightarrow \exists \text{ C.O.B. } M \ni M^{-1}AM = J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{bmatrix}$$

Each J_i block has only a single EW λ_i and 1s above the main diag of J_i and is associated with only 1 EV $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

[a value λ_i can appear in more than 1 Jordan block if it matches to more than 1 EV – see examples to follow]

2 matrices are similiar $\Leftrightarrow$ the have same JNF J is uniq up to ordering of blocks. a convention is to write them in decreasing size

I worked thru the pf in Schaums LA ch 10

Here I will just give some examples:

(1) Consider $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

all share $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ That means 2 EWs $\lambda_1 = 1 = \lambda_2$ and 1 EV

For T $\quad M^{-1}TM = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = J$

For B we can do it with a perm matrix (accomplish transpose to J)

$$P^{-1}BP = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = J$$

For A they don't give calculation, just say by Schur Lemma, $\exists\ U \ni$ $U^{-1}AU = T$ and we already did T.

(2) $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ $\lambda = 0$ is a triple EW for both - look at the main diag since both triangular

The possible Jordan forms are $J = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$ or $J = \left[\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right]$ or $J = \begin{bmatrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{0} \end{bmatrix}$

1 EV $\qquad$ 2 EVs $\qquad$ 3 EVs (or ∞ EVs)

This one is a class of one matrix $M\,\underset{J}{0}\,M^{-1} = 0$

For A, there is only 1 EV: $A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Thus it's Jordan form must be the J with only 1 block. $J = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$

For B there are 2 EVs for $\lambda = 0$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Rightarrow$ J has 2 blocks, must be $\left[\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right]$

But this counting trick only works when nothing more than a triple EW

Prob 5.6.31 Write out all (4x4) J forms of a 4x4 matrix with quad $\lambda = 0$:

By convention we start with largest block and decrease down main diag

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \text{ or } \left[\begin{array}{ccc|c} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ \hline & & & 0 \end{array}\right] \text{ or } \left[\begin{array}{cc|cc} 0 & 1 & & \\ & 0 & & \\ \hline & & 0 & 1 \\ & & & 0 \end{array}\right] \text{ or } \left[\begin{array}{cc|c|c} 0 & 1 & & \\ & 0 & & \\ \hline & & 0 & \\ \hline & & & 0 \end{array}\right] \text{ or } \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

1 EV $\qquad$ 2 EVs $\qquad$ 3 EVs $\qquad$ 4 EVs (∞ EVs)

So if 2 EVs, there are 2 choices for J and a priori we don't know.

example from Schaums LA ch 9 prob 9.8 p. 206-207

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & 1 \\ -6 & 6 & -2 \end{bmatrix}$$

Characteristic poly

$\Delta(\lambda) = (\lambda+2)^2(\lambda-4)$ $\qquad$ $\Delta(\lambda) = (\lambda+2)^2(\lambda-4)$ same poly, same EWs

But different EVs $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}$ $\qquad$ EVs $\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}$ only 2

$$J = \begin{bmatrix} -2 & & \\ & -2 & \\ & & 4 \end{bmatrix} \qquad J = \begin{bmatrix} -2 & 1 & \\ & -2 & \\ & & 4 \end{bmatrix}$$

matrices NOT similiar.

Strang Appendix B p. 454

consider $J = \begin{bmatrix} 8 & 1 & & & \\ 0 & 8 & & & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$

$\lambda = 8$ is double EW
only single EV $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = \hat{e}_1$

triple $\lambda = 0$
2 EVs $\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} = \hat{e}_3$ and $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \hat{e}_5$

For some 5x5 matrix A, when will there exist $M \ni M^{-1}AM = J$? $\quad$ one requirement is same EVs

we need vectors $x_1, x_2, x_3, x_4, x_5 \ni$

$$A \underbrace{\begin{bmatrix} | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ | & | & | & | & | \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ | & | & | & | & | \end{bmatrix}}_{M} \begin{bmatrix} 8 & 1 & & & \\ & 8 & & & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

$\Rightarrow A\vec{x}_1 = 8\vec{x}_1 \;\Big|\; A\vec{x}_2 = 8\vec{x}_2 + \vec{x}_1 \;\Big|\; A\vec{x}_3 = 0\vec{x}_3 \;\Big|\; A\vec{x}_4 = 0\vec{x}_4 + \vec{x}_3 \;\Big|\; Ax_5 = 0x_5$

So we have 3 genuine EVs x_1, x_3, x_5

There must be 2 "generalized EVs" $\vec{x}_2, \vec{x}_4$

x_2 belongs to a "string of vectors" headed by x_1

In general to find J we must search for these <u>strings of vectors</u>

Strang gives Filippov's Jordan form pf, which I shall skip.