7/3/2013

①

Spivak DG
vol II

Ch 1

$c : [a,b] \longrightarrow \mathbb{R}^2$ immersion $c'(t) \neq 0 \ \forall t$

$t \longmapsto$

arc len $s(t) = \int_a^t |c'(t)|\, dt$

$s : [a,b] \longrightarrow \mathbb{R}$ diffeo $[a,b] \to [0,L]$

$t \longmapsto s = \int_a^t |c'(t)|\,dt$

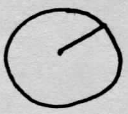
$\gamma = c \circ s^{-1}$ reparam of c

$\|\gamma'(s)\| = 1$

convention here $\gamma'(s) \in \mathbb{R}^3$ not in $\mathbb{R}^3_{\gamma(s)} = T_{\gamma(s)}\mathbb{R}^3$

(curvature of a circle of radius r) $= \frac{1}{r}$

Bigger circle gets $\Rightarrow$ less it curves.



p.6 **Thm 1** $c : [a,b] \longrightarrow \mathbb{R}^2$ C^2 smooth, param by arc len

$c''(s) \neq 0 \Rightarrow$ for s_1, s_2, s_3 suff close to s, $c(s_1), c(s_2), c(s_3)$ do not lie on straight line.

As $s_1, s_2, s_3 \to s \Rightarrow$ uniq circle thru pts $c(s_i)$ approaches circle passing thru $c(s)$ with radius $1/c''(s)$ — osculating circle

and whose center lies on the line thru $c(s)$ $\perp$ to the tangent line $c(s)$.

planar curves

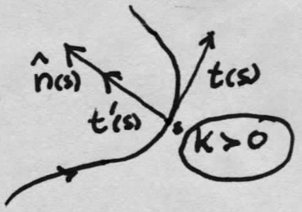
Define $\hat{t}(s) := c'(s)$

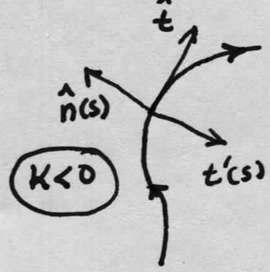
Define $\hat{n}(s)$ to be perp to $\hat{t}$ and order basis $\{t_s, n_s\}$ is pos $\Rightarrow \hat{n} = \begin{bmatrix} -t^{2} \\ t^{1} \end{bmatrix}$

$t'(s) = \kappa(s)\,\hat{n}(s)$ cf. Do Carmo p.21

Signed curvature $\kappa(s)$ by $t'(s) = \kappa(s)\,\hat{n}(s)$

Note $|\kappa(s)| = |t'(s)| = \|c''(s)\|$ just take norms above

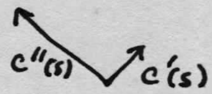




let $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ lower subscript

$|\kappa(s)|$ is length of $c''(s)$, which is perp to $\hat{c}'(s)$

$|\kappa(s)|$ is area of rectangle



$$c' \times c'' = \begin{vmatrix} i & j & k \\ c_1' & c_2' & 0 \\ c_1'' & c_2'' & 0 \end{vmatrix} = +\hat{k}(c_1'c_2'' - c_2'c_1'')$$

p.9

$$\kappa(s) = \det \begin{bmatrix} c_1' & c_1'' \\ c_2' & c_2'' \end{bmatrix} = c_1'c_2'' - c_2'c_1''$$

Now derive formulas not param by arc len:

~~~
~~~

Thm 2 let $K : [a,b] \to \mathbb{R}$ cont $\Rightarrow$ · $\exists$ curve $C : [a,b] \to \mathbb{R}^2$, param by arc len, $\ni$ curvature of C at s is $K(s)$ $\forall s \in [a,b]$

· C is unique up to proper Euclidean motions. (translation, followed by rotation).

p.26 **Thm 7** (Hopf) The rotation index of a simple clsd planar curve is ± 1.

Thm 8 A simple clsd planar curve is convex $\Leftrightarrow$ curvature $K \geq 0$ (or $K \leq 0$) always

Thm 9 4 vertex thm Every simple, clsd, convex, planar curve has (at least) 4 vertexes.

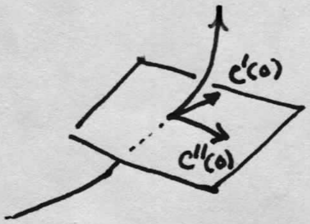
Now curves in $\mathbb{R}^3$

osculating Plane

Prop 10 $C : [a,b] \to \mathbb{R}^3$ C^2 smooth

$C''(s) \neq 0$

p.32 For s_1, s_2, s_3 close to s, pts $C(s_1), C(s_2), C(s_3)$ do not lie on a line. $\Rightarrow$ As $s_i \to s$ the unique plane through pts $C(s_i)$ approaches plane P spanned by $C'(s)$ and $C''(s)$.



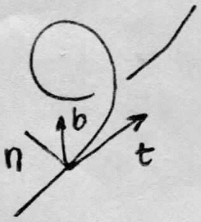
t

Define curvature $K(s) := \|t'(s)\|$ even when $C''(s) = t'(s) = 0$

p.37 when $K(s) \neq 0$, define $\hat{n}(s)$ by $t'(s) = K(s)\,\hat{n}(s)$ $\hat{n}(s)$ = normalized $t'(s)$.

Principal normal

Define binormal $\hat{b}(s) := \hat{t}(s) \times \hat{n}(s)$ (so $[t, n, b]$ pos oriented O.N. basis)



Note $\langle b, b \rangle = 1 \Rightarrow \langle b', b \rangle = 0$

classic arg
$\left(\frac{d}{ds}\text{ both sides}\right)$
$\langle b', b \rangle + \langle b, b' \rangle = 0$
$2\langle b', b \rangle = 0$

$\Rightarrow b'$ is a LC of t & n.

We also have $\langle b, t \rangle = 0$

p.37 $\frac{d}{ds}\langle b, t \rangle = \langle b', t \rangle + \langle b, t' \rangle = 0 \Rightarrow \langle b', t \rangle = -\langle b, t' \rangle = -\langle b, n \rangle = 0$

$\Rightarrow b'$ is actually a multiple of $\hat{n}$

So write $b' = -\tau n$ where scalar τ is the torsion (only defined when n exists, i.e. $C''(s) \neq 0$).

Why $-\tau$? you'll see...

p.41

" a standard and possibly illuminating way of examining the geo significance of κ and τ is to examine the proj of c onto the planes spanned by "

$\{t, n\}$, "osculating p"; $\{n, b\}$, "normal p"; $\{t, b\}$ "rectifying p"

i.e. Taylor expand

$\tau = 0$ everywhere $\Rightarrow$ plane curve

$$\tau = 0 \Rightarrow b' = 0 \Rightarrow b(s) = b_0 \text{ const vector}$$
$$\Rightarrow \langle t, b_0 \rangle = 0$$
$$\Rightarrow \frac{d}{ds}\langle c(s), b_0 \rangle = 0$$
$$\Rightarrow \langle c(s), b_0 \rangle = a \text{ const}$$
$$\Rightarrow c \text{ lies in plane } \left\{ \text{all } x \in \mathbb{R}^3 \,\middle|\, \langle x, b_0 \rangle = a \right\}$$

Helix $\kappa > 0$ τ const

p.44 **Do κ and τ determine c up to Euclidean motion?**

recall
$$t' = \kappa n$$
$$b' = -\tau n$$

we get

Frenet-Serret

$$\{t' \ n' \ b'\} = \{t \ n \ b\} \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

$t' = \kappa n$ (def of κ)

$b' = -\tau n$ (def of τ)

Thm 11
$\kappa : [a,b] \to \mathbb{R}$ cont
$\tau : [a,b] \to \mathbb{R}$
$\kappa > 0$ on $[a,b]$

$\Rightarrow$ $\exists$ curve $c : [a,b] \to \mathbb{R}^3$, param by arc len, $\ni$ curvature$(c) = \kappa$, torsion$(c) = \tau$

any 2 such curves differ by proper Euclidean motion.

Supposedly I worked thru this in DoCarmo p.309 but cl can't find it.

Spivak DG v2
p. 26-27

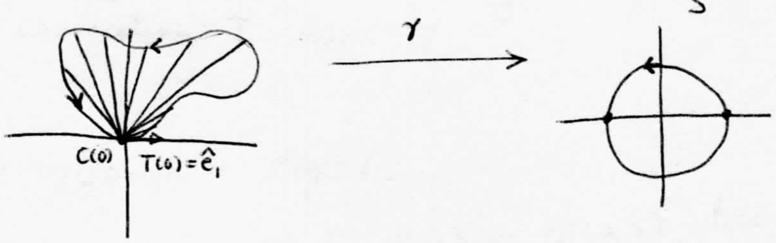
Thm 7 (Hopf umlaufsatz) The $\overbrace{\text{rotation index}}^{\deg(T)}$ of a simple closed, planar curve is ± 1 (depending on the direction in which it is traversed).

Pf. The idea is to construct another "circle map" γ, which is homotopic to T, and for which we can easily compute the degree. [we know that homotopic maps have the same degree [cf e.g. G&P p. 108-109]

Let $c : [0,L] \longrightarrow \mathbb{R}^2$ be the curve, parameterized by arc length.

Step 1 Define $\gamma : [0, 2L] \longrightarrow S^1$ (If I wanted to really do this right, I'd reparam to get the domain to $[0,L]$.

First note that the rotation index doesn't change if we rotate or translate c, so we can assume c lies in the upper half plane, with $c(0) = \langle 0,0\rangle$ and the tangent line at $c(0)$ is the x-axis. We also assume c is traversed in the direction $\ni$ $T(0) = \hat{e}_1 = \langle 1,0\rangle$.



$$\text{Let} \quad \gamma(s) := \begin{cases} \gamma_1(s) := \dfrac{c(s) - c(0)}{\| c(s) - c(0) \|} & s \in [0, L) \\[2ex] \gamma_2(s) := \dfrac{c(L) - c(s-L)}{\| c(L) - c(s-L) \|} & s \in (L, 2L) \\[2ex] \quad = \dfrac{c(0) - c(s)}{\| c(0) - c(s) \|} & s \in (0, L) \\[2ex] \quad = -\gamma_1(s) \end{cases}$$

This traces out the top half of the circle.

since $c(0) = c(L)$ closed curve. and we can extend c to be periodic of period L $c : \mathbb{R} \longrightarrow \mathbb{R}^2$

[This traces out bottom half of circle]

$\gamma = \gamma_1 * \gamma_2$
join of curves
Munkres T p.

Spivak p. 27–30 } Elements of both proofs incorporated. { Notation from Spivak / ideas from DoCarmo. }

DoCarmo p. 397–399

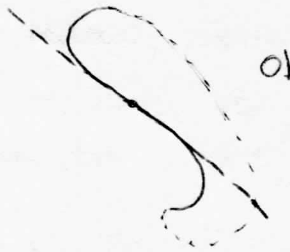
3/21/97

Thm: a planar, regular, closed curve is convex $\Leftrightarrow$ it is simple and its curvature K does not change sign.

regular: $c'(s) \neq 0 \;\forall s$

simple: no self intersections

<u>Intuitive Idea:</u>



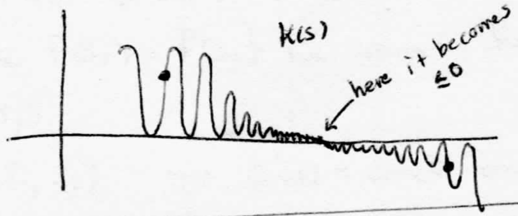
Obviously, this curve is not convex.

However, it seems to be difficult to make this into a rigorous proof.

For $K : [0, l] \longrightarrow \mathbb{R}$
$s \longmapsto (c_1'(s)\, c_2''(s) - c_2'(s)\, c_1''(s))$ (p. 9)

Even if we let K (and c) be very smooth, we still can have pathalogical curves that do not allow us to find an interval where K is pos, next to one where $K = 0$, next to one where K is neg.

Here K changes sign

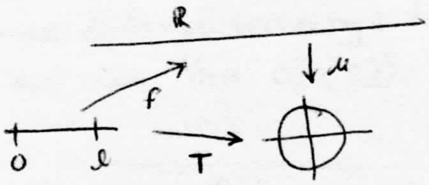


Based on $x^2 \sin(1/x)$ but modified. Ross EATTOC

accumulation pts of zeroes.

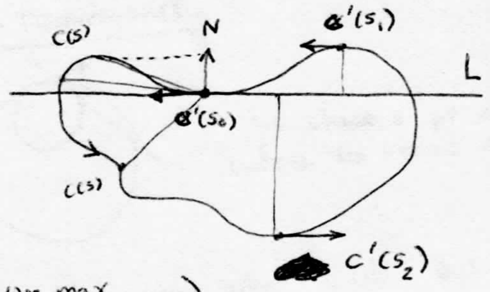
<u>Pf</u> we let $c : [0, l] \longrightarrow \mathbb{R}^2$ be our curve, paramed by arc length.

We also refer to the familiar diagram of the lift of T:



($\Leftarrow$) K is simple and K does not change sign in $[0, l]$;
w.l.o.g. lets assume $K \geq 0$.
Assume c is <u>NOT</u> convex and derive a contradiction.

<u>Step 1</u> Since c not convex, $\exists$ a pt $s_0 \ni c$ lies on both sides of the tan line (L) thru $c(s_0)$.



Define a "height" fcn of the curve from this line L:
$h(s) := [c(s) - c(s_0)] \circ \hat{N}(s_0)$ ← unit normal vector.

Since $[0, l]$ is a cpt set, the cont fcn h attains both its max (at $c(s_1)$ say) and its min (negative max value at $c(s_2)$).
Note $s_0 \neq s_1$ or s_2 by def of L.

Cont'd →

Spivak 27-30

DoCarmo 397-399

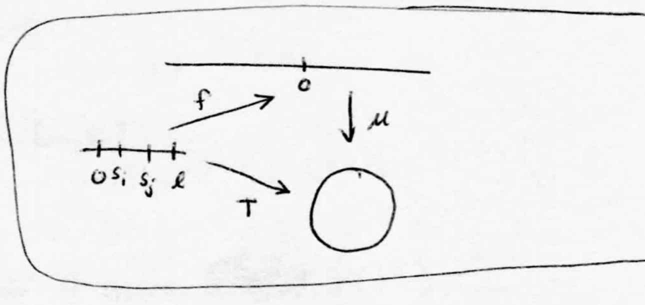
at s_1 and s_2 we know $h' = 0 \Rightarrow$ $c'(s_1) \cdot N(s_0) = 0$

$c'(s_2) \cdot N(s_0) = 0$

Since we're in $\mathbb{R}^2$, we see $c'(s_0)$, $c'(s_1)$, $c'(s_2)$ are all parallel.

step 2 With 3 parallel vectors, some group of 2 must point in the same direction — say it's s_i and s_j for some $i, j \in \{0, 1, 2\}$.

Also we can take $s_i < s_j$.



step 3 So $T(s_i) = T(s_j)$.

$\Rightarrow f(s_i) = f(s_j)$ since we are considering a particular lift of T with $f(0) = 0$.

Now we use $K \geq 0$. From Spivak p. 24 we have $K(s) = f'(s)$.

Thus $f' \geq 0$. But since $f(s_i) = f(s_j)$ we must have $f' \equiv 0$ on $[s_i, s_j]$, and f = const on $[s_i, s_j]$.

$\Rightarrow T$ const on $[s_i, s_j]$ $\Rightarrow c'(s) =$ const on $[s_i, s_j]$

$\Rightarrow c([s_i, s_j]) \subset L$

But this is a contradiction: either $c(s_1)$ or $c(s_2)$ is NOT the max distance (pos or neg) from L or else L does not cut thru $c([0, l])$ $\Rightarrow\Leftarrow$.

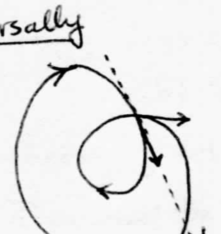
($\Rightarrow$) Now we assume c is convex and try to prove (i) c is free of self intersection

(ii) K does not change sign in $[0, l]$.

Lets dispose of (i) with some hand waving [DoCarmo p. 398]:

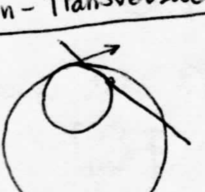
c can intersect its self 2 ways:

Transversally



Here the fact that $c'(s_1) \neq \pm c'(s_2)$ lets us show that a small arc around $c(s_2)$ is on both sides of the line L.

Non-transversally



we choose a pt near $c(s_1)$ where tan vectors do not coincide.

The picture gives the idea but I want to skip the details.

cont'd →

Spivak 27-30 ③

Docarmo 397-399

Now show C convex and simple $\Rightarrow$ K does not change sign in $[0, \ell]$.

$\triangleright$ Again, proof by contradiction:

Step 1 — $K(s) = f'(s)$. If f' changes sign in $[0,\ell]$, $\exists$ interior pts $a<b$

Prove this claim

$\ni f'(a) > 0$ or < 0

$f'(b) < 0$ $\quad$ > 0

CLAIM: $\exists$ pts $s_1, s_2 \ni \begin{cases} s_1 < s_2 \\ f(s_1) = f(s_2) \\ f \text{ is NOT const on } [s_1, s_2] \end{cases}$

Pf — Lets just do $f'(a)>0$, $f'(b)<0$.

ⓐ f is a cont fcn on cpt set $[a,b] \Rightarrow \exists$ max $f(s^*)$.

ⓑ $s^* \neq a$ or b: $\displaystyle \lim_{s \searrow a} \frac{f(s)-f(a)}{s-a} = f'(a) > 0$

take $\varepsilon = \frac{f'(a)}{2}$, $s>a$: $\quad -\varepsilon < \dfrac{f(s)-f(a)}{s-a} - f'(a) < \varepsilon$ $\quad$ for $s-a<\delta_\varepsilon$

$\Rightarrow f(a) + \underbrace{(s-a)}_{pos}\underbrace{(f'(a)-\varepsilon)}_{pos} < f(s) < \cdots$

$\Rightarrow f(a)$ is not the max

$\displaystyle \lim_{s \nearrow b} \frac{f(s)-f(b)}{s-b} = f'(b) < 0$

$-\varepsilon < \dfrac{f(b)-f(s)}{b-s} - f'(b) < \varepsilon$

$\vdots$

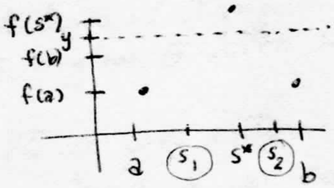
$f(b) - (b-s)(f'(b)-\varepsilon) > f(s) > f(b) - \underbrace{(f'(b)+\varepsilon)}_{\text{neg for } \varepsilon \text{ suff small}}\underbrace{(b-s)}_{pos}$

$\Rightarrow f(s) > f(b) + pos$

$\Rightarrow f(b)$ is not the max.

ⓒ Thus $f(a) < f(s^*)$

$f(b) < f(s^*)$ and w.l.o.g. we can take $f(a) \le f(b)$.



choose $y \in (f(b), f(s^*))$

There must be at least one pt $s_1 \in (a, s^*)$ $\ni f(s_1) = y$ or else we violate the Intermediate value thm.

Similiarly, there must be $s_2 \in (s^*, b)$.

Then we are done! $f(s_1) = f(s_2)$ and f is <u>not</u> const on $[s_1, s_2]$ ▨

cont'd →

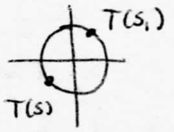
Step 2 $\quad f(s_1) = f(s_2) \Rightarrow T(s_1) = T(s_2)$.

Since the winding number of T is ± 1 [Hopf Umlaufsatz, Spivak p. 26],

we see that T maps Onto S^1 (nothing said about One-to-One)

$T(s_1) = T(s_2)$
$T(s) = -T(s_1)$.

$\Rightarrow \exists\ s_0 \ni T(s_0) = -T(s_1)$



Step 3 $\quad$ So we have 3 parallel tangent vectors $c'(s_1)$, $c'(s_2)$ and $c'(s_0)$. and thus we have defined 3 parallel lines in the plane.

Claim: 2 of these lines must co-incide.

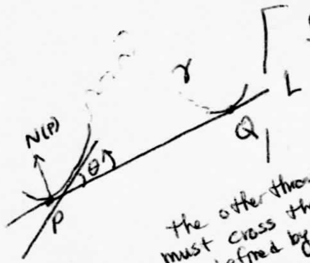
Call the common line L.

Suppose not. But the curve $C([0,\ell])$ is Connected, and since it touches the outer lines, it must cross the center one $\Rightarrow$ curve not convex $\Rightarrow\!\Leftarrow$

Step 4 $\quad$ Claim: In fact, the tangent lines that coincide must be the ones thru $C(s_1)$ and $C(s_2)$.

Pf $\quad$ For a contrad., lets assume the ones that co-incide are thru $C(s_0)$ and $C(s_1)$.

Lemma: If a convex planar curve is tangent to a line L at 2 pts, P and Q say, then the curve actually coincides with L between those pts.



the other thread must cross the line defined by the normal vector at P.

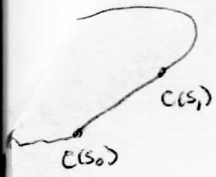
pf. At P, the thread of the curve in the direction $\overrightarrow{PQ}$ is well defined, even if the flow along the curve is not in this direction. This thread eventually reaches Q. Call this thread γ.

- We know $P \in L$ and the curve's tangent vector c' is colinear with L at P. Call $c(t) = P$
- The tangent vector must turn continuously from $c'(t)$; no sharp corners
- The curve c can't dip below L [or we violate convexity]

▷ We know γ ~~agrees~~ coincides with L at P. If γ would happen to deviate from $\overrightarrow{PQ}$, then in a nbhd of any pt where γ starts to deviate from PQ, the collection of tangent lines must make only small pos angles θ with L.
Fix one of these lines.
It cuts thru the line segment $\overline{PQ}$ at an interior pt and hence P and Q lie on opposite sides $\Rightarrow c$ is not convex $\Rightarrow\!\Leftarrow$

END LEMMA

cont'd →



From the Lemma, we see $C(s_0)$ and $C(s_1)$ are joined by a straight line segment.

$\Rightarrow$ $C'(s_0)$ and $C'(s_1)$ must point in the same direction $\Rightarrow\!\Leftarrow$

This proves the step 4 claim:

it is $C(s_1)$ and $C(s_2)$ which are joined by the straight line segment.

$\Rightarrow$ $T(s)$ is const on $[s_1, s_2]$

$\Rightarrow$ f is const on $[s_1, s_2]$

But this contradicts what we established in step 1 and we must conclude that f' does <u>not</u> change sign

$\Rightarrow$ K does not change sign on $[0, \ell]$

$\boxed{\text{QED!}}$