5-31 $\omega = \dfrac{x}{r^3}\, dy \wedge dz - \dfrac{y}{r^3}\, dx \wedge dz + \dfrac{z}{r^3}\, dx \wedge dy = \text{"}d\Theta\text{"}$ ← It is called this, not because there is a fcn Θ on $\mathbb{R}^3 - \{0\} \ni d\Theta = \omega$, but because the Solid Angle $\Theta = \int_S \omega$

Defined on $\mathbb{R}^3 - \{0\}$. $r = \sqrt{x^2+y^2+z^2}$

IMPORTANT

The def $\Theta = \int_{\pi(S)} dA$ only really makes sense when S is pierced only once by each radial line from 0. However, $\int_S \omega$ is perfectly well defined for any surface S; It is the flux thru S from the vector field $\frac{1}{r^2}\hat{e}_r$

(a) Show $d\omega = 0$

This is a straightforward calculation which I won't bother to recopy here.

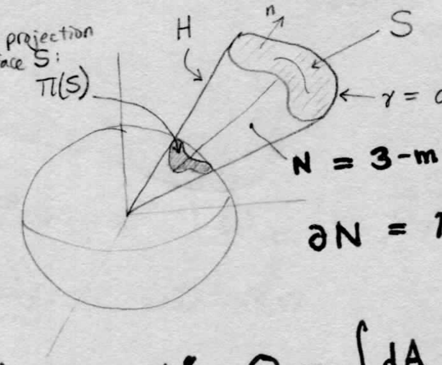
(b) Show $\omega_{\vec{r}}(v,w) = \overrightarrow{v \times w} \cdot \dfrac{1}{r^2}\hat{r}$ or $\overrightarrow{v \times w} \cdot \dfrac{1}{r^3}\vec{r}$ $\quad \vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

observe directly that $\omega = \dfrac{1}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} dx^2 \wedge dx^3 \\ -dx^1 \wedge dx^3 \\ dx^1 \wedge dx^2 \end{bmatrix}$

and observe that

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \leftarrow dx^2 \wedge dx^3 (v,w) \qquad \square$$

I'm going to deviate from Spivak's (a), (b), (c) sequence and just solve the problem:



Radial flux from the origin is what is the same thru the 2 end surfaces, and 0 thru H, so ω must measure that. See next page.

ω is the analog of the vf $F = \frac{1}{r^2}\hat{e}_r$ which is exactly the vf that has the same flux thru sphere $S^2(0,R)$ for any R.

N = 3-mfd (solid) bordered by S, H, and $\pi(S)$.

$\partial N = \pi(S) \cup H \cup S$ and even though this surface has sharp creases, we assume that we can apply Stokes Thm.

We define "Solid Angle" $\Theta = \int_{\pi(S)} dA$ and we want to show $\Theta = \int_S \omega$

First relate $\int_{\pi(S)} \omega$ to $\int_S \omega$ via Stokes' Thm applied to N:

We are implicitly assuming that S is oriented with an outward pointing normal wrt the solid "cone" N. See prob 5-33

$$\int_{\partial N} \omega = \int_N d\omega = \int_N 0 = 0$$

$$\underbrace{\int_{-\pi(S)} \omega}_{} + \int_H \omega + \int_S \omega$$

opposite orientation: outward normal to N is inward normal to $S^2(0,R)$

Thus if we can show $\int_H \omega = 0$ then I'd have $\int_S \omega = -\int_{-\pi(S)} \omega = \int_{\pi(S)} \omega$

▷ By construction of surface H, for any $x \in H$, outward normal $\hat{n}_x \perp \hat{e}_r$ indeed $\hat{e}_r = \hat{x}$

Then since for any $\vec{v}, \vec{w} \in T_x H$, $\vec{v} \times \vec{w} = \lambda \hat{n}_x$ for some $\lambda \neq 0$ $\quad$ v, w not colinear!

Then $\omega_x(v,w) = \underbrace{\dfrac{1}{\|x\|^3}\vec{x}}_{\frac{1}{r^2}\hat{e}_r} \cdot \underbrace{(\vec{v} \times \vec{w})}_{\lambda \hat{n}} = 0$ and thus $\omega|_H = 0$

and thus $\int_H \omega = 0$

cont'd →

5-31 cont'd

So now we have $\displaystyle\int_{\pi(S)} \omega = \int_S \omega$

and we only need $\displaystyle\int_{\pi(S)} \omega = \int_{\pi(S)} dA$

The sphere $S^2_{(0,R)}$ is parameterized (except for a set of ⓜ 0) by

$$\alpha : (0, 2\pi) \times (0,\pi) \longrightarrow \mathbb{R}^3$$
$$(\theta, \varphi) \longmapsto \begin{bmatrix} R\cos\theta\sin\varphi \\ R\sin\theta\sin\varphi \\ R\cos\varphi \end{bmatrix}$$

although $\vec{\alpha}_\theta \times \vec{\alpha}_\varphi$ does give an inward pointing normal.

Marsden & Tromba VC ch 6 ?.

By def

area $\displaystyle\int_{\alpha(D)} 1\, dS = \int_D 1\, \|\vec{\alpha}_\theta \times \vec{\alpha}_\varphi\|\, d\theta\, d\varphi$

and we also know from Marsden & Tromba Flux $= \displaystyle\int_{\alpha(D)} \vec{F}\cdot d\vec{S} = \int \vec{F}\cdot\hat{n}\, dS$ But if $\vec{F} = \hat{n}$ we'd recover the area form

$$\int_{\alpha(D)} \hat{n}\cdot\hat{n}\, dS = \int_{\alpha(D)} 1\, dS.$$

$\triangleright$ $\displaystyle\int_{\pi(S)} \omega = \int_{\underbrace{\alpha^{-1}(\pi(S))}_{D}} \alpha^*\omega = \int_D \omega_{\alpha(x)}(\vec{\alpha}_\theta, \vec{\alpha}_\varphi)$

$\langle\theta,\varphi\rangle$

$\hat{n}\,dS$ for $S = S(0,R)$

$$= \int \frac{1}{\|\alpha(x)\|^3}\vec{\alpha}(x) \cdot \overbrace{R^2\left(\widehat{\alpha_\theta \times \alpha_\varphi}\right) d\theta\, d\varphi}^{\hat{n}\,dS}$$

Maybe glossing over a sign error due to $\alpha_\theta \times \alpha_\varphi$ inward normal...

now $S = S(0,1)$

$$= \int_D \frac{1}{\cancel{R^3}}\vec{R}\cdot \cancel{R^2}\,\hat{n}\, dS$$

Probably should be $-\hat{n}$ here, which means I should change my α param to be $\begin{bmatrix} -\alpha^1 \\ \alpha^2 \\ \alpha^3 \end{bmatrix}$ to get outward $\hat{n}$.

$$= \int_D \hat{n}\cdot\hat{n}\, dS$$

$$= \int_D 1\, dS = \text{area}(D) \text{ on } S^2_{(0,1)} = \int_{\pi(S)} dA$$ $\square$

more specifically, no Θ can exist on any sphere $S(0,r)$ for $r>0$.

$\triangleright$ observe there is no fcn Θ on $\mathbb{R}^3 - \{0\}$ $\ni$ $d\Theta = \omega$:

we just found $\displaystyle\int_{\pi(S)} \omega = \int_{\pi(S)} dA$. Take $\pi(S) = S^2(0,r)$. Then $\displaystyle\int_{S^2_r} \omega = \text{area}(S^2_r) = 4\pi r^2 \neq 0$.

same

But if $\omega = d\Theta$:

$$4\pi r^2 \neq \int_{S^2_r} d\Theta \overset{\text{Stokes}}{=} \int_{\partial S^2_r} \Theta = \int_{\emptyset} \Theta = 0 \quad \Rightarrow\Leftarrow$$

and this holds $\forall\, r>0$

Thus no Θ can exist on $\mathbb{R}^3 - \{0\}$

$$\mathbb{R}^3 - \{0\} = \bigcup_{r>0} S(0,r)$$ $\square$

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$\omega_0^A = \frac{x}{r^3} dy\wedge dz - \frac{y}{r^3} dx\wedge dz + \frac{z}{r^3} dx\wedge dy$

5-32

$f, g : [0,1] \hookrightarrow \mathbb{R}^3$ non-intersecting closed curves.

linking number $\ell(f,g) := \frac{-1}{4\pi} \int_{C_{f,g}} d\Theta = \frac{-1}{4\pi} \int \omega_0^A$ $\quad d\omega^A = 0$

$C_{fg} \to \Gamma(fg)$

ⓐ Let (F, G) be a homotopy of non-intersecting closed curves

show $\ell(F_0, G_0) = \ell(F_1, G_1)$

Stokes for chains

$0 = \int_\Sigma d\omega = \int_{\partial\Sigma = C_{F_0G_0} - C_{F_1G_1}} \omega$ and we have the result.

ⓑ Show $\ell(f,g) = \frac{-1}{4\pi} \int_0^1 \int_0^1 \frac{1}{\| f(u) - g(v) \|^3} A(u,v)\, du\, dv$

where $A(u,v) := \det \begin{bmatrix} Df_u^{1} & Df_u^{2} & Df_u^{3} \\ Dg_v^{1} & Dg_v^{2} & Dg_v^{3} \\ f^1(u) - g^1(v) & f^2 - g^2 & f^3 - g^3 \end{bmatrix}$

ⓒ Show $\ell(f,g) = 0$ if f, g both lie in x-y plane

This, at least, is easy

$\det \begin{vmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{vmatrix} = 0$

thus $\ell(f,g) = -\frac{1}{4\pi} \iint 0 = 0$

$\ell(f,g) = -\frac{1}{4\pi} \int_{\Delta_{fg}} \frac{x}{r^3} dy\wedge dz - \frac{y}{r^3} dx\wedge dz + \frac{z}{r^3} dx\wedge dy$

$= -\frac{1}{4\pi} \int_{[0,1]\times[0,1]} \frac{\Delta^1}{\rho^3} \det\begin{pmatrix} 2,3 \\ u,v \end{pmatrix} - \frac{\Delta^2}{\rho^3} \det\left(\frac{\partial(1,3)}{\partial(u,v)}\right) + \frac{\Delta^3}{\rho^3} \det\left(\frac{\partial(1,2)}{\partial u,v}\right) du\, dv$

Fubini $\to \int_0^1\int_0^1$

$\begin{vmatrix} \Delta_u^2 & \Delta_v^2 \\ \Delta_u^3 & \Delta_v^3 \end{vmatrix} \quad \begin{vmatrix} \Delta_u^1 & \Delta_v^1 \\ \Delta_u^3 & \Delta_v^3 \end{vmatrix} \quad \begin{vmatrix} \Delta_u^1 & \Delta_v^1 \\ \Delta_u^2 & \Delta_v^2 \end{vmatrix}$

$\Delta_u^1 = f_u^1 - g_u^1 = f_u^1$

$\Rightarrow (f^1 - g^1)\begin{vmatrix} f_u^2 & -g_v^2 \\ f_u^3 & -g_v^3 \end{vmatrix} - (f^2 - g^2)\begin{vmatrix} f_u^1 & -g_v^1 \\ f_u^3 & -g_v^3 \end{vmatrix} + (f^3 - g^3)\begin{vmatrix} f_u^1 & -g_v^1 \\ f_u^2 & -g_v^2 \end{vmatrix}$

f is at u

g is at v

$$A(u,v) = \det \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1' & g_2' & g_3' \\ \Delta^1 & \Delta^2 & \Delta^3 \end{vmatrix} \qquad \begin{aligned} \Delta_1 &= f_1 - g_1 \\ \Delta_2 &= f_2 - g_2 \\ \Delta_3 &= f_3 - g_3 \end{aligned}$$

$$= f_1'\left(\cancel{g_2'\Delta_3} - \cancel{g_3'\Delta_2}\right) - f_2'\left(\cancel{g_1'}\Delta_3 - g_3'\Delta_1\right) + f_3'\left(\cancel{g_1'\Delta_2} - g_2'\Delta_1\right)$$

$$= \left(f_1'g_2' - f_2'g_1'\right)\Delta_3 + \left(f_3'g_1' - f_1'g_3'\right)\Delta_2 + \left(f_2'g_3' - f_3'g_2'\right)\Delta_1$$

$$= \begin{vmatrix} f_1' & g_1' \\ g_2 & g_2' \end{vmatrix}\Delta_3 + -\Delta_2 \begin{vmatrix} f_u^1 & g_v^1 \\ f_u^3 & g_v^3 \end{vmatrix} + \Delta_1 \begin{vmatrix} f_u^2 & g_v^2 \\ f_u^3 & g_v^3 \end{vmatrix}$$

if d mult a row in the det by α, the det is changed by α^n [multilinearity]

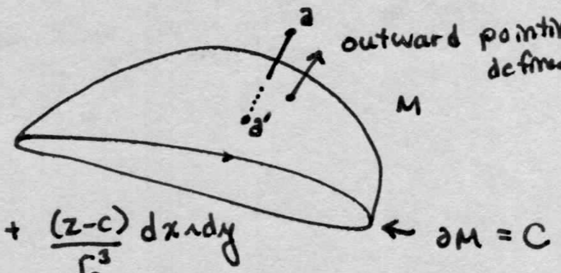
thus

$$\begin{vmatrix} f_u^2 & g_v^2 \\ f_u^3 & g_v^3 \end{vmatrix} = (-1)^2 \begin{vmatrix} f_u^2 & -g_v^2 \\ f_u^3 & -g_v^3 \end{vmatrix}$$

Spivak COM 9/18/2003

(5-33) ①

Let M be a cpt oriented 2-mfd-∂ in $\mathbb{R}^3$ (a surface!)
(Indeed, M is intended to be a surface spanning a fixed curve C; If C is too knotted or otherwise pathalogical, it is not obvious to me that such an oriented spanning surface, like a soap film, can be found. Therefore I will draw C "nice")



Let $\mathbf{a} = \langle a, b, c \rangle \notin M$

Let $\overset{\mathbf{a}}{\omega}_x := \frac{(x-a)}{r_{\mathbf{a}}^3} dy \wedge dz - \frac{(y-b)}{r_{\mathbf{a}}^3} dx \wedge dz + \frac{(z-c)}{r_{\mathbf{a}}^3} dx \wedge dy$

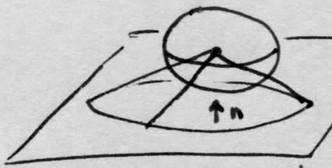
where $r_{\mathbf{a}} = \left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{1/2}$

Let $\Omega_M : (\mathbb{R}^3 - M) \longrightarrow \mathbb{R}$
$\mathbf{a} \longmapsto \int_M \overset{\mathbf{a}}{\omega}$

This is the solid angle of M wrt a little sphere centered at $\mathbf{a}$.
(I guess the symbol "Θ" of prob 5-31 only refered to a sphere centered at 0.)

ⓐ Let $\mathbf{a}$ be a pt on the pos side of M and $\mathbf{a}'$ a pt on the neg side.
Show that if $\|\mathbf{a}-\mathbf{a}'\|$ is small enough, then $\Omega_M(\mathbf{a}) - \Omega_M(\mathbf{a}') = -4\pi + \Theta(\|\mathbf{a}-\mathbf{a}'\|)$.

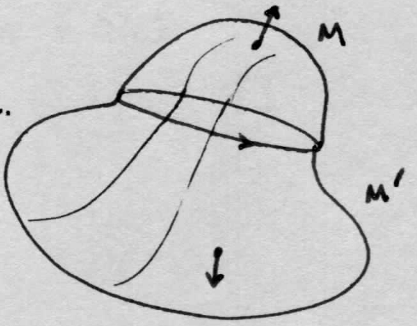
[Heuristically what I'd like is that when $\mathbf{a}$ is very close to M on the outside, then $\Omega(\mathbf{a}) = -2\pi$ since from inside the sphere at $\mathbf{a}$, M looks like an infinite plane and $\Omega(\mathbf{a}') = +2\pi$



unfortunately, I can't quite establish this stronger relation]

Let us complete M to a solid cpt 3-mfd-∂ by closing M with another surface M' spanning the curve $C = \partial M$

Pf we must preserve the outward normal of M by building M' from the neg side.



The solid is N and $\partial N = M \cup M'$

$\mathbf{a}$ is outside N ↳ Apply Stokes

$$\int_{\partial N} \overset{\mathbf{a}}{\omega} = \int_N d\overset{\mathbf{a}}{\omega} = \int_N 0 = 0$$

$$\parallel$$

$$\int_M \overset{\mathbf{a}}{\omega} + \int_{M'} \overset{\mathbf{a}}{\omega}$$

Thus $\int_M \overset{\mathbf{a}}{\omega} + \int_{M'} \overset{\mathbf{a}}{\omega} = 0$ ⊛

Cont'd →

Now let a' be in the interior of N (i.e. on the neg side of M)

Enclose a' by a small ball B (the classic arg) Apply Stokes

$$\int_{\partial N \cup -\partial B} \overset{a'}{\omega} = \int_{N-B} d\overset{a'}{\omega} = 0$$

$$\Rightarrow \int_{\partial N} \overset{a'}{\omega} = \int_{+\partial B} \overset{a'}{\omega} = 4\pi$$

we know this for a sphere; we can directly calculate the $\frac{1}{r^2}\hat{e}_r$ flux from a' thru a sphere centered at a'.

$$\Rightarrow \int_M \overset{a'}{\omega} + \int_{M'} \overset{a'}{\omega} = 4\pi \quad (\star\star)$$

$\triangleright$ Now $(\star) - (\star\star)$

$$= \int_M \overset{a}{\omega} + \int_{M'} \overset{a}{\omega} - \int_M \overset{a'}{\omega} - \int_{M'} \overset{a'}{\omega} = -4\pi$$

$$= \int_M \overset{a}{\omega} - \int_M \overset{a'}{\omega} = -4\pi + \int_{M'} \overset{a'}{\omega} - \int_{M'} \overset{a}{\omega}$$

both a and a' are on the neg side of M'.

$$= \Omega_M(a) - \Omega_M(a') = -4\pi + \underbrace{\int_{M'} (\overset{a'}{\omega} - \overset{a}{\omega})}_{\Omega_{M'}(a') - \Omega_{M'}(a)}$$

we only care about a and a' sliding along a fixed short curve γ and meeting at a pt $m \in M$.

$\Omega_{M'}$ is cont at m (it is Ω_M that is not!)

$\Omega_{M'}$ is cont in an open ball around m also, but we don't need that. By using the triangle ineq, we can establish: for any given $\varepsilon > 0$

$$\|a - a'\| < \delta \Rightarrow |\Omega_{M'}(a) - \Omega_{M'}(a')| < \varepsilon$$

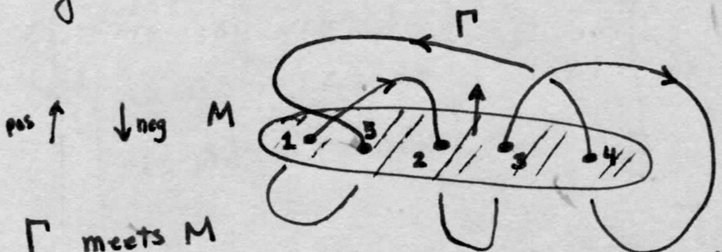
Thus

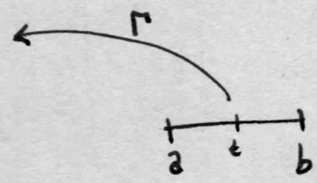
$$\Omega_M(a) - \Omega_M(a') = -4\pi + O(\delta)$$

$\square$

Cont'd $\longrightarrow$

(b) Now, following the notation of Courant, let C be a fixed curve and M a oriented 2-mfd-∂ spanning C (like a soap film). Let Γ be another curve.





Assume that if Γ meets M then Γ transversally intersects M there. (no bouncing off)

Let $p :=$ # of intersections where Γ comes out on the pos side of M.
$n :=$ " " " " " " " " " neg " " ".

Show $\displaystyle 4\pi(p-n) = \int_\Gamma d\Omega_M$ where "$d\Omega_M$" means $\overset{a}{\omega}$; $a := \Gamma(t)$.

Pf $\displaystyle \int_\Gamma d\Omega_M = \int_\Gamma \nabla\Omega_M \cdot ds = \Omega(\Gamma(b)) - \Omega(\Gamma(a))$ $\left[\text{Marsden \& Tromba VC notation } \int_\sigma F\cdot ds = \int_a^b F(\sigma(t))\cdot\sigma'(t)\,dt\right]$

But the integrand blows up every time Γ meets M, say at the param values $t_1, \dots, t_N$ (N finite!). So do it piecewise on good intervals.

Thus we write $\displaystyle \int_\Gamma d\Omega = \lim_{\delta\downarrow 0} \sum_{i=1}^{N}\left(\int_{t_i+\delta}^{t_{i+1}-\delta} d\Omega\right)$ with the convention that $t_{N+1} = t_1$

[Param Γ so that $\Gamma(a) = \Gamma(b) \notin M$ so all $t_i \in (a,b)$.]

Then writing "Ω" for $\Omega_M \circ \Gamma$:

δ must be so small we don't overlap any other intervals or end pts!

$$= \lim_{\delta\downarrow 0} \sum_{i=1}^{N} \Omega(t_{i+1}-\delta) - \Omega(t_i+\delta)$$

$$= \Omega(t_2-\delta) - \Omega(t_1+\delta) + \Omega(t_3-\delta) - \Omega(t_2+\delta) + \dots + \Omega(t_1-\delta) - \Omega(t_N+\delta)$$

re arranging:

$$= \lim_{\delta\downarrow 0}\left[-\big(\Omega(t_1+\delta) - \Omega(t_1-\delta)\big) + -\big(\Omega(t_2+\delta) - \Omega(t_2-\delta)\big) + \dots + -\big(\Omega(t_N+\delta) - \Omega(t_N-\delta)\big)\right]$$

From Part (a) In the limit, each term is either $+4\pi$ if Γ is going thru to pos side or -4π in the other case.

Thus $\displaystyle \int_\Gamma d\Omega = 4\pi(p-n)$

$\square$

cont'd →

ⓒ Spivak wants us to compute partial derivs of Ω_M, but in fact, as hinted in Courant, we really want to establish

sign discrepancy with Courant

$$\nabla\Omega_M(\mathbf{a}) = \int_C \frac{1}{r_{\mathbf{a}}^3}\begin{bmatrix} -(z-c)\,dy + (y-b)\,dz \\ (z-c)\,dx - (x-a)\,dz \\ -(y-b)\,dx + (x-a)\,dy \end{bmatrix} = \int_C \frac{1}{r_{\mathbf{a}}^3}\begin{bmatrix} (x-a) \\ (y-b) \\ (z-c)\end{bmatrix} \times \begin{bmatrix} dx \\ dy \\ dz\end{bmatrix}$$

$\mathbf{a} = \langle a,b,c\rangle$

$$= \int_C \frac{1}{r_{\mathbf{a}}^3}\vec{r}_{\mathbf{a}} \times d\vec{l}$$

and thus we have the Biot-Savart law from E&M

$$\vec{B}(\mathbf{a}) = \frac{\mu_0 i}{4\pi}\vec{\nabla}\Omega_M(\mathbf{a})$$

i.e. we express the familiar integral as the grad of some fcn, namely $\Omega_M : (\mathbb{R}^3 - M) \longrightarrow \mathbb{R}$, $\mathbf{a} \longmapsto \int_M \overset{\mathbf{a}}{\omega}$

Note: (5-31) established there is no fcn $f : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}$ such that $df = \omega$. There is no contradiction here because Ω_M is a "surface dependent" "fcn" just like the angle fcn $\int_\gamma \omega$ in $\mathbb{R}^2$.

▷ we shall only compute $\frac{\partial}{\partial a}\Omega_M(\mathbf{a})$; the other terms are analogous.

$$\frac{\partial}{\partial a}\Omega_M(\mathbf{a}) = \frac{\partial}{\partial a}\int_M \underbrace{\frac{(x-a)}{r_{\mathbf{a}}^3}}_{A} dy\wedge dz - \underbrace{\frac{(y-b)}{r_{\mathbf{a}}^3}}_{B} dx\wedge dz + \underbrace{\frac{(z-c)}{r_{\mathbf{a}}^3}}_{C} dx\wedge dy$$

$$= \int_M \frac{\partial}{\partial a}A\, dy\wedge dz - \frac{\partial}{\partial a}B\, dx\wedge dz + \frac{\partial}{\partial a}C\, dx\wedge dy$$

we can interchange $\frac{\partial}{\partial a}$ and $\int$ because integrand is cont at $\mathbf{a}$, $\mathbf{a} \notin M$.

From the solution in Courant, we know we want to apply the classic Stokes thm:

$$\int_{\partial M} P\,dx + Q\,dy + R\,dz = \int_M (R_y - Q_z)\,n^{(1)} + (P_z - R_x)\,n^{(2)} + (Q_x - P_y)\,n^{(3)}\,dS$$

Thus we seek a solution to:

$$R_y - Q_z = A_a$$
$$P_z - R_x = -B_a$$
$$Q_x - P_y = C_a$$

Cont'd →

5-33

Here I know the soln, and I try to fake an arg as to how I would "derive" it:

TRICK $\frac{\partial}{\partial x} r_a^{-3} = -\frac{\partial}{\partial a} r_a^{-3}$ due to $(x-a)$. Drop r_a subscript

$$A_a = -\frac{1}{r^3} + \frac{3(x-a)^2}{r^5}$$

$$B_a = (y-b)\frac{\partial}{\partial a} r^{-3} = \frac{3(y-b)(x-a)}{r^5}$$

$$C_a = \frac{3(z-c)(x-a)}{r^5}$$

So since $B = K r^{-3}$ where K does not depend on x or a

$B_a = K \frac{\partial}{\partial a} r^{-3}$ thus take $P_z = 0$ and $R_x = \frac{\partial}{\partial x} B = K \frac{\partial}{\partial x} r^{-3}$

and we have $0 - K\frac{\partial}{\partial x} r^{-3} = K \frac{\partial}{\partial a} r^{-3}$ by TRICK

Same idea for $Q_x - P_y = C_a$

Take $P_y = 0$ then we have $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial a}$ so use the trick.

Now for $R_y - Q_z = A_a$

we already found $R_x = B_x$ $\Rightarrow$ $R = B + \varphi(y,z)$ ← take these fcns 0.

$Q_x = C_x$ $\qquad Q = C + \lambda(x,y)$ ←

and verify $B_y - C_z = A_a$ and we take $P = 0$ since $P_z = 0$, $P_y = 0$

Thus the solution is: $P = 0$, $Q = C$, $R = B$ sign error? Should be $P = 0$, $Q = C$, $R = -B$

□