Returning to Real cannonical forms

Recall p. 286 $T^* = T$ self-adj $\Rightarrow$ In a ON basis $[T]$ "$A = A^T$ $\mathbb{R}$ vector space

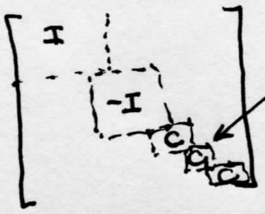
$A = A^H$ $\mathbb{C}$ vector space

$T^* = T^{-1}$ unitary $\Rightarrow$ In a ON basis $Q^T = Q^{-1}$ $\mathbb{R}$ – Here we call T an "OG operator"

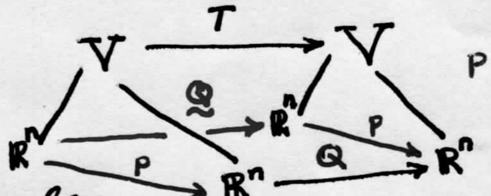
$U^H = U^{-1}$ $\mathbb{C}$

Cannonical form for Real ON mattrices

Thm 13.15 $T: V \to V$ OG operator on Real VS $\Rightarrow$ $\exists$ an ON basis for V where T has the cannonical form

$\underset{\sim}{Q} =$  $C_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$

For example, in 3x3 case $\underset{\sim}{Q} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & c & -s \\ 0 & s & c \end{array}\right]$



P O.N. CoB.

pf. I couldn't follow Schaum's pf so I give this one, very similar to Thm 13.14 and Thm 13.16 pfs.

Step 1 allow Q to operate in extended domain $Q: \mathbb{C}^n \to \mathbb{C}^n$ and show we can diagonalize.

We just apply the "Orthog subsp splitting arg" a.k.a. key Arg from sheets (16), (19):

We know $\exists$ one EW $\lambda \in \mathbb{C}$ and $w_n \in \mathbb{C}^n$ $Tw_n = \lambda w_n$ $T^* w_n = T^{-1} w_n = \frac{1}{\lambda} w_n$

Let $W_n := \text{Span}\{w_n\}$. choose $v \in W^\perp$ show $Tv \in W^\perp$ so $T: W^\perp \to W^\perp$

$$\langle Tv, w_n \rangle = \langle v, T^* w_n \rangle = \langle v, T^{-1} w_n \rangle = \langle v, \tfrac{1}{\lambda} w_n \rangle = \tfrac{1}{\lambda} \langle v, w_n \rangle = 0$$

If we have to write this in terms of Q:

$$\langle Qv, w_n \rangle = \langle v, Q^H w_n \rangle = \langle v, Q^{-1} w_n \rangle = \langle v, \tfrac{1}{\lambda} w_n \rangle = 0$$

$$w_n^H Q v = (Q^H w_n)^H v \quad (Q^{-1} w_n)^H v \quad (\tfrac{1}{\lambda} w_n)^H v = \tfrac{1}{\lambda} w_n^H v = \tfrac{1}{\lambda} 0 = 0$$

Then by the same iterative arg of restricting to lower dim $\perp$ subspaces that we used previously, we get $Q \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ so Q is diagonalized over $\mathbb{C}^n$.

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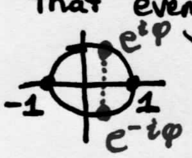
Step 2 Now use this for $Q: \mathbb{R}^n \to \mathbb{R}^n$ use the fact Q has all Real elts.

We know for any unitary matrix (here O.N.) that every EW satisfies $|\lambda| = 1$

That means it lies on unit circle in $\mathbb{C}$ 

$\det(Q - \lambda I) = 0$

This is a poly with Real coeffs $\lambda^n + c_n \lambda^{n-1} + \ldots + c_1 \lambda + c_0 = 0$ If λ is soln, so is $\bar{\lambda}$

On the unit circle $\bar{\lambda} = \frac{1}{\lambda}$

- If $\lambda = 1$ or $\lambda = -1$, since Q is Real, we'd get EV $\vec{w} = \vec{u} + i\vec{0}$ i.e. $\vec{w} = \vec{u} \in \mathbb{R}^n$
- If $\lambda = e^{i\varphi}$ otherwise, we'd get $\vec{w} = \vec{u} + i\vec{v}$ where $u, v \in \mathbb{R}^n$

cont'd →

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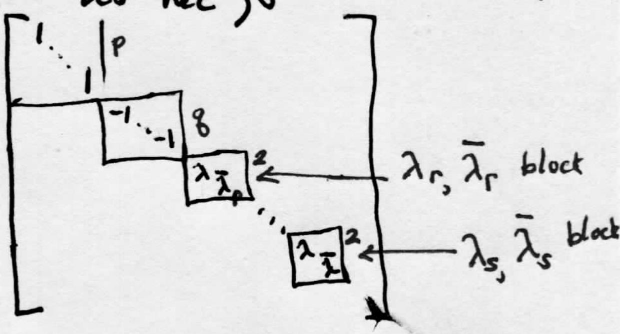
(23)

If $Aw = \lambda w$ then $\overline{Aw} = \overline{\lambda w} \Rightarrow A\bar{w} = \bar{\lambda}\bar{w}$

if $w = u + iv$ then conj $\bar{w} = u - iv$

That means that the result of Step 1 can be written (by re-ordering and re-numbering as nec)

$$Q\begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & | & & | & | \\ u_1 & \cdots & u_{p+q} & w_r & \bar{w}_r & \cdots & w_s & \bar{w}_s \\ | & & | & | & | & & | & | \end{bmatrix}$$



These vectors have no imag component $u_j \in \mathbb{R}^n$

conj pairs $\vec{u}_r + i\vec{v}_r$, $\vec{u}_r - i\vec{v}_r$

$p + q + 2 + 2 + \dots + 2 = n$

<u>Step 3</u> Now we want to utilize this to express $Q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with no reference to anything Complex.

For a $\mathbb{C}$-conj pair of EVs $w, \bar{w}$ $\quad w = \vec{u} + i\vec{v} \quad \bar{w} = \vec{u} - i\vec{v}$

$\{w, \bar{w}\}$ span a 2-dim subsp in $\mathbb{C}^n$, we will show $\{u, v\}$ span 2dim subsp in $\mathbb{R}^n$

and if EW $\lambda = \alpha + i\beta$, we show $\begin{bmatrix} \lambda & \\ & \bar{\lambda} \end{bmatrix}$ becomes $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

In fact, $|\lambda| = 1$ so $\alpha = \cos\theta$, $\beta = \sin\theta$ for some angle θ

(idea from Hirsch & Smale p.68)

$$Q(\{w, \bar{w}\}) = \{w \; \bar{w}\}\begin{bmatrix} \lambda & \\ & \bar{\lambda} \end{bmatrix}$$

$$Qw = Q(u + iv) = (\alpha + i\beta)(u + iv) = \alpha\vec{u} - \beta\vec{v} + i(\alpha\vec{v} + \beta\vec{u})$$

$$Qu + iQv$$

Re part: $Qu = \alpha u - \beta v$

Im part: $Qv = \beta u + \alpha v$

$$\Rightarrow Q(\{v, u\}) = \{v, u\}\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

(switch order to get this std form)

This looks good. We'd need to show $\{v, u\}$ is LI <u>over Real scalars</u> (which we can do following Hirsch & Smale DEDSALA p.68, written up in 'Systems of ODEs' sheets 3/12/2016)

We'd also need to show $\langle u, v \rangle = 0$ as vectors in $\mathbb{R}^n$ (maybe some Gram-Schmidt?)

Let $w_r = \vec{u} + i\vec{v}$ and $w_s = \vec{f} + i\vec{g}$

a priori $\langle w_r, \bar{w}_r \rangle \overset{!}{=} 0$ but that only gives $\langle u + iv, u - iv \rangle = 0 \Rightarrow \langle u, u \rangle = -\langle v, v \rangle$ NOT $\langle u, v \rangle = 0$

(This is impossible! $|u|^2 = -|v|^2$)

$\langle w_r, w_s \rangle \overset{!}{=} 0 \Rightarrow \langle u + iv, f + ig \rangle \overset{!}{=} 0 \Rightarrow$ $\langle u, f \rangle \overset{!}{=} -\langle v, g \rangle$, $\langle u, g \rangle \overset{!}{=} \langle v, f \rangle$ so I seem to be stuck.

[Horn & Johnson MA p.105-106 give a harder and more general pf for normal matrices, then trivially specialize to Sym and O.N.]

My arg here seems too good to fail because it does give me the correct canonical form!