13.20 (i) $\boxed{I^* = I}$ because $\forall u,v\ \langle Iu, v\rangle = \langle u, v\rangle = \langle u, Iv\rangle$

So $I = I^*$ by uniqueness.

(ii) $\boxed{0^*_{E\to E} = 0_{E\to E}}$ because $\langle 0(u), v\rangle = \langle 0_E, v\rangle = 0_K = \langle u, 0_E\rangle = \langle u, 0^*(v)\rangle$

(iii) $\boxed{(T^{-1})^* = (T^*)^{-1}}$ because $I = I^* = (TT^{-1})^* = (T^{-1})^* T^*$ ← by (3) prev page.

$$\Rightarrow I(T^*)^{-1} = (T^{-1})^*$$
$$(T^*)^{-1} = (T^{-1})^* \quad \square$$

13.21 $T: V \to V$ linear

$\left.\begin{array}{l} W \text{ is a } T\text{-invariant subsp of } V: \\ \qquad T|_W : W \to W \end{array}\right\} \Rightarrow W^\perp$ is invariant under T^*

Let $u \in W^\perp$

If $w \in W$ then $T(w) \in W$ so $\langle w, T^* u\rangle = \langle Tw, u\rangle = 0$

Thus $T^* u \in W^\perp \Rightarrow W^\perp$ is invariant under T^* $\square$

13.78 Counter-ex to show the adjoint does not always exist — although the internet seems to disagree.

Let V = vs of polys over $\mathbb{R}$, presumably all polys defined on $[0,1]$

(arb C^∞ fcns might also work?)

$\langle p, q\rangle := \int_0^1 p(t)q(t)\,dt$. Let linear operator $D = \frac{d}{dt}$ $\left[\text{so } Dp = \frac{d}{dt}p = p'\right]$

<u>CLAIM</u>: $\nexists\ D^*: V \to V$ such that $\langle Dp, q\rangle = \langle p, D^* q\rangle$

Renaming D^* as L, what we want is a linear map $L \ni \int_0^1 p' q\,dt = \int_0^1 p\, L(q)\,dt$ ★

and L must be the same fixed map for all p, q — it can't depend on q or p, just like $D = \frac{d}{dt}$ is fixed.

<u>IDEA</u>: ★ $\Rightarrow \int_0^1 (p'q - p\,L(q(t)))\,dt = 0$ for any p, q. Does this force $p'(t)q(t) - p(t)\,L(q(t)) = 0$?

No, because we can't rule out

<u>IDEA 2</u> Integration by parts $\int_0^1 \underbrace{p}_{u}\,\underbrace{q\,dt}_{dv} = p(t)q(t)\Big]_0^1 - \int_0^1 p\,q'\,dt$

$= (p(1)q(1) - p(0)q(0)) + \int_0^1 p(-q')\,dt$

So for any polys p, q where $p(1)q(1) = 0$ and $p(0)q(0) = 0$ say $p(1) = 0$ and $q(0) = 0$ we have it, or even $[p(1)q(1) - p(0)q(0)] = 0$. More usefully, we could restrict V to only polys that have zeroes at 0 and 1.

In that case $L = D^* = -\frac{d}{dt}$

But in general, we would need $\int_0^1 p\,L(q)\,dt \overset{!}{=} p_1 q_1 - p_0 q_0 + \int_0^1 p(-q')\,dt$

No way this is possible if L can operate only on q and have no dependence on p $\square$

ASIDE: If T <u>self-adj</u> $(T = T^*) \Rightarrow A = A^H$ (or $A = A^T$ in Real case)

Thus we could call T '<u>Hermetian</u>' or '<u>Symmetric</u>' respectively.

we know, from sheet 10 $\quad b_{ij} = \langle T^* e_j, e_i\rangle \overset{\text{transpose}}{=} \overline{\langle e_i, T^* e_j\rangle} \overset{\text{adjoint}}{=} \overline{\langle Te_i, e_j\rangle} = \bar{a}_{ji}$ $\quad$ cf sheet 17

T self-adj $\longrightarrow \ \parallel$

$\underbrace{\langle Te_j, e_i\rangle}_{a_{ij}} \Rightarrow a_{ij} = \bar{a}_{ji} \Rightarrow A = A^H$

$A = A^T$ in Real case $\square$

Analogy between Linear operators and $\mathbb{C}$

$\mathbb{C}$	operation	Linear Operators
Unit circle $\lvert z\rvert=1$	$\bar{z}=\frac{1}{z}$ ⟵⋯⟶ $T^*=T^{-1}$	T is <u>O.N. Operator</u> over $\mathbb{R}$ T is <u>Unitary</u> op over $\mathbb{C}$
$z\in\mathbb{R}$ axis	$\bar{z}=z$ ⋯ $T^*=T$	T <u>Self-Adjoint</u> i.e. <u>Symmetric</u> $\mathbb{R}$ <u>Hermitian</u> $\mathbb{C}$
$z\in\mathbb{I}\mathrm{m}$ axis	$\bar{z}=-z$ ⋯ $T^*=-T$	T skew-adjoint skew-symm $\mathbb{R}$ skew-Herm $\mathbb{C}$
$z\in$ pos $\mathbb{R}$ axis $(0,\infty)$	$z=\bar{w}w$, $w\neq 0$ ⋯ $T=S^*S$, S nsing	<u>T Pos def</u>

Thm 13.8 Consider $Tx=\lambda x \quad x\neq 0$

For EV results see prob 13.38 p.303

all special cases of $T^*T=TT^*$ Normal operators

(i) [Unitary] $T^*=T^{-1} \Rightarrow$ all EWs $|\lambda|=1$

(ii) [self-adj] $T^*=T \Rightarrow$ all EWs $\lambda\in\mathbb{R}$

(iii) ["Skew-adj"] $T^*=-T \Rightarrow$ all EWs λ imaginary $\mathrm{Re}(\lambda)=0$

(iv) [Pos def] $T=S^*S$ for S nsing $\Rightarrow$ all EWs λ Real and pos. $(0,\infty)$

Pf.

$Tx=\lambda x \Rightarrow T^{-1}Tx=\lambda T^{-1}x \Rightarrow \frac{1}{\lambda}x=T^{-1}x$

(i) $\langle Tx,x\rangle=\langle\lambda x,x\rangle=\lambda\langle x,x\rangle$

$\quad \overset{\shortparallel}{\langle x,T^*x\rangle} \overset{\text{hypoth}}{=} \langle x,T^{-1}x\rangle = \langle x,\frac{1}{\lambda}x\rangle=\frac{1}{\bar\lambda}\langle x,x\rangle \Rightarrow \frac{1}{\bar\lambda}=\lambda \Rightarrow \bar\lambda\lambda=1 \Rightarrow |\lambda|^2=1 \Rightarrow |\lambda|=1$ □

$(x-iy)(x+iy)$

(ii) $\langle Tx,x\rangle=\lambda\langle x,x\rangle$

$\quad \overset{\shortparallel}{\langle x,T^*x\rangle} \overset{\text{hyp}}{=} \langle x,Tx\rangle=\overline{\langle Tx,x\rangle}=\bar\lambda\langle x,x\rangle$ since $\langle x,x\rangle$ Real $\Rightarrow \lambda=\bar\lambda \Rightarrow \lambda\in\mathbb{R}$

(iii) From (ii) $\lambda\langle x,x\rangle=\overline{\langle -Tx,x\rangle}=-\bar\lambda\langle x,x\rangle \Rightarrow \bar\lambda=-\lambda \quad (x-iy)=-x-iy \Rightarrow x=0$

$\mathrm{Re}(\lambda)=0$

λ pure imag.

(iv) $T=S^*S$ $\quad S$ nsing so $x\neq 0 \Rightarrow Sx\neq 0$

$\underbrace{\langle Sx,Sx\rangle}_{\text{Real and pos}}=\langle x,S^*Sx\rangle=\langle x,Tx\rangle=\langle x,\lambda x\rangle=\bar\lambda\underbrace{\langle x,x\rangle}_{\text{Real and pos}}$

$\Rightarrow \bar\lambda$ must be Real and pos

so $\mathrm{Im}(\lambda)=0$ and $\lambda\in(0,\infty)$ □

Prob 13.22

(i) show $\langle Tu,v\rangle=0 \quad \forall u,v \Rightarrow T=0$

Take $v=Tu$ then $\langle Tu,Tu\rangle=0 \Rightarrow Tu=0$ but u is arb

$\Rightarrow T=0$

cont'd →

13.22 cont'd (ii) $\boxed{V \text{ w/ } \mathbb{C}\text{-scalars, } \langle Tu, u\rangle = 0\ \forall u \Rightarrow T = 0}$

(not true for $\mathbb{R}$-scalars, see below)

Consider $\langle T(u+w), u+w\rangle = 0$ by hypoth

$$= \underbrace{\cancel{\langle Tu, u\rangle}}_{0} + \langle Tu, w\rangle + \langle Tw, u\rangle + \overset{0}{\cancel{\langle Tw, w\rangle}} = 0$$

$$\Rightarrow \langle Tu, w\rangle + \langle Tw, u\rangle = 0$$

By the same arg applied to $v + iw$ $\longrightarrow$ $\langle Tu, iw\rangle + \langle T(iw), u\rangle = 0$

$$\bar{i}\langle Tu, w\rangle + \cancel{i}\langle Tw, u\rangle$$

add them: $\Rightarrow \overset{-\cancel{i}(-1)}{}\ \langle Tw, u\rangle = 0$ but w, u arb $\Rightarrow T = 0$ by (i)

▷ If we have V with $\mathbb{R}$ scalars, this doesn't work:

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$

Then $\langle T\vec{u}, \vec{u}\rangle = 0$ but $T \neq 0$

$[x\ y]\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x\ y]\begin{bmatrix} y \\ -x \end{bmatrix} = 0$

what happens if we try this arg for $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$?

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longmapsto \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix}$

$\Rightarrow [\bar{z}_1\ \bar{z}_2]\begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} = \bar{z}_1 z_2 - \bar{z}_2 z_1 \neq 0$

(iii) $\boxed{\text{Show } \left.\begin{array}{l} T = T^* \\ \langle Tu, u\rangle = 0\ \forall u \end{array}\right\} \Rightarrow T = 0}$

with no need for $T = T^*$

pf. If V has $\mathbb{C}$-scalars, we already have this result by (ii)

Assume V has only $\mathbb{R}$ scalars.

Just like above, $\langle T(u+w), u+w\rangle = 0 \Rightarrow \langle Tu, w\rangle + \langle Tw, u\rangle = 0$

Observe $\langle Tw, u\rangle \overset{\text{adj}}{=} \langle w, T^* u\rangle \overset{\text{hypoth}}{=} \langle w, Tu\rangle = \overline{\langle Tu, w\rangle} = \langle Tu, w\rangle$ since Real valued

$\Rightarrow 2\langle Tu, w\rangle = 0$ but u, w arb $\Rightarrow T = 0$ by (i) □

▷ <u>O.G. and Unitary operators</u>

$U: V \longrightarrow V$ satisfying $U^* = U^{-1}$

U is <u>Unitary</u> if V has $\mathbb{C}$ as its scalar field

<u>Orthogonal</u> (really O.N.) if field is $\mathbb{R}$

Thm 13.9 For Unitary Ops TFAE:

(i) $U^* = U^{-1}$

(ii) U preserves inner prods: $\langle Ux, Uy\rangle = \langle x, y\rangle\ \forall x, y$

(iii) U preserves lengths: $\langle Ux, Ux\rangle = \langle x, x\rangle$ ie $\|Ux\| = \|x\|$

<u>pf</u> (i) $\Rightarrow$ (ii) $\langle Ux, Uy\rangle = \langle x, U^* U y\rangle \overset{(i)}{=} \langle x, U^{-1} U y\rangle = \langle x, y\rangle$

(ii) $\Rightarrow$ (iii) $\langle Ux, Ux\rangle = \langle x, y\rangle$ thus $\|Ux\|^2 = \|x\|^2$ thus $\|Ux\| = \|x\|$

cont'd →

Thm 13.9 cont'd

(iii) $\Rightarrow$ (i) $\langle Ux, Ux\rangle = \langle x, x\rangle \quad \forall x$

Real

so Real $\rightarrow \langle x, U^*Ux\rangle = \overline{\langle U^*Ux, x\rangle} = \langle U^*Ux, x\rangle$

$\Rightarrow \langle U^*Ux, x\rangle - \langle x, x\rangle = 0 \quad \Rightarrow \langle (U^*U - I)x, x\rangle = 0 \quad \forall x$

To use prob 13.22 (iii), we need $(U^*U - I)$ to be self-adj : $(U^*U-I)^*$
$= (U^*U)^* - I^* = U^*U - I$
self adj ✓

Thus we know $U^*U - I = 0$ and this is $U^*U = I$
$U^* = U^{-1}$ $\square$

example 13.14 $T\left(\begin{bmatrix} x \\ y \\ z\end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1\end{bmatrix}\begin{bmatrix} x \\ y \\ z\end{bmatrix}$

This is O.N. rotation matrix
we can verify it preserves lengths.

example 13.15 $T : \ell_2 \longrightarrow \ell_2$ Hilbert sp.
$(a_1, a_2, a_3, \dots) \mapsto (0, a_1, a_2, a_3, \dots)$ Right shift operator

$\|Ta\| = \|a\|$
$\langle Ta, Tb\rangle = \langle a, b\rangle$

This is supposed to be a counter ex to Thm 13.9 in ∞ dim spaces, but Cheney (for example) doesn't worry.

T is not onto ℓ_2 because no pre-image for $(1, 0, 0, 0, \dots)$ so T^{-1} can't be defined.
Not sure what T^* would even be...

▷ O.G. and Unitary Matrices

From Thm 13.6
see sheet 9

$T : E \longrightarrow E$ linear, fin dim
$A = [T]_{\{e\}}$ in O.N. basis

$\Rightarrow \exists!$ linear $T^* : E \longrightarrow E \ni$
$\langle Tu, v\rangle = \langle u, T^*v\rangle$
and downstairs $[T^*]_{\{e\}} = A^H$

$$\begin{tikzcd} E \arrow[r, shift left, "T"] \arrow[d] & E \arrow[l, shift left, "T^*"] \arrow[d] \\ K^n \arrow[r, shift left, "A"] & K^n \arrow[l, shift left, "A^H"] \end{tikzcd}$$

Now let $T = U$ unitary
then $T^* = T^{-1}$

So $A^H = A^{-1}$ unitary matrix
and if the entries of A are Real, $A^H = A^T$
so $A^T = A^{-1}$ O.G. matrix (really O.N.)

Prob 13.21 $T : V \longrightarrow V$
Show $T|_W : W \longrightarrow W$ invariant $\Rightarrow T^*|_{W^\perp} : W^\perp \longrightarrow W^\perp$ invariant

Let $u \in W^\perp$ If $w \in W$ $Tw \in W$ $\langle w, T^*u\rangle = \langle \underset{\in W}{Tw}, \underset{\in W^\perp}{u}\rangle = 0$
Thus $T^*u \perp w$ so $T^*u \in W^\perp$, since u is arb, $W^\perp$ is invariant under T^* $\square$

Prob 13.24 unitary $U : V \longrightarrow V$
Show $U|_W : W \longrightarrow W$ invar $\Rightarrow U|_{W^\perp} : W^\perp \longrightarrow W^\perp$ invariant (no star)

From prev prob, we know $U^{-1} : W^\perp \longrightarrow W^\perp$ invar.

By hypoth $U(W) \subseteq W$, but actually $U(W) = W$ since U^{-1} exists (one-to-one, onto)
Let $v \in W^\perp$ and $w \in W$ $\langle Uv, w\rangle = \langle Uv, Uw'\rangle = \langle v, w'\rangle = 0$ Thus $Uv \in W^\perp$
Preimage of W
$W^\perp$ invar under U $\square$

13.25 A is arb matrix with rows $\vec{r}_i$ and cols $\vec{c}_i$

(i) Show the ij-elt of AA^H is $\langle r_i, r_j\rangle$

$$AA^H = \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix}\begin{bmatrix} | & | & | \\ \bar{r}_1 & \bar{r}_2 & \bar{r}_3 \\ | & | & | \end{bmatrix} \qquad (elt)_{23} = r_2 \cdot \bar{r}_3 = \langle r_2, r_3\rangle \left(= r_3^H r_2\right)$$

so $(elt)_{ij} = \langle r_i, r_j\rangle$

(ii) Show ij-elt of A^HA is $\langle c_j, c_i\rangle$:

$$A^HA = \begin{bmatrix} - \bar{c}_1 - \\ - \bar{c}_2 - \\ - \bar{c}_3 - \end{bmatrix}\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \qquad (elt)_{23} = c_2^H c_3 = \langle c_3, c_2\rangle$$

$$(elt)_{ij} = \langle c_j, c_i\rangle$$

13.26 **Thm 13.11** TFAE
(i) A is unitary matrix
(ii) Rows $\{\vec{r}_i\}$ form an ON set
(iii) Cols $\{\vec{c}_i\}$ form an ON set

(i) $\Leftrightarrow$ (ii) we want $\langle r_i, r_j\rangle = \delta_{ij}$.
we can form AA^H = matrix of elts $\left[\langle r_i, r_j\rangle\right]$ by above prob 13.25
and since A is unitary, $AA^H = I \Rightarrow \langle r_i, r_j\rangle = \delta_{ij}$
To go backwards, if we know $\langle r_i, r_j\rangle = \delta_{ij}$ then $AA^H = I \Rightarrow A$ unitary.

(i) $\Leftrightarrow$ (iii) very similiar. We want $\langle c_i, c_j\rangle = \delta_{ij}$
We can form $A^HA = \left[\langle c_j, c_i\rangle\right] = I$ since A unitary Thus $\langle c_j, c_i\rangle = \delta_{ij}$
or $\langle c_i, c_j\rangle = \bar{\delta}_{ij} = \delta_{ij}$ Real
$\langle c_i, c_j\rangle = \delta_{ij}$ □

▷ Remark: In Ch 3, by a <u>much</u> more involved argument, (prob 3.37) we showed $AB = I \Rightarrow BA = I$ for fin dim matrices
I used this in Strang LAAIA write up to show $Q^TQ = I \Rightarrow QQ^T = I$ for O.N. matrices. Using Thm 13.11 the result is virtually trivial.

Thm 13.12 Given the standard COB formula $\{f\} = \{e\}P$
Let $\{e\}$ be ON basis
Then
$\{f\}$ is ON basis $\Leftrightarrow$ P is a unitary matrix
(O.N. matrix if V is over $\mathbb{R}$)

pf. ($\Rightarrow$) $\{f_1 f_2 f_3\} = \{e_1 e_2 e_3\}\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \qquad f_i = p_{1i}e_1 + p_{2i}e_2 + p_{3i}e_3 = \{e\}\vec{p}_i$

Then since $\{f\}$ ON, $\delta_{ij} = \langle f_i, f_j\rangle = \langle \{e\}\vec{p}_i, \{e\}\vec{p}_j\rangle =$
$= \langle e_1 p_{1i} + e_2 p_{2i}, e_1 p_{1j} + e_2 p_{2j}\rangle$
$= p_{1i}\bar{p}_{1j}\langle e_1, e_1\rangle + p_{2i}\bar{p}_{2j}\langle e_2, e_2\rangle + \ldots$

$\delta_{ij} = \langle \vec{p}_i, \vec{p}_j\rangle$ Thus by Thm 13.11 P is Unitary (or O.N.)

($\Leftarrow$) Here we know $\delta_{ij} = \langle p_i, p_j\rangle$ and we reverse the steps to get $\langle f_i, f_j\rangle = \delta_{ij}$ □

aside: We can compute the values of the elts of P

$\{f_1 \dots f_n\} = \{e_1 \dots e_n\}[P]$ and for co-ord vectors $v_f = P^{-1} v_e$

$$f_j = \sum_k p_{kj} e_k = p_{1j} e_1 + p_{2j} e_2 + \dots + p_{nj} e_n$$

$$\langle f_j, e_i \rangle = 0 + \dots + p_{ij} \langle e_i, e_i \rangle = p_{ij} \qquad \text{row } i\text{, col } j$$

$$P = \begin{bmatrix} \langle f_1, e_1 \rangle & \langle f_2, e_1 \rangle & \langle f_3, e_1 \rangle \\ \langle f_1, e_2 \rangle & \langle f_2, e_2 \rangle & \langle f_3, e_2 \rangle \\ \langle f_1, e_3 \rangle & \langle f_2, e_3 \rangle & \langle f_3, e_3 \rangle \end{bmatrix}$$

and if V was over $\mathbb{R}$
$\langle f_j, e_i \rangle = \langle e_i, f_j \rangle$
(not symm, just order does not matter)

Now in the special case where $\{f\}$ and $\{e\}$ are co-ord vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} | & & | \\ f_1 & \cdots & f_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} \begin{bmatrix} P \end{bmatrix} \Rightarrow E^T F = P$$

Thm 13.14 **Real Spectral Thm**

V over $\mathbb{R}$

V fin dim vs with inner prod $\langle \cdot, \cdot \rangle$
$T: V \to V$ self-adj : $T^* = T$
('symm')

$\Rightarrow$
- $\exists$ ON basis for V consisting of EVs of T
- Thus T can be represented by a diag matrix Λ in this basis

Alternatively:
A is a Real symm matrix $\Longrightarrow \exists$ O.N. matrix $Q \ni Q^T A Q = \Lambda$

Pf. Schuams wants to do an induction pf, building up from $n=1$ dim vs. It doesn't quite make sense to me how these vector spaces join together, so I am following a different approach, kind of like pf of sequentially deflating down from V, ch 5.6 writeup sheets. Schur's Lemma in Strang LAAIA

$$\begin{tikzcd}
 & V \arrow[rr, "T"] \arrow[dl, "\varphi_w"'] \arrow[dd, "\varphi_e"] & & V \arrow[dl, "\varphi_e"'] \arrow[ddr, "\varphi_e"] & \\
\mathbb{R}^n \arrow[rr, "\Lambda"] \arrow[dr, "Q"'] & & \mathbb{R}^n & & \\
 & \mathbb{R}^n \arrow[rrr, "A"'] & & & \mathbb{R}^n \arrow[ull, "Q^T"']
\end{tikzcd}$$

φ_e is co-ord mapping from ON basis $\{e\}$

Step 1 We know $T: V \to V$ has at least one non-zero EV w_n
because $\det(T - \lambda I) = 0$ must have atleast one root λ_n (possibly repeated) and $\lambda_n \in \mathbb{R}$ by Thm 13.8
$(T - \lambda I)w = 0$ has a soln, call it w_n $\qquad T w_n = \lambda_n w_n$

Let $W_n := \operatorname{Span}\{w_n\}$. choose any $v \in W_n^\perp$ thus $v \perp w_n$

Claim $Tv \in W_n^\perp$ so $T|_{W_n^\perp} : W_n^\perp \to W_n^\perp$ This is the key property

prob 13.21

to see this $\langle Tv, w_n \rangle = \langle v, T^* w_n \rangle \overset{\text{sym}}{=} \langle v, T w_n \rangle = \langle v, \lambda_n w_n \rangle = \lambda_n \underbrace{\langle v, w_n \rangle}_{v \perp w_n} = 0$

What if $\lambda_n = 0$? No problem, that would only give another reason why $\langle Tv, w_n \rangle = 0$.

cont'd →

Thm 13.14 cont'd

Step 2 Repeat this process $W_n^\perp$ is a vs $T: W_n^\perp \to W_n^\perp$ $V = W_n \oplus W_n^\perp$

$\exists$ EW λ_{n-1} and EV w_{n-1} $Tw_{n-1} = \lambda_{n-1} w_{n-1}$

Let $W_{n-1} = \text{Span}\{w_{n-1}\}$

choose any $v \in (W_n \oplus W_{n-1})^\perp$ $v \neq 0$

again $Tv \in (W_n \oplus W_{n-1})^\perp$ because $\langle Tv, w_n\rangle = \langle v, Tw_n\rangle = \langle v, \lambda_n w_n\rangle = 0$

$\langle Tv, w_{n-1}\rangle = \langle v, Tw_{n-1}\rangle = 0$

Step 3 At the i^{th} step of this iteration:

$Tw_{n-i} = \lambda_{n-i} w_{n-i}$ choose $v \in (W_n \oplus \cdots \oplus W_{n-i})^\perp$ $v \neq 0$

Same arg gives $T: (W_n + \cdots + W_{n-i})^\perp \to (W_n + \cdots + W_{n-i})^\perp$ $\langle Tv, w_n\rangle = 0$ $\vdots$ $\langle Tv, w_{n-i}\rangle = 0$

Step 4 This process terminates when $(W_n + \cdots + W_1)^\perp = \{0\}$

and we are left with $T(\{w_1 \dots w_n\}) = \{w_1 \dots w_n\}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = \{w\}\Lambda$ $\square$

For the <u>alternatively</u> part, let's <u>recap sym matrices:</u>

a linear map $T: V \to V$ can be self adj $(T = T^*)$ and that is indep of basis.

But for T to have a sym matrix (or Hermetian) we need an <u>ON basis</u> $\{e_1, \dots e_n\}$

Sheet 7: $T(e_j) = \{e_1 \dots e_n\}\begin{bmatrix} a_j^1 \\ \vdots \\ a_j^n \end{bmatrix} = \sum_k a_j^{k} e_k$

col j of A

To isolate a_j^{i}: $\langle Te_j, e_i\rangle = \sum_k a_j^k \overbrace{\langle e_k, e_i\rangle}^{\delta_{ki}} = a_j^{i} = a_{ij}$ so $A = [T\langle e_j, e_i\rangle]$

Sheet 10: T^* has matrix rep A^H: applying above result to map T^*: $b_{ij} = \langle T^* e_j, e_i\rangle$

$b_{ij} = \langle T^* e_j, e_i\rangle = \overline{\langle e_i, T^* e_j\rangle} = \overline{\langle Te_i, e_j\rangle} = \bar{a}_{ji}$ so $B = A^H$

sheet 11: $T = T^* \; \overset{//}{\longrightarrow}$ $\langle Te_j, e_i\rangle = a_{ij}$ $\Rightarrow a_{ij} = \bar{a}_{ji} \Rightarrow A = A^H$ or $A = A^T$ in Real case.

So we see how a ON basis always leads to a sym matrix

Just because a matrix A is sym in one basis does <u>not</u> mean it is in an arb basis

$B = P^{-1}AP$ $A = A^T \not\Rightarrow B = B^T$. BUT if we make a COB with a ON matrix

$B = Q^TAQ \Rightarrow B^T = (Q^TAQ)^T = Q^TA^TQ^{TT} = Q^TAQ = B \Rightarrow B$ is sym. $\square$

Let's go out of sequence a little and continue generalizing Spectral thm.

<u>Def</u> T is a Normal operator if $T^*T = TT^*$ commutes with adj

Prob 13.38 (i) $Tv = 0 \Leftrightarrow T^*v = 0$

we show $\langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle$

$$\langle Tv, Tv\rangle = \langle v, T^*Tv\rangle = \langle v, TT^*v\rangle = \langle T^*v, T^*v\rangle$$

Thus $Tv = 0 \Rightarrow \langle Tv, Tv\rangle = 0 \Rightarrow \langle T^*v, T^*v\rangle = 0$

$\Rightarrow T^*v = 0$ and vice versa

Axiom 3 for IP $\langle u,u\rangle = 0 \Leftrightarrow u = 0$

(ii) $(T-\lambda I)^*(T-\lambda I) = (T-\lambda I)(T-\lambda I)^*$

$$\begin{aligned}(T-\lambda I)(T-\lambda I)^* &= (T-\lambda I)(T^* - \bar\lambda I)\\ &= TT^* - \lambda T^* - \bar\lambda T + \lambda\bar\lambda I\\ &\overset{N}{=} T^*T - \bar\lambda T - \lambda T^* + \lambda\bar\lambda I\\ &= T^*(T-\lambda I) - \bar\lambda(T-\lambda I) = (T^* - \bar\lambda I)(T-\lambda I)\\ &= (T-\lambda I)^*(T-\lambda I) \quad \square\end{aligned}$$

(iii) $Tv = \lambda v \Rightarrow T^*v = \bar\lambda v$ (same EVs)

$(T-\lambda I)v = 0$ and $(T-\lambda I)$ is Normal by (ii) $\Rightarrow (T-\lambda I)^*v = 0$ by (i)

That means $T^*v = \bar\lambda v$ and thus T and T^* share EVs

(iv) EVs of T belonging to distinct EWs λ are OG (this was previously shown for A Hermetian cf Strang LAAIA ch 5.6 writeup)

$Tv = \lambda v \quad Tw = \mu w$ show $\lambda\langle v,w\rangle = \mu\langle v,w\rangle$

$$\lambda\langle v,w\rangle = \langle \lambda v, w\rangle = \langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, \bar\mu w\rangle = \mu\langle v,w\rangle$$

$\Rightarrow \lambda\langle v,w\rangle - \mu\langle v,w\rangle = 0$

$\Rightarrow \lambda \neq \mu \Rightarrow \langle v,w\rangle = 0 \quad \square$

<u>Thm 13.16</u> <u>Complex Spectral Thm for Normal Operators</u>

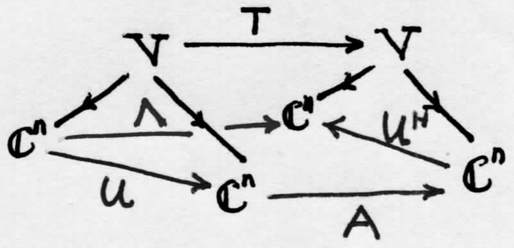
V fin dim vs over $\mathbb{C}$ with inner prod $\langle\cdot,\cdot\rangle$
$T: V \to V$ satisfies $T^*T = TT^*$ (commutes) $\Rightarrow$ $\exists$ ON basis for V consisting of EVs of T. In this basis, T is represented by a diag matrix Λ

Alternately: A is Normal matrix $A^HA = AA^H$ $\Rightarrow$ $\exists$ unitary matrix U such that
$$U^HAU = \Lambda$$

This thm encompases $T^* = T$ (self-adj) $A = A^H$ Hermetian skew-Hermetian
$T^* = T^{-1}$ unitary $U^{-1} = U^H$ and others

Pf. again, the pf given by Schuams (S. Lipschutz) seems a bit questionable, so I am deviating from it.

This is almost identical to Real Spectral pf



cont'd →

pf cont'd **Step 1** $T: V \to V$ has at least one EW $\lambda_n \in \mathbb{C}$ (if $\lambda = 0$ no problem)

$(T - \lambda I)$ is sing so $\exists$ at least one nonzero EV w_n

$Tw_n = \lambda_n w_n$ and since T is Normal $T^* w_n = \bar{\lambda}_n w_n$ (prob 13.38 iii sheet 18)

$W_n := \text{Span}\{w_n\}$ KEY

Choose $v \in W_n^\perp$, $v \neq 0$ $\quad \langle Tv, w_n \rangle = \langle v, T^* w_n \rangle = \langle v, \bar{\lambda}_n w_n \rangle = \lambda \langle v, w_n \rangle = 0$ because we know $v \perp w_n$

what if $\lambda = 0$? That would only give another reason why $\langle Tv, w_n \rangle = 0$

Thus $T: W^\perp \longrightarrow W^\perp$ and we can repeat the procedure

Step 2 Proceding inductively down i times:

$Tw_{n-i} = \lambda_{n-i} w_{n-i}$ choose $v \in (W_n \oplus \dots \oplus W_{n-i})^\perp$ $v \neq 0$ until $(W_n \oplus \dots \oplus W_1)^\perp = \{0\}$

Same arg gives $T: (W_n \oplus \dots \oplus W_{n-i})^\perp \longrightarrow (W_n \oplus \dots \oplus W_{n-i})^\perp$

Step 3 Process terminates when $(W_n \oplus \dots \oplus W_1)^\perp = \{0\}$ and we are left with

$$T(\{w_1 \dots w_n\}) = \{w_1 \dots w_n\} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

□

Corollary may be a better term

Thm 13.18 Additional Formulation of Spectral Thm

$T = \sum^m \lambda_i P_i$ where P_i is projection onto eigenspace of λ_i where $\{\lambda_1 \dots \lambda_m\}$ are the distinct EWs. So eigenspace may be greater than 1 dim

$P_1 + P_2 + \dots + P_m = Id$

$P_i P_j = 0$ for $i \neq j$

$$A = U \Lambda U^H = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} - u_1^H - \\ - u_2^H - \\ - u_3^H - \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \lambda_3 u_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} - u_1^H - \\ - u_2^H - \\ - u_3^H - \end{bmatrix} = \sum_{i=1}^n \lambda_i u_i u_i^H$$

$$Ax = \sum^n \lambda_i u_i u_i^H x = \sum^n \lambda_i u_i \langle x, u_i \rangle = \sum^n \lambda_i P_i(x)$$

Recall Prob 13.69 sheet 7
OG Proj onto subsp $W = \text{Span}\{u_1 \dots u_p\}$
$P_W: V \longrightarrow V_p$
$x \longmapsto \sum \langle x, u_i \rangle u_i$

But in fact the group the projections together for equal λs

Say $n = 5$ and $\lambda_1 = \lambda_2 = \lambda_3$ $\quad \sum^5 \lambda_i u_i \langle x, u_i \rangle = \lambda_1 \underbrace{(u_1 \langle \cdot, u_1 \rangle + u_2 \langle \cdot, u_2 \rangle + u_3 \langle \cdot, u_3 \rangle)}_{P_1(x)}(x) + \underbrace{\lambda_4 \langle \cdot, u_4 \rangle u_4}_{P_4(x)} + \lambda_5 P_5$

$= \lambda_1 P_1(x) + \lambda_2 P_2(x) + \lambda_3 P_3(x)$ re-indexing

▷ $P_1 + P_2 + P_3 = I$ because the set $\{u_1 \dots u_5\}$ is a basis so any $x = \underbrace{\langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + \langle x, u_3 \rangle u_3}_{P_1(x)} + \underbrace{\langle x, u_4 \rangle u_4}_{P_2(x)} + \underbrace{\langle x, u_5 \rangle u_5}_{P_3(x)}$

$I(x) = (P_1 + P_2 + P_3)(x)$

▷ $P_i P_j = 0$ because

P_i projects onto W_i but every $v \in W_i$ is $\perp$ to W_j (except 0)

Thus $P_j(P_i(x)) = 0$

□

Thm 13.17 Schur Triangularization Lemma

$$\left.\begin{array}{l} V \text{ fin dim vs over } \mathbb{C} \text{ with innerprod} \\ T: V \to V \text{ arb linear map} \end{array}\right\} \Rightarrow \begin{array}{l} \exists \text{ ON basis } \{u_1 \dots u_n\} \ni \\ T(\{u_1 \dots u_n\}) = \{u_1 \dots u_n\} \begin{bmatrix} \lambda_1 & a_{12} & \cdots & a_{1n} \\ & \lambda_2 & * & \vdots \\ 0 & & \ddots & \lambda_n \end{bmatrix} \end{array}$$

(As for any triang matrix, EWs are on main diag)

Pf. I find some of this pf questionable. I did another version in Strang LAAIA ch 5.6 writeup.

Proof is by induction on the dimension

▷ $\dim(V) = 1$. Thm trivially true since T is just scalar mult.

▷ $\dim(V) = n$. We assume thm is true for any vector space with $\dim = n-1$

That is for any subsp H of dim $n-1$ and any map $T: H \to H$

$$\exists \text{ O.N. basis } \{h_1 \dots h_{n-1}\} \ni T(h_i) = \{h_1 \dots h_{n-1}\} \begin{bmatrix} a_{i1} \\ \vdots \\ \lambda_i \\ 0 \\ 0 \end{bmatrix}$$

Since the field is $\mathbb{C}$, the fund thm of algebra guarantees $\exists$ at least one EW λ_1 and EV u_1 (take as a unit vector) $Tu_1 = \lambda u_1$

Take $W := \text{Span}\{u_1\}$ and then $V = W \oplus W^\perp$

$W^\perp$ is a perfectly fine $(n-1)$ dim vs and let $\pi: V \longrightarrow W^\perp$ be O.G. proj

Then $(\pi \circ T): W^\perp \longrightarrow W^\perp$ falls under the induction hypoth

[Here we do not have $W^\perp$ invariant under T^* or T as in Spectral Thm; instead we use $W^\perp$ invariant under $\pi \circ T$]

$$\Rightarrow \exists \text{ ON basis } \{u_2, \dots, u_n\} \ni T(u_i) = \{u_2 \dots u_n\} \begin{bmatrix} a_{i2} \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\{u_1, u_2 \dots u_n\}$ is ON basis for all of V

$$\begin{aligned} T(u_i) &= (\text{Piece in } W) + (\text{Piece in } W^\perp) \\ &= a_{i1} u_1 + \{u_2 \dots u_n\} \begin{bmatrix} a_{i2} \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \{u_1 u_2 \dots u_n\} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

□

Thm 13.13 P **Pos Def** (we only this for $P^* = P$ self-adj symm)

TFAE (i) $\langle Pu, u\rangle > 0 \ \forall u \neq 0$

P has 'Square root' $\longrightarrow$ (ii) $P = T^2$ for T nsing, self-adj $T = T^*$

(iii) $P = S^*S$ for nsing S (not uniq)

If we take out the parts circled in red, we get Pos semi def

Strang LAAIA ch 6 has different conds
- All EWs $\lambda > 0$
- all pivots pos
- subdets pos
- $A = R^T R$ where R has LI cols

(i) $\Rightarrow$ (ii) $P^* = P$ want: $P = T^2$ for T nsing, self-adj

$\langle Pu, u\rangle > 0$

$P = P^* \Rightarrow \exists \{u_1 \ldots u_n\}$ ON consisting of EVs of P, say $Pu_i = \lambda_i u_i$

By $\boxed{\text{Thm 13.8}}$ $\lambda_i \in \mathbb{R}$ and $\lambda_i > 0$ because $0 < \langle Pu_i, u_i\rangle = \lambda_i \underbrace{\langle u_i, u_i\rangle}_{1} = \lambda_i$

$\Rightarrow \sqrt{\lambda_i} \in \mathbb{R}$

Let T be defined on basis $\{u\}$ by $T(u_i) := \sqrt{\lambda_i}\, u_i$

$$T^2 u_i = T(Tu_i) = T(\sqrt{\lambda_i}\, u_i) = \sqrt{\lambda_i}\, Tu_i = \sqrt{\lambda_i}\sqrt{\lambda_i}\, u_i = \lambda_i u_i = P(u_i)$$

Thus T^2 and P agree on a basis for V, by linearity $P = T^2$ on V

T is the uniq pos operator such that $P = T^2$

T is self-adj because it has matrix rep

$$T(\{u\}) = \{u\}\begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$$

(ii) $\Rightarrow$ (iii) $P = T^2 \Rightarrow P = S^*S$

$P = T^2 = TT = T^*T$ so just take $S = T$

(iii) $\Rightarrow$ (i) $P = S^*S$

$\Rightarrow P^* = (S^*S)^* = S^*S^{**} = S^*S = P$ self-adj

Furthermore $\langle Pu, u\rangle = \langle S^*Su, u\rangle = \langle Su, S^{**}u\rangle$

$= \langle Su, Su\rangle = \|Su\|^2 \geq 0$

and S nsing $\Rightarrow Su \neq 0$ if $u \neq 0$

$\square$