Inner Prod Spaces

Let V be a vector sp. over $\mathbb{R}$ or $\mathbb{C}$, finite dim.

Def $\langle \cdot,\cdot \rangle : V \times V \longrightarrow K$ (Field $K = \mathbb{R}$ or $\mathbb{C}$) is an *Inner Product* if

1. Linear in 1st position
2. $\mathbb{C}$-conj symmetric: $\langle u,v\rangle = \overline{\langle v,u\rangle}$
3. Pos definite: $\begin{cases} \langle u,u\rangle > 0 & \text{if } u \neq 0 \\ \langle u,u\rangle = 0 & \text{if } u = 0 \end{cases}$

(1) and (2) mean $\langle u, ax+y\rangle = \overline{\langle ax+y, u\rangle} = \bar{a}\overline{\langle x,u\rangle} + \overline{\langle y,u\rangle} = \bar{a}\langle u,x\rangle + \langle u,y\rangle$

we might call this phenomena "$\mathbb{C}$-conj linear" in 2nd position.

Why require this weird thing? One reason is to preserve pos definiteness for $\mathbb{C}$ scalars.

Suppose IP was just linear in both positions:

$\langle iu, iu\rangle = i\langle u, iu\rangle = i^2\langle u,u\rangle = -\langle u,u\rangle$ negative so cond (3) violated.

In language of ch 12, IP is pos def, bilinear, symm form if $K=\mathbb{R}$

pos def, Hermetian form if $K=\mathbb{C}$

Define norm $\|u\| := {}^{+}\sqrt{\langle u,u\rangle}$ 　 Distance fcn $d(u,v) := \|u-v\|$

example 13.2 "dot prod" in $\mathbb{C}^n$ is a little wierd. Lets take $n=2$ to save writing

$\vec{u} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$u \cdot v = \langle z_1\hat{i} + z_2\hat{j}, w_1\hat{i} + w_2\hat{j}\rangle = \langle z_1\hat{i}, w_1\hat{i}+w_2\hat{j}\rangle + \langle z_2\hat{j}, w_1\hat{i}+w_2\hat{j}\rangle$$

$$= z_1\bar{w}_1\underbrace{\langle\hat{i},\hat{i}\rangle}_{1} + z_1\bar{w}_2\underbrace{\langle\hat{i},\hat{j}\rangle}_{0} + z_2\bar{w}_1\underbrace{\langle\hat{j},\hat{i}\rangle}_{0} + z_2\bar{w}_2\underbrace{\langle\hat{j},\hat{j}\rangle}_{1}$$

$$= z_1\bar{w}_1 + z_2\bar{w}_2$$

I think I give this discussion better in my writeup of Cheney Apl M I ch 3

What if we naively tried it like the regular dot prod, but with $\mathbb{C}$ values?

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1i \\ a_2+b_2i \end{bmatrix} \cdot \begin{bmatrix} c_1+d_1i \\ c_2+d_2i \end{bmatrix} = (a_1+b_1i)(c_1+d_1i) + (a_2+b_2i)(c_2+d_2i)$$

$$= (a_1c_1 - b_1d_1) + i(a_1d_1 + b_1c_1) + (a_2c_2 - b_2d_2) + i(a_2d_2 + b_2c_2)$$

$$= (a_1c_1 - b_1d_1 + a_2c_2 - b_2d_2) + i(a_1d_1 + b_1c_1 + a_2d_2 + b_2c_2)$$

$$\neq z_1\bar{w}_1 + z_2\bar{w}_2$$

$$= (a_1c_1 + b_1d_1 + a_2c_2 + b_2d_2) - i(a_1d_1 - b_1c_1 + a_2d_2 - b_2c_2)$$

example 13.3

$V = m \times n$ matrices over K

$$\langle A,B\rangle := \begin{cases} \mathrm{Tr}(B^TA) & K=\mathbb{R} \\ \mathrm{Tr}(B^HA) & K=\mathbb{C} \end{cases}$$

Note also from p. 292 $u,v \in \mathbb{R}^n$

$\langle u,v\rangle := u^TAv$

where A is symm, pos def matrix

example 13.4

$V = C[a,b]$

$$\langle f,g\rangle := \begin{cases} \int_a^b f(t)g(t)\,dt & f,g\ \mathbb{R}\text{ valued} \\ \int_a^b f(t)\overline{g(t)}\,dt & f,g\ \mathbb{C}\text{ valued} \end{cases}$$

Thm 13.1 Cauchy-Schwartz Ineq $\boxed{|\langle u,v\rangle| \leq \|u\|\,\|v\|}$

pf. see Strang LAHIA ch 3 write-up or Cheney Apl M I ch 3 writeup

Def u, v **Orthogonal** (OG) if $\langle u,v\rangle = 0$

Def $W^\perp := \left\{ \begin{smallmatrix}\text{all}\\ v\in V\end{smallmatrix} \;\middle|\; \langle v,w\rangle = 0 \ \forall w \in W \right\}$

Claim $W^\perp$ is a Subsp of V

pf: $0 \in W^\perp$ because $\langle 0,w\rangle = 0 \ \forall w\in W$ ✓

$W^\perp$ clsd under vector addition?

Let $u, v \in W^\perp$ $\quad \langle \alpha u + \beta v, w\rangle \overset{?}{=} 0$

$= \alpha\langle u,w\rangle + \beta\langle v,w\rangle$

$= \alpha\cdot 0 + \beta\cdot 0 = 0$ ✓

We want to prove **Thm 13.2** $V = W \oplus W^\perp$

But the pf uses the Gram-Schmidt process and projections

So first we must discuss that.

(ch 4 p.69) Given subspaces U, W of V we can form the subsp $U+W := \left\{ \begin{smallmatrix}\text{all}\\ u+w\end{smallmatrix} \;\middle|\; \begin{smallmatrix}u\in U\\ w\in W\end{smallmatrix}\right\}$

Def **Direct Sum** $V = U \oplus W$ if any $v\in V$ is $v = u+w$ for uniq u, w

$$\boxed{\textbf{Thm 4.9}\quad V = U\oplus W \iff \begin{cases} V = U+W & (i)\\ U\cap W = \{0\} & (ii)\end{cases}}$$

($\Rightarrow$) (i) is obvious. For (ii) if $\exists a \in U\cap W$, $a\neq 0$ then $v = u+w$ and also $v = u'+w'$ where $u' = u-a$ $\quad w' = w+a$ $\Rightarrow\!\!\!\!\!/$ not uniq

($\Leftarrow$) choose $v\in V$ so we have $v = u+w$

if also $v = u'+w'$ $\Rightarrow u+w = u'+w' \Rightarrow \underbrace{u-u'}_{\in U} = \underbrace{w'-w}_{\in W}$

then $u'-u \in U\cap W = \{0\} \Rightarrow u'-u = 0 \Rightarrow u = u'$ and likewise $w = w'$ $\square$

(ch 10 p.243) Let $W_1, \dots, W_r$ be subsps of V and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$

thus any $v = w_1 + w_2 + \dots + w_r$ uniquely.

Define **Projection** $P_k : V \longrightarrow W_k$, $\quad v \longmapsto w_k$

we want to show ① P_k is linear and ② $P_k^2 = P_k$

① Lets write $v = w_1^v + \dots + w_r^v$ and $u = w_1^u + \dots + w_r^u$

Since each W_i is a subsp, $w_i^u + w_i^v \in W_i$ and $\lambda w_i^v \in W_i$ for scalar λ

Then $P_k(\lambda v + u) = P_k\left(\sum_i \lambda w_i^v + w_i^u\right) = \lambda w_k^v + w_k^u = \lambda P_k(v) + P_k(u)$

② $P_k^2(v) = P_k(P_k v) = P_k(w_k) = w_k = P_k(v)$ so $P^2 = P$ $\square$

(Prob 10.34) show $\left.\begin{matrix} P: V \xrightarrow{\text{linear}} V \\ P^2 = P\end{matrix}\right\} \Rightarrow$

ⓐ $P(u) = u \ \forall u \in Im(P)$ thus $P|_{Im(P)} = Id$

ⓑ $V = Image(P) \oplus ker(P)$

ⓒ $P: V \longrightarrow Im(P)$ this is a priori obvious!

pf

ⓐ $u \in Im(P) \Rightarrow \exists\, V_u \ni P(V_u) = u$ so $P(u) = P(P(V_u)) = P^2(V_u) \underset{\text{hypoth}}{=} P(V_u) = u$

ⓑ Let $v\in V$ Trick: $v = \underset{Im P}{Pv} + \underset{ker P}{(v - Pv)}$ observe $P(v-Pv) = Pv - P^2v = Pv - Pv = 0$ so $(v-Pv)\in ker$

Now to complete the application of thm 4.9 above, we must show $Im(P)\cap ker(P) = \{0\}$

Let $w \in Im(P)\cap ker(P)$ $\quad Im(P) \Rightarrow Pw = w \quad ker(P) \Rightarrow Pw = 0$ can only mean $w = 0$

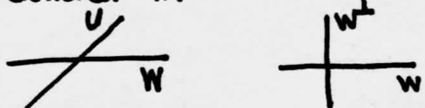
$\Rightarrow V = Im(P) \oplus ker(P)$

$\square$

p.281 Let W subsp V and let $P: V \to W$ be a proj. (so we know $V = \mathrm{Im}(P) \oplus \ker(P) = W \oplus \ker(P)$)

If $\ker(P) = W^{\perp}$ the P is <u>O.G. Proj</u>

consider $\mathbb{R}^2$



$\mathbb{R}^2 = W \oplus U$ $\qquad$ $\mathbb{R}^2 = W \oplus W^{\perp}$

A non-OG proj is called an <u>oblique proj</u>. Here is an example: (wikipedia)

$P = \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix}$ $\quad$ Image$(P) = \mathrm{Span}\{e_2\}$ (so my picture should have had $W = y$ axis)

$\ker(P) = \mathrm{Span}\left\{ \begin{bmatrix} -1/\alpha \\ 1 \end{bmatrix} \right\}$ take $\alpha = 1$

$P^2 = \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix} = P$

If $\alpha = 0$ then O.G. Proj

$P^T = P$ (justified later)

p.308 **13.66** Let $w \in V$, $w \neq 0$ $\quad W = \mathrm{Span}\{w\}$

Let $P_w : V \to W$ $\quad$ Show P_w is an O.G. Proj

$v \longmapsto \dfrac{\langle v, w\rangle}{\langle w, w\rangle} w$ $\quad$ [This is Proj of v along w]

(1) P is linear since $\langle \cdot, w\rangle$ is linear

(2) $P^2 = P$ because $P^2(v) = P\left(\dfrac{\langle v, w\rangle w}{\langle w, w\rangle}\right) = \dfrac{\left\langle \dfrac{\langle v, w\rangle w}{\langle w, w\rangle}, w\right\rangle w}{\langle w, w\rangle}$

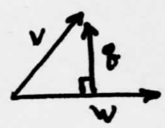
$= \dfrac{\langle v, w\rangle \cancel{\langle w, w\rangle}}{\cancel{\langle w, w\rangle}\langle w, w\rangle} w = P(v)$

(3) $\ker(P) = \left\{ \text{all } x \mid \langle x, w\rangle = 0 \right\} = (\mathrm{Span}\{w\})^{\perp} = W^{\perp}$ □

<u>Gram-Schmidt Precursor:</u> let $q := v - P_w(v)$

<u>claim</u>: $q \perp w$ that is q O.G. to W; $\langle q, w\rangle = 0$

$\langle q, w\rangle = \left\langle v - \dfrac{\langle v, w\rangle}{\langle w, w\rangle} w, w\right\rangle = \langle v, w\rangle - \dfrac{\langle v, w\rangle}{\cancel{\langle w, w\rangle}} \cancel{\langle w, w\rangle} = 0$ ✓



13.67 Example of O.G. Proj in $\mathbb{C}^2$

Let $v = \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$ $\quad w = \begin{bmatrix} 2-i \\ 3 \end{bmatrix}$ $\quad$ Find proj of $\vec{v}$ along $\vec{w}$

$P_w(\vec{v}) = \dfrac{\langle v, w\rangle}{\langle w, w\rangle} \vec{w}$

$\langle \vec{v}, \vec{w}\rangle = v_1 \bar{w}_1 + v_2 \bar{w}_2 = (1-i)(2+i) + (2+3i)3$

$= 2 + i - 2i - i^2 + 6 + 9i$

$= 9 + 8i$

$\langle w, w\rangle = (2-i)(2+i) + 3 \cdot 3 = 4 + 1 + 9 = 14$

Then $P_w(v) = \dfrac{9+8i}{14} \begin{bmatrix} 2-i \\ 3 \end{bmatrix}$ $\quad$ If <u>for some reason</u> we didn't want a $\mathbb{C}$ scalar we could mult and get

$\dfrac{1}{14}\begin{bmatrix} 26+7i \\ 27+24i \end{bmatrix}$ $\quad$ I don't know why the ans on p.312 has $\dfrac{1}{\sqrt{14}}$

This is <u>NOT</u> a <u>Real</u> scalar multiple of $\vec{w}$ — so my intuition of "colinear" does not apply!

<u>Lemma 13.3</u> Let $\{u_1 \dots u_r\}$ O.N. vectors $\Rightarrow$ $\{u_1 \dots u_r\}$ is LI

$W := v - \underbrace{\sum_{i=1}^{r} \langle v, u_i\rangle u_i}_{P_U(v) \text{ O.G. Proj as we see later}}$ is O.G. to each u_i

13.69 on sheet 7

pf. $\$\ \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r = 0$ we must show this implies each $\alpha_i = 0$ for LI

Fix i and take IP of both sides with u_i:

$\langle \alpha_1 u_1 + \dots + \alpha_r u_r, u_i\rangle = \langle 0, u_i\rangle = 0$

$\alpha_1 \underbrace{\langle u_1, u_i\rangle}_{0} + \dots + \alpha_i \underbrace{\langle u_i, u_i\rangle}_{1} + \dots + \alpha_r \underbrace{\langle u_r, u_r\rangle}_{0} \Rightarrow \alpha_i \cdot 1 = 0$ and this holds for any $i = 1, \dots, r$ □

Now show w is O.G. to all u_i ⟶

Lemma 13.3 pf cont'd $\quad w = v - \sum_{j=1}^{r} \langle v, u_j \rangle u_j$ Again fix any $i \in \{1,\dots,r\}$ and IP both sides $\langle \cdot, u_i \rangle$:

$$\langle w, u_i \rangle = \left\langle v - \sum \langle v, u_j \rangle u_j , u_i \right\rangle$$
$$= \langle v, u_i \rangle - \sum \langle v, u_j \rangle \underbrace{\langle u_j, u_i \rangle}_{\delta_{ij}} = \langle v, u_i \rangle - \langle v, u_i \rangle \cancel{\langle u_i, u_i \rangle}^{1} = 0 \quad \square$$

Thm 13.4 Gram-Schmidt

[changing notation to parallel Strang LAANA ch 3.4 my writeup]

$\{a_1 \dots a_n\}$ arb basis for V $\Rightarrow$ $\cdot$ $\exists$ O.N. basis $\{\hat{q}_1, \dots \hat{q}_n\}$ constructed by process below. $\hat{q}_j$ means unit vector

$\cdot$ $\{a\} = \{\hat{q}\} R$ $\quad R$ is upper triang matrix

Pf Define $q_1 := a_1$ then $\hat{q}_1 = \frac{1}{\|q_1\|} q_1$

$q_2 := a_2 - \langle a_2, \hat{q}_1 \rangle \hat{q}_1 \qquad \hat{q}_2 = \frac{1}{\|q_2\|} q_2$

$q_3 := a_3 - \langle a_3, \hat{q}_1 \rangle \hat{q}_1 - \langle a_3, \hat{q}_2 \rangle \hat{q}_2$

$\vdots$

$q_{i+1} = a_{i+1} - \langle a_{i+1}, \hat{q}_1 \rangle \hat{q}_1 - \dots - \langle a_{i+1}, \hat{q}_i \rangle \hat{q}_i$

By Lemma 13.3, at each step, q_{i+1} is O.G. to span$\{q_1 \dots q_i\}$

How do we know $q_{i+1} \neq 0$ for any $i+1 \leq n$? Because $a_{i+1} \notin \text{Span}\{a_1 \dots a_i\} = \text{Span}\{q_1 \dots q_i\}$

Thus we get the basis $\{\hat{q}_1 \dots \hat{q}_n\}$ of O.N. vectors we seek.

To show the COB formula $\{a_1 \dots a_n\} = \{\hat{q}_1 \dots \hat{q}_n\} R$ let's note

Fix j $\quad q_j = a_j - \sum_{k=1}^{j-1} \langle a_j, \hat{q}_k \rangle \hat{q}_k$

(★) re-arranging $\quad a_j = \sum^{j-1} \langle a_j, \hat{q}_k \rangle \hat{q}_k + q_j \qquad q_j = \|q_j\| \hat{q}_j$

Now show $\|q_j\| = \langle a_j, \hat{q}_j \rangle$ by taking IP of both sides $\langle \cdot, \hat{q}_j \rangle$ of above eq:

$$\langle a_j, \hat{q}_j \rangle = \sum^{j-1} \langle a_j, \hat{q}_k \rangle \cancel{\langle \hat{q}_k, \hat{q}_j \rangle}^{0} + \underbrace{\langle q_j, \hat{q}_j \rangle}_{= \langle \|q_j\| \hat{q}_j, \hat{q}_j \rangle = \|q_j\| \, \|\hat{q}_j\|^2 = \|q_j\|}$$

so (★) becomes:

$$a_j = \langle a_j, \hat{q}_1 \rangle \hat{q}_1 + \langle a_j, \hat{q}_2 \rangle \hat{q}_2 + \dots + \langle a_j, \hat{q}_j \rangle \hat{q}_j$$

$$\begin{aligned} a_1 &= \langle a_1, \hat{q}_1 \rangle \hat{q}_1 \\ a_2 &= \langle a_2, \hat{q}_1 \rangle \hat{q}_1 + \langle a_2, \hat{q}_2 \rangle \hat{q}_2 \\ a_3 &= \langle a_3, \hat{q}_1 \rangle \hat{q}_1 + \langle a_3, \hat{q}_2 \rangle \hat{q}_2 + \langle a_3, \hat{q}_3 \rangle \hat{q}_3 \end{aligned} \Rightarrow \{a_1\ a_2\ a_3\} = \{\hat{q}_1\ \hat{q}_2\ \hat{q}_3\} \underbrace{\begin{bmatrix} \langle a_1, \hat{q}_1 \rangle & \langle a_2, \hat{q}_1 \rangle & \langle a_3, \hat{q}_1 \rangle \\ 0 & \langle a_2, \hat{q}_2 \rangle & \langle a_3, \hat{q}_2 \rangle \\ 0 & 0 & \langle a_3, \hat{q}_3 \rangle \end{bmatrix}}_{R}$$

Note that if the vectors a_i are the col vectors of matrix A then this yields the QR factorization: $A = QR$ $\quad \square$

COR

We also see that, at any step p, the ON set $\{\hat{q}_1 \dots \hat{q}_p\}$ has same span as $\{a_1 \dots a_p\}$.

13.10 Example of Gram-Schmidt in $\mathbb{C}^3$: Let $a_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ 2 \\ 1-i \end{bmatrix}$

we know $\hat{q}_1 = \frac{1}{\|a_1\|} a_1$ $\quad \|a_1\|^2 = \langle a_1, a_1 \rangle = a_1^H a_1$ (in Strang's notation; for $\mathbb{C}^n$ I have $\langle \cdot, y \rangle = y^H(\cdot)$)

$$\Rightarrow \boxed{\hat{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}} \qquad = [1 \ -i \ 0] \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = 1 - i^2 = 2$$

$q_2 := a_2 - \langle a_2, \hat{q}_1 \rangle \hat{q}_1$ $\quad$ so compute $\langle a_2, \hat{q}_1 \rangle = \hat{q}_1^H a_2 = \frac{1}{\sqrt{2}} [1 \ -i \ 0] \begin{bmatrix} 1 \\ 2 \\ 1-i \end{bmatrix} = \frac{1}{\sqrt{2}}(1-2i)$

$$= \begin{bmatrix} 1 \\ 2 \\ 1-i \end{bmatrix} - \frac{1}{\sqrt{2}}(1-2i) \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1-i \end{bmatrix} - \frac{(1-2i)}{2} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + i \\ 1 - \frac{1}{2} i \\ 1 - i \end{bmatrix}$$

Then $\|q_2\|^2 = \langle q_2, q_2 \rangle = q_2^H q_2 = \left[\frac{1}{2} - i \quad 1 + \frac{1}{2} i \quad 1 + i\right] \begin{bmatrix} \frac{1}{2} + i \\ 1 - \frac{1}{2} i \\ 1 - i \end{bmatrix} = (\frac{1}{2})^2 + 1 + 1 + (\frac{1}{2})^2 + 1 + 1 = 9/2$

$$\boxed{\hat{q}_2 = \frac{1}{3/\sqrt{2}} \begin{bmatrix} \frac{1}{2} + i \\ 1 - \frac{1}{2} i \\ 1 - i \end{bmatrix}} = \text{book's ans } \frac{1}{\sqrt{18}} \begin{bmatrix} 1 + 2i \\ 2 - i \\ 2 - 2i \end{bmatrix} \qquad \|q_2\| = \sqrt{9/2} = 3/\sqrt{2}$$

13.12 Lemma: W subsp V $\Rightarrow \exists$ an O.N. Basis for W which is part of an ON basis for V

Pf. we know W has some basis $\{w_1 \dots w_r\}$ and we can extend it to a basis $\{w_1, \dots, w_r, v_{r+1}, \dots v_n\}$ for V. By Gram-Schmidt, we can make an O.N. basis $\{\hat{q}_1 \dots \hat{q}_r\}$ that spans W, and by continuing the process, $\{\hat{q}_1 \dots \hat{q}_r, \hat{q}_{r+1}, \dots \hat{q}_n\}$ ON basis for V $\square$

13.13 Thm 13.2 $\quad W$ subsp $V \Rightarrow V = W \oplus W^\perp$

[Dropping the hats $\hat{q}$]

Pf. From the 13.12 Lemma, we have basis $\{q_1, \dots q_r, \dots q_n\}$ where $\{q_1 \dots q_r\}$ spans W

By Thm 4.9 (given on sheet 2 here) $V = W \oplus U$ iff (i) Every $v = w + u$
(ii) $W \cap U = \{0\}$

we have ON basis, so for any v, $v = \underbrace{\sum_{i=1}^r \alpha_i q_i}_{w \in W} + \underbrace{\sum_{j=r+1}^n \alpha_j q_j}_{u \in U}$ $\quad$ Let $U := \text{Span}\{q_{r+1} \dots q_n\}$ and in fact, this decomp is uniq

To show $W \cap U = \{0\}$, let $h \in W \cap U$. Then LC of $\{q_i\}_{1 \dots r}$ and LC of $\{q_j\}_{r+1 \dots n}$

$$\Rightarrow \langle h, h \rangle = 0 \Rightarrow h = 0$$

$\square$

13.14 Let W subsp V. show $(W^\perp)^\perp = W$ for V fin dim (It is always true that $W^{\perp\perp} \subseteq W$ and for an ∞ dim topological v.s. $W^{\perp\perp} = \overline{W}$ closure)

we know $W^\perp = \{ \text{all } x \mid \langle x, w \rangle = 0 \ \forall w \}$

$(W^\perp)^\perp = \{ \text{all } y \mid \langle y, x \rangle = 0 \ \forall x \in W^\perp \}$

choose any $w \in W$. Then $\langle w, x \rangle = 0 \ \forall x \in W^\perp$. Thus this x qualifies as a "y" belonging to $W^{\perp\perp}$
$\Rightarrow w \in W^{\perp\perp}$ and since this holds for any w $\Rightarrow W \subseteq W^{\perp\perp}$

Now observe: W is subsp $\Rightarrow V = W \oplus W^\perp$ by Thm 13.2 $\quad \Rightarrow \dim W = \dim V - \dim W^\perp$

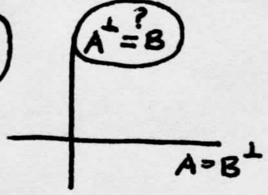
And $W^\perp$ is also a subsp $\Longrightarrow V = W^\perp + W^{\perp\perp}$ " $\quad \Rightarrow \dim W^{\perp\perp} = \dim V - \dim W^\perp$

$\Rightarrow \dim W = \dim W^{\perp\perp}$ and $W \subseteq W^{\perp\perp}$

$\Rightarrow W = W^{\perp\perp}$ $\square$

ASIDE Over the years I have made use of this result: (which seems 'obvious')

$$\boxed{A = B^\perp \Rightarrow A^\perp = B}$$



Essentially taking $\perp$ of both sides:
$A = B^\perp \overset{?}{\Rightarrow} A^\perp = B^{\perp\perp} = B$
is such an operation even well-defined?

Pf. Let A subsp V

$A = B^\perp$ and $B^\perp := \{ \text{all } x \in V \mid \langle x, b \rangle = 0 \ \forall b \}$

in other words $\quad [\langle a, b \rangle = 0 \ \forall b \in B] \ \forall a \in A$ ⊛

And $A^\perp := \{ \text{all } y \in V \mid \langle y, a \rangle = 0 \ \forall a \in A \}$ ← Is this B?

We can show $B \subseteq A^\perp$:

⊛ gives us: $\langle a, b \rangle = 0 \Rightarrow \overline{\langle a, b \rangle} = \bar{0} = 0 \Rightarrow [\langle b, a \rangle = 0 \ \forall a \in A] \ \forall b \in B$

$\overline{\langle a, b \rangle} = \langle b, a \rangle$

$\Rightarrow B \subseteq A^\perp$

But how to show $B \supseteq A^\perp$? I never defined B

Lets show $\dim B = \dim A^\perp$

By Thm 13.2 $\quad V = A \oplus A^\perp$ and $A = B^\perp$ so $B^\perp$ is a subsp and $V = B^\perp \oplus A^\perp$

$\dim A^\perp = \dim V - \dim B^\perp$

Likewise $V = B^\perp \oplus \underbrace{B^{\perp\perp}}_{B} \Rightarrow \dim B = \dim V - \dim B^\perp \Rightarrow \dim A^\perp = \dim B$

I think this pf relies on V fin dim and A, B both subsp $\quad \Rightarrow B = A^T$ □

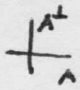
ASIDE 2: A is also a set and V is the universe

It is true in a set theoretic sense $A = B^c \Rightarrow A^c = B$ but $A^c \neq A^\perp$!!

Pf. $A = B^c$ means $z \in A \Rightarrow z \notin B$

logically negate both sides: $z \notin A \Leftarrow z \in B$

But this Satement is $A^c = B$ $\quad$ QED



but A^c is everything in plane not on x axis.

$A \cup A^c = V$

$A \cap A^c = \emptyset$ not $\{0\}$

▷ Before continuing with the problems, lets do example 13.9 p. 282

This is another pf of Rank + Nullity Thm

This is revisiting Ch 11 p. 252 but that discussion was very unclear.

Consider $A: \mathbb{R}^n \underset{A^T}{\overset{}{\rightleftarrows}} \mathbb{R}^m$ and homog sys $Ax = 0$

soln's are $x \in \ker(A)$

$Ax = 0$ iff $x \in \ker(A)$

$\Rightarrow x^T A^T = 0^T$ so for any $y \in \mathbb{R}^m$ $\quad x^T A^T y = 0^T y = 0$ and $\mathrm{Im}(A^T) = \{ \text{all } A^T y \mid y \in \mathbb{R}^m \}$

$\Rightarrow x \perp \mathrm{Im}(A^T)$ for any $x \in \ker(A)$

$\Rightarrow \ker A = [\mathrm{Im}(A^T)]^\perp$ or, by my above discussion, $(\ker A)^\perp = \mathrm{Im}(A^T)$

step 2 $\quad \mathbb{R}^n = \ker(A) \oplus \ker(A)^\perp$

$\qquad = \ker(A) \oplus \mathrm{Im}(A^T)$

[Strang LAAIA Fund Thm of LA II]

$\dim \mathbb{R}^n = \dim(\ker(A)) + \dim(\mathrm{Im}(A^T))$

$n = \text{Nullity}(A) + \text{rank}(A^T)$ $\quad$ and we know $\text{rank}(A) = \text{rank}(A^T)$

$$\boxed{n = \text{nullity}(A) + \text{rank}(A)}$$

□

13.15 Let $\{e_1 \dots e_n\}$ be O.N. basis for V.

(i) show: any $u \in V$ can be expressed $\boxed{u = \langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2 + \dots + \langle u, e_n\rangle e_n}$

(This is ch 11 Thm 11.3 because fcnals $\varphi_i := \langle \cdot, e_i\rangle$ are a dual basis for $\{e\}$)

But to show it directly here: $u = \sum \mu^i e_i$ and $\langle u, e_j\rangle = \sum \mu^i \underbrace{\langle e_i, e_j\rangle}_{\delta_{ij}} = \mu^j$ for each j □

(ii) $\boxed{\left\langle \sum_i \alpha_i e_i, \sum_j \beta_j e_j \right\rangle = \sum \alpha_i \overline{\beta_i}}$ [This was used in example 13.2 in sheet 1]

$$\left\langle \sum_i \alpha_i e_i, \sum_j \beta_j e_j \right\rangle = \sum_i \alpha_i \left\langle e_i, \sum_j \beta_j e_j \right\rangle = \sum_i \alpha_i \overline{\left\langle \sum \beta_j e_j, e_i \right\rangle}$$

$$= \sum_i \alpha_i \sum_j \overline{\beta_j} \overbrace{\langle e_i, e_j\rangle}^{\delta_{ij}} = \sum \alpha_i \overline{\beta_i} \quad \square$$

(iii) $\boxed{\langle u, v\rangle = \langle u, e_1\rangle \overline{\langle v, e_1\rangle} + \dots + \langle u, e_n\rangle \overline{\langle v, e_n\rangle}}$

This follows directly from (i) and (ii)

(iv) $\boxed{\text{Linear map } T: V \to V \text{ has matrix rep } A. \text{ show } a_{ij} = \langle Te_j, e_i\rangle.}$

Follow ch 11 sheet 7 (which followed ch 7 p. 156)

For any vector v, $T(v) = T(\sum v^j e_j) = \sum v^j T(e_j)$ and Te_j is LC of $\{e\}$

$$\Rightarrow T(e_j) = \{e_1 \dots e_n\} \underbrace{\begin{bmatrix} a^1_j \\ \vdots \\ a^n_j \end{bmatrix}}_{\text{col } j \text{ of } A} = \sum_k a^k_j e_k$$

Fix i

To isolate a^i_j, apply $\langle \cdot, e_i\rangle$ to both sides, $\langle Te_j, e_i\rangle = \sum_k a^k_j \overbrace{\langle e_k, e_i\rangle}^{\delta_{ki}} = a^i_j = a_{ij}$ □

13.69 Let $\{u_1 \dots u_r\}$ be O.N. basis for $W \subseteq V$.

$\boxed{\begin{array}{l} \text{Define } P_W: V \longrightarrow V \\ \quad v \longmapsto \sum^r \langle v, u_i\rangle u_i \\ \text{show } P_W \text{ is O.G. Proj onto } W \end{array}}$

we must show
- P is linear
- $P^2 = P$
- $\ker(P) = W^\perp$

► P is linear: $P(\lambda v + u) = \sum \langle \lambda v + u, u_i\rangle u_i = \sum (\lambda \langle v, u_i\rangle + \langle u, u_i\rangle) u_i$

$= \lambda \sum \langle v, u_i\rangle u_i + \sum \langle u, u_i\rangle u_i$

$= \lambda P(v) + P(u)$ ✓

► $P^2 = P$: $P^2(v) = P(\sum \langle v, u_i\rangle u_i) \overset{P \text{ linear}}{=} \sum \langle v, u_i\rangle P(u_i) = \sum_i \langle v, u_i\rangle \sum_j \underbrace{\langle u_i, u_j\rangle}_{\delta_{ij}} u_j$

$= \sum_i \langle v, u_i\rangle u_i = P(v)$ ✓

► $\ker(P) = W^\perp$

$\ker(P) \subseteq W^\perp$: Let $Px = 0$ so $x \in \ker(P)$

Then $\sum \langle x, u_i\rangle u_i = 0$ and since $\{u_i\}$ is LI, a LC $= 0$ iff all coeffs are 0

$\Rightarrow$ each $\langle x, u_i\rangle = 0 \Rightarrow x \perp u_i \ \forall i \Rightarrow x \in W^\perp \Rightarrow \ker(P) \subseteq W^\perp$

$\ker(P) \supseteq W^\perp$: Let $y \in W^\perp \Rightarrow \langle y, w\rangle = 0 \ \forall w \in W$

in particular, $\langle y, u_i\rangle = 0 \ \forall u_i$

$\Rightarrow P(y) = 0 \Rightarrow y \in \ker(P)$

□

Linear Fcnals and Adjoints

schuams calls this $\hat{v}$

For any $v \in V$, we can define a fcnal $\varphi_v : V \longrightarrow K$ ← always it is put in the back (so φ_v is linear in x)

$x \longmapsto \langle x, v \rangle$

So with an IP, we can map all elts of V to fcnals $\varphi \in V^*$

We can also map the other way!

$\boxed{\text{Thm 13.5 "Baby Riesz Rep Thm"} \quad \left.\begin{array}{l} V \text{ fin dim IP space} \\ \varphi \in V^* \end{array}\right\} \Rightarrow \exists!\, u_\varphi \in V \ni \varphi(\cdot) = \langle \cdot, u_\varphi \rangle}$

Pf. Given φ, we will exhibit the corresponding u_φ. Let V have O.N. basis $\{e_1 \dots e_n\}$

Create vector $u_\varphi := \sum^n \overline{\varphi(e_i)}\, e_i$

observe $\langle e_j, u_\varphi \rangle = \langle e_j, \sum_i \overline{\varphi(e_i)} e_i \rangle = \overline{\langle \sum \varphi(e_i) e_i, e_j \rangle}$

$= \sum_i \overline{\overline{\varphi(e_i)}} \underbrace{\langle e_i, e_j \rangle}_{\delta_{ij}} = \varphi(e_j)$

So for an arb vector $x = \sum \alpha^i e_i$

$\varphi(x) = \varphi(\sum \alpha^i e_i) = \sum \alpha^i \varphi(e_i)$

and $\langle x, u_\varphi \rangle = \langle \sum \alpha^i e_i, u_\varphi \rangle = \sum \alpha^i \langle e_i, \sum_j \overline{\varphi(e_j)} e_j \rangle = \sum \alpha^i \varphi(e_i)$

SAME

□

Is this u_φ uniq?

we assume $\varphi \neq 0$

\$ $\varphi(v) = \langle v, u_\varphi \rangle$ and $\varphi(v) = \langle v, w_\varphi \rangle$ for $u_\varphi \neq w_\varphi$

$\Rightarrow \langle v, u_\varphi \rangle = \langle v, w_\varphi \rangle$ for any v

$\Rightarrow \langle v, u_\varphi - w_\varphi \rangle = 0$

Now choose $v = u_\varphi - w_\varphi \Rightarrow \langle u_\varphi - w_\varphi, u_\varphi - w_\varphi \rangle = 0$

$\Rightarrow u_\varphi - w_\varphi = 0$ or $u_\varphi = w_\varphi$ $\Rightarrow\Leftarrow$ □

Prob 13.45 Counterexample in ∞ dim

Let V = polys over $\mathbb{R}$ – can go to infinitely high degree. [This space is counter-ex to many things]

$\langle p, q \rangle = \int_0^1 p(t) q(t)\, dt$

Consider the fcnal $\delta_0 : V \longrightarrow \mathbb{R}$

$p \longmapsto p(0)$ evaluation at 0 – this is nonzero for any poly with const term.

Suppose $\exists$ some fixed poly $h \ni \delta_0(p) = \langle p, h \rangle \; \forall p \in V$

For any poly p, define another poly q_p by $q_p(t) = t\, p(t)$ ← in effect, raising every exponent $t^n \to t^{n+1}$

So for any p, $\delta_0(q_p) = 0$

that is $\int_0^1 t\, p(t) h(t)\, dt = 0$

[This is why we need ∞ deg of polys, or else we could leave the set V]

Now choose $p(t) := t\, h(t)$

Then $\int_0^1 t^2 h^2(t)\, dt = 0$ and since h is cont and integrand always ≥ 0

this result is only possible if $h \equiv 0$

$\Rightarrow \delta_0(p) = \langle p, 0 \rangle = 0 \;\forall p$

$\Rightarrow\Leftarrow$ So no h exists where $\delta_0(\cdot) = \langle \cdot, h \rangle$

□

Let's compare Reisz Rep Thm in Hilbert sp X

Cheney Apl M ch 3

Thm (Riesz Rep Thm)

Linear fcnal $\varphi : X \to (\mathbb{R} \text{ or } \mathbb{C})$ is Cont (Bdd Linear) $\Rightarrow \exists!\ v_\varphi \in X \ni \varphi(\cdot) = \langle \cdot, v_\varphi \rangle$

pf. Trivial Case: If $\ker(\varphi) = X$, then $\varphi(x) = 0\ \forall x$ so take $v_\varphi = 0$: $\varphi(\cdot) = \langle \cdot, 0 \rangle = 0$

So suppose $\ker(\varphi) \neq X$

Then $\ker(\varphi)^\perp \neq \emptyset$ or $\{0\}$. Since $\ker(\varphi)$ is topologically closed ($\varphi^{-1}(0)$, φ is cont), we can write $X = \ker(\varphi) \oplus \ker(\varphi)^\perp$

Let $u \in \ker(\varphi)^\perp$. By rescaling, we can take $\varphi(u) = 1$

__Claim__ $\boxed{\ker(\varphi)^\perp = \operatorname{span}\{u\}}$

since $\operatorname{Image}(\varphi) = \mathbb{R}$ or $\mathbb{C}$, $\operatorname{Im}(\varphi)$ has dim 1

So $\ker(\varphi)$ is really BIG – takes up almost all of X in some sense.

Let us write any x as $x = \underbrace{x - \varphi(x)u}_{\text{in } \ker(\varphi)} + \underbrace{\varphi(x)u}_{\text{in span}\{u\}}$

because $\varphi(x - \varphi(x)u) = \varphi(x) - \varphi(x)\underbrace{\varphi(u)}_{1} = \varphi(x) - \varphi(x) = 0$ ✓ □

Define $v_\varphi := \dfrac{1}{\|u\|^2} u$

Then
$$\begin{aligned}\langle x, v_\varphi \rangle &= \langle x - \varphi(x)u + \varphi(x)u, v_\varphi \rangle = \langle x - \varphi(x)u, v_\varphi \rangle + \langle \varphi(x)u, v_\varphi \rangle \\ &= 0 + \langle \varphi(x)u, v_\varphi \rangle \quad \text{because } (x - \varphi(x)u) \in \ker(\varphi) \text{, thus } \perp \text{ to } u \text{ and hence } v_\varphi. \\ &= \varphi(x) \langle u, v_\varphi \rangle = \frac{\varphi(x)}{\|u\|^2} \langle u, u \rangle = \varphi(x)\end{aligned}$$

QED

▷ Following what I did in ch 11 sheet 7, I am changing the name "V" to "E".

Given a linear map $T : E \to E$

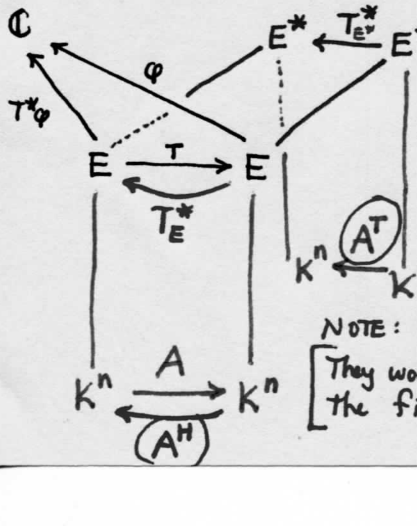
__If__ $\exists$ linear map $T^* : E \to E \ni \langle Tu, v \rangle = \langle u, T^* v \rangle \ \forall u, v \in E$

Then we call T^* the __Adjoint of T__. Prob 13.78 later will show it does not always exist. sheet 11

We must also show how this T^* is "the same as" the T^* of ch 11.

In ch 11 $T^*\varphi = \varphi \circ T$ (pullback). ch 11 was more general: $T : E \to F$

$T^* : F^* \to E^*$ dual spaces

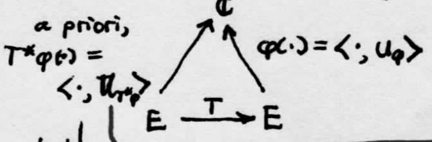


$T^*_{E^*}$ vs T^*_E

See Thm 13.6 next page →

NOTE: A^T vs A^H (is this correct?)

[They would be the same if the field was $\mathbb{R}$.]



but $T^*\varphi = \varphi \circ T$ so $(\varphi \circ T)(g) = \varphi(Tg) = \langle Tg, u_\varphi \rangle$

i.e. $(T^*\varphi)(\cdot) = \langle T(\cdot), u_\varphi \rangle$

But by the def of Adjoint:

$\langle T(\cdot), u_\varphi \rangle = \langle \cdot, T^* u_\varphi \rangle$ $\quad u_{T^*\varphi} = T^* u_\varphi$

$\boxed{\text{So we can say } T^*\varphi(\cdot) = \langle \cdot, T^* u_\varphi \rangle}$

This is why people use the same T^* for ch 11 and ch 13

BUT more accurate would be $\boxed{(T^*_{E^*}\varphi)(\cdot) = \langle \cdot, T^*_E u_\varphi \rangle}$

Thm 13.6

E fin dim IP space
$T: E \to E$ linear
$A := [T]_e$ in O.N. basis $\{e\}$

$\Rightarrow$ $\exists!$ a linear map $T^*_E : E \to E \ni$
$\langle Tu, v\rangle = \langle u, T^*_E v\rangle$
and downstairs $[T^*]_e = A^H$

Pf. (I am going to drop the subscript: T^* means T^*_E here.)

Step 1 Define $T^*: E \to E$

Fix $v \in E$ and define $f_{(T,v)}(\cdot)$ by $f_{T,v}: E \to \mathbb{C}$, $u \mapsto \langle Tu, v\rangle$ Thus it is a linear fcnal

By Thm 13.5 Baby Riesz $f_{T,v}(\cdot) = \langle \cdot, u_{f_{T,v}}\rangle$ Just call this u_v because T is fixed and v is the thing we vary. Given T, v, u_v is uniquely specified.

Define $T^*(v) := u_v$

Then we have it: $\langle Tu, v\rangle = \langle u, u_v\rangle = \langle u, T^*v\rangle$ and this holds $\forall v$

Step 2 T^* is linear:

Let u be arb

$$\langle u, T^*(av_1+bv_2)\rangle = \langle Tu, av_1+bv_2\rangle = \bar{a}\langle Tu, v_1\rangle + \bar{b}\langle Tu, v_2\rangle$$
$$= \bar{a}\langle u, T^*v_1\rangle + \bar{b}\langle u, T^*v_2\rangle$$
$$= \langle u, aT^*v_1\rangle + \langle u, bT^*v_2\rangle$$
$$= \langle u, (aT^*v_1 + bT^*v_2)\rangle$$

Thus $T^*(av_1+bv_2) = aT^*v_1 + bT^*v_2$

[Because $\langle u,p\rangle = \langle u,q\rangle \Rightarrow \langle u, p-q\rangle = 0$ and if this holds $\forall u$, then $p-q=0$].

Step 3 Matrix representation

we know T has representation $A = [a_{ij}]$ in O.N. basis $\{e\}$
and since T^* is just another linear map $E \to E$, it has matrix $B = [b_{ij}]$ in basis $\{e\}$

From prob 13.15(iv) [sheet 7] we know $a_{ij} = \langle Te_j, e_i\rangle$
$b_{ij} = \langle T^*e_j, e_i\rangle$

Hence $b_{ij} = \langle T^*e_j, e_i\rangle = \overline{\langle e_i, T^*e_j\rangle} = \overline{\langle Te_i, e_j\rangle} = \bar{a}_{ji}$

Thus $B = A^H$

QED

Thm 13.7 Let u, v be arb

1. $\langle u, (S+T)^*v\rangle = \langle (S+T)u, v\rangle = \langle Su + Tu, v\rangle = \langle Su, v\rangle + \langle Tu, v\rangle = \langle u, S^*v\rangle + \langle u, T^*v\rangle = \langle u, S^*v + T^*v\rangle$
$= \langle u, (S^*+T^*)v\rangle \Rightarrow \boxed{(S+T)^* = S^* + T^*}$
2. $\langle u, (kT)^*v\rangle = \langle kTu, v\rangle = k\langle u, T^*v\rangle = \langle u, \bar{k}T^*v\rangle \Rightarrow \boxed{(kT)^* = \bar{k}T^*}$
3. $\langle u, (ST)^*v\rangle = \langle STu, v\rangle = \langle S(Tu), v\rangle = \langle Tu, S^*v\rangle = \langle u, T^*S^*v\rangle \Rightarrow \boxed{(ST)^* = T^*S^*}$
4. $\langle u, (T^*)^*v\rangle = \langle T^*u, v\rangle = \overline{\langle v, T^*u\rangle} = \overline{\langle Tv, u\rangle} = \langle u, Tv\rangle \Rightarrow \boxed{(T^*)^* = T}$

□

Thm 13.6

E fin dim IP space
$T: E \to E$ linear
$A := [T]_e$ in O.N. basis $\{e\}$

$\Rightarrow$ $\exists!$ a linear map $T_E^*: E \to E \ni$
$\langle Tu, v\rangle = \langle u, T_E^* v\rangle$
and downstairs $[T^*]_e = A^H$

pf. (I am going to drop the subscript: T^* means T_E^* here.)

Step 1 Define $T^*: E \to E$

Fix $v \in E$ and define $f_{(T,v)}(\cdot)$ by $f_{T,v}: E \to \mathbb{C}$, $u \mapsto \langle Tu, v\rangle$ — Thus it is a linear fcnal

By Thm 13.5 Baby Riesz $f_{T,v}(\cdot) = \langle \cdot, u_{f_{T,v}}\rangle$ — Just call this u_v because T is fixed and v is the thing we vary. Given T, v, u_v is uniquely specified.

Define $T^*(v) := u_v$

Then we have it: $\langle Tu, v\rangle = \langle u, \overset{u_v}{\cancel{u_v}}\rangle = \langle u, T^*v\rangle$ and this holds $\forall v$

Step 2 T^* is linear:

Let u be arb

$$\begin{aligned}\langle u, T^*(av_1+bv_2)\rangle = \langle Tu, av_1+bv_2\rangle &= \bar a\langle Tu, v_1\rangle + \bar b\langle Tu, v_2\rangle \\ &= \bar a\langle u, T^*v_1\rangle + \bar b\langle u, T^*v_2\rangle \\ &= \langle u, aT^*v_1\rangle + \langle u, bT^*v_2\rangle \\ &= \langle u, (aT^*v_1 + bT^*v_2)\rangle\end{aligned}$$

Thus $T^*(av_1+bv_2) = aT^*v_1 + bT^*v_2$

[Because $\langle u, p\rangle = \langle u, q\rangle \Rightarrow \langle u, p-q\rangle = 0$ and if this holds $\forall u$, then $p-q=0$].

Step 3 Matrix representation

We know T has representation $A = [a_{ij}]$ in O.N. basis $\{e\}$ and since T^* is just another linear map $E \to E$, it has matrix $B = [b_{ij}]$ in basis $\{e\}$

From prob 13.15(iv) [sheet 7] we know $a_{ij} = \langle Te_j, e_i\rangle$
$b_{ij} = \langle T^*e_j, e_i\rangle$

Hence $b_{ij} = \langle T^*e_j, e_i\rangle = \overline{\langle e_i, T^*e_j\rangle} = \overline{\langle Te_i, e_j\rangle} = \bar a_{ji}$

Thus $B = A^H$

QED

Thm 13.7 Let u, v be arb

1. $\langle u, (S+T)^*v\rangle = \langle (S+T)u, v\rangle = \langle Su+Tu, v\rangle = \langle Su, v\rangle + \langle Tu, v\rangle = \langle u, S^*v\rangle + \langle u, T^*v\rangle = \langle u, S^*v + T^*v\rangle = \langle u, (S^*+T^*)v\rangle \Rightarrow \boxed{(S+T)^* = S^* + T^*}$
2. $\langle u, (kT)^*v\rangle = \langle kTu, v\rangle = k\langle u, T^*v\rangle = \langle u, \bar k T^*v\rangle \Rightarrow \boxed{(kT)^* = \bar k T^*}$
3. $\langle u, (ST)^*v\rangle = \langle STu, v\rangle = \langle S(Tu), v\rangle = \langle Tu, S^*v\rangle = \langle u, T^*S^*v\rangle \Rightarrow \boxed{(ST)^* = T^*S^*}$
4. $\langle u, (T^*)^*v\rangle = \langle T^*u, v\rangle = \overline{\langle v, T^*u\rangle} = \overline{\langle Tv, u\rangle} = \langle u, Tv\rangle \Rightarrow \boxed{(T^*)^* = T}$

$\square$

13.20 (i) $\boxed{I^* = I}$ because $\forall u,v \quad \langle Iu, v\rangle = \langle u, v\rangle = \langle u, Iv\rangle$
So $I = I^*$ by uniqueness.

(ii) $\boxed{0^*_{E\to E} = 0_{E\to E}}$ because $\langle 0(u), v\rangle = \langle 0_E, v\rangle = 0_K = \langle u, 0_E\rangle = \langle u, 0^*(v)\rangle$

(iii) $\boxed{(T^{-1})^* = (T^*)^{-1}}$ because $I = I^* = (TT^{-1})^* = (T^{-1})^* T^*$ ← by (3) prev page.

$$\Rightarrow I(T^*)^{-1} = (T^{-1})^*$$
$$(T^*)^{-1} = (T^{-1})^* \quad \square$$

13.21 $T: V \to V$ linear

$\left.\begin{array}{l} W \text{ is a } T\text{-invariant subsp of } V: \\ T|_W : W \to W \end{array}\right\} \Rightarrow W^\perp$ is invariant under T^*

Let $u \in W^\perp$
If $w \in W$ then $T(w) \in W$ so $\langle w, T^*u\rangle = \langle Tw, u\rangle = 0$
Thus $T^*u \in W^\perp \Rightarrow W^\perp$ is invariant under T^* $\square$

13.78 Counter-ex to show the adjoint does not always exist — although the internet seems to disagree.
Let V = vs of polys over $\mathbb{R}$, presumably all polys defined on $[0,1]$
(arb C^∞ fcns might also work?)

$\langle p, q\rangle := \int_0^1 p(t)q(t)\,dt$. Let linear operator $D = \frac{d}{dt}$ $\left[\text{so } Dp = \frac{d}{dt}p = p'\right]$

CLAIM: $\not\exists$ $D^*: V \to V$ such that $\langle Dp, q\rangle = \langle p, D^*q\rangle$

Renaming D^* as L, what we want is a linear map $L \ni \int_0^1 p'q\,dt = \int_0^1 p\,L(q)\,dt$ ⊛
and L must be the same fixed map for all p, q — it can't depend on q or p, just like $D = \frac{d}{dt}$ is fixed.

IDEA: ⊛ $\Rightarrow \int_0^1 (p'q - p\,L(q(t)))\,dt = 0$ for any p, q. Does this force $p'(t)q(t) - p(t)\,L(q(t)) = 0$?
No, because we can't rule out

IDEA 2 Integration by parts $\int_0^1 p'q\,dt = p(t)q(t)\Big]_0^1 - \int_0^1 pq'\,dt$

$= (p(1)q(1) - p(0)q(0)) + \int_0^1 p(-q')\,dt$

So for any polys p, q where $p(1)q(1) = 0$ and $p(0)q(0) = 0$ say $p(1) = 0$ and $q(0) = 0$
we have it, or even $[p(1)q(1) - p(0)q(0)] = 0$. More usefully, we could restrict V to only polys that have zeroes at 0 and 1.

In that case $L = D^* = -\frac{d}{dt}$

But in general, we would need $\int_0^1 p\,L(q)\,dt \overset{!}{=} p_1q_1 - p_0q_0 + \int_0^1 p(-q')\,dt$

No way this is possible if L can operate only on q and have no dependence on p $\square$