Schuam's LA
Ch 11

2/14/2023
♥ ①

Linear Functionals and the Dual Space

Let $\mathbb{V}$ be a vector space over a field of Scalars K (think $K = \mathbb{R}$ or $\mathbb{C}$)
Consider linear $\varphi : \mathbb{V} \rightarrow K$ this is called a linear fcnal or linear form or 1-form since vacuously alternating.

example 11.2 $\mathbb{V}$ = polys in t over $\mathbb{R}$ (∞-dim)

$$\varphi : \mathbb{V} \longrightarrow \mathbb{R}$$
$$p \longmapsto \int_0^1 p(t)dt$$

ex 11.3 Let $A = n \times n$ matrix
Let φ = trace : $GL(n) \longrightarrow \mathbb{R}$

$$A \longmapsto tr(A)$$

Define Dual Sp of $\mathbb{V}$: $\mathbb{V}^* := \left\{ \begin{matrix} \text{all linear fcnals} \\ \varphi : \mathbb{V} \rightarrow K \end{matrix} \right\}$ $\mathbb{V}^*$ is a v.s. with $(\varphi + \sigma)(v) = \varphi(v) + \sigma(v)$

$$(a\varphi)(v) = a\varphi(v)$$

example 11.4 Let $V = \mathbb{R}^n$ col vectors $\vec{x}$
Then V^* = row vectors $\varphi_a = [a_1 \dots a_n]$ so $\varphi_a(x) = a^T x$

Thm 11.1 Existence of Dual Basis

Let $\{v_1 \dots v_n\}$ be a basis for $\mathbb{V}$ $\Longrightarrow$ ① we can define linear $\varphi_j : V \rightarrow K$ for $j = 1, \dots, n \ni \varphi_j(v_i) = \delta_{ij}$
② $\{\varphi_1 \dots \varphi_n\}$ is a basis for $\mathbb{V}^*$

Pf. To prove this we need some Lemmas

p.135 **Lemma 1:** Thm 6.2 Linear Map is determined on basis
$\left. \begin{matrix} \text{Given basis } \{v_1 \dots v_n\} \text{ for } \mathbb{V} \text{ and} \\ n \text{ arb vectors } u_1, \dots, u_n \text{ (not nec distinct!) in v.s. } \mathbb{U} \end{matrix} \right\} \Longrightarrow \exists!$ linear $F : \mathbb{V} \longrightarrow \mathbb{U} \ni F(v_i) = u_i$

pf. Step 1 show we can define $F \ni F(v_i) = u_i$ for arb u_i, $i = 1 \dots n$
Firstly, we know every $v \in V$ can be expressed $v = \sum a_i v_i$
Define F by taking the coeffs $\{a_i\}$ and making a LC with $\{u_i\}$:

$$F(v) := \sum a_i u_i$$

Then if we plug in v_j $F(v_j) = 0u_1 + \dots + 1u_j + \dots 0u_n = u_j$

Step 2 F is linear because

$$v + w = \sum (a_i + b_i) v_i \text{ so } F(v+w) = \sum (a_i + b_i) u_i \neq \sum a_i u_i + \sum b_i u_i = F(v) + F(w)$$

Similiarly $F(\lambda v) = \lambda F(v)$

Step 3 An F meeting this criteria is unique because
If $\exists$ linear $G \ni F(v_i) = u_i = G(v_i)$ for $i = 1 \dots n$, does $F(v) = G(v)$ for any v?
Yes. Choose arb $v = \sum a_i v_i$

$$G(v) = G(\sum a_i v_i) = \sum a_i G(v_i) = \sum a_i F(v_i) = F(v)$$

□

Lemma 2 Prob 11.7 $\left. \begin{matrix} \text{Fin dim } \mathbb{V} \\ \text{nonzero } v \in V \end{matrix} \right\} \Rightarrow \exists \varphi \in V^* \ni \varphi(v) = 1$ (i.e. $\neq 0$)

Pf. Extend $\{v\}$ to a basis $\{v = v_1, v_2, \dots, v_n\}$.
Take vs $\mathbb{U} := \mathbb{R}$ with $u_1 = 1$, $u_2 = 0, \dots, u_n = 0$
Apply above Lemma 1 and we get a uniq linear $F : V \longrightarrow \mathbb{R}$ with $F(v_1) = 1$
$F(v_i) = 0 \quad i > 1$

▷ So to prove ① we rename "F" as φ_1
we can repeatedly apply Lemma 2 with $u_i = 1$ and generate $\{\varphi_1 \dots \varphi_n\}$ with $\varphi_i(v_j) = \delta_{ij}$

cont'd →

▷ Now we must show $\{\varphi_1 \dots \varphi_n\}$ is a basis for V^*
which means we must show that set is LI and spans V^*

Easy to show <u>LI</u> : we need $\sum a_i \varphi_i(\cdot) = 0 \Leftrightarrow$ each $a_i = 0$
we only need to show $(\Rightarrow)$ Let $\sum a_i \varphi_i = 0$
Fix j. Apply both sides to v_j : $\sum a_i \varphi_i(v_j) = 0(v_j) = 0$

$$\overset{\shortparallel}{a_j \cdot 1} \Rightarrow a_j = 0$$

To show <u>Span</u> , we need another lemma:

<u>Lemma 3</u> (Thm 11.2) Given basis $\{v_i\}$ and Set of fcnals $\{\varphi_j\} \ni \varphi_j(v_i) = \delta_{ij}$ $\Big\} \Rightarrow$ ① we can write any $v \in V$ $\quad v = \sum^n \varphi_i(v) v_i$ ⊛
② we can write any $\sigma \in V^*$
(★★) $\sigma = \sum^n \sigma(v_i) \varphi_i$

Pf. To see ⊛, observe any $v = \sum a_i v_i = \sum \varphi_i(v) v_i$ because

$$\varphi_i(v) = \varphi_i(a_1 v_1 + \dots + a_i v_i + \dots + a_n v_n) = a_1 0 + \dots + a_i 1 + \dots + 0 = a_i$$

To see (★★) choose any $\sigma \in V^*$
apply σ to ⊛ : $\sigma(v) = \sigma\left(\sum \varphi_i(v) v_i\right) = \sum \varphi_i(v) \sigma(v_i)$

$$= \left(\sum \sigma(v_i) \varphi_i(\cdot)\right)(v) \quad \square$$

▷ So this Lemma shows $\{\varphi_1 \dots \varphi_n\}$ spans V^* and we are done!

$\square$

<u>observe this fact:</u>
$\varphi: V \rightarrow K$ linear fcnal.
Since $\dim(K) = 1$, $\dim(\mathrm{Im}(\varphi)) = 1$ (if not 0)
Rank ⊕ Nullity $= n \Rightarrow$ Nullity$(\varphi) = n-1$
So <u>most</u> of V gets mapped to 0 by any φ. Think of φ as $[\cdot \cdots \cdot]$ $1 \times n$ row vector to also see this.

Thm 11.3 How C.O.B. in V induces COB in Dual basis for V^*

Recall $\{f\} = \{e\}P$
$\Rightarrow [v]_f = P^{-1}[v]_e$

Let $\{e\}_1^n$ and $\{f\}_1^n$ be bases for V

$\{f\} = \{e\}P$

Let $\{e^*\}$ and $\{f^*\}$ denote corresponding dual bases

$$\Longrightarrow \{f^*\} = \{e^*\}(P^{-1})^T$$

Thus the co-ord vectors would be
$[\varphi]_{f^*} = P^T[\varphi]_{e^*}$

See co-ord charts diagram below

pf. Lets say $\{f_1\, f_2\, f_3\} = \{e_1\, e_2\, e_3\}\begin{bmatrix} p_1^{①} & p_2^{①} & p_3^{①} \\ p_1^{②} & p_2^{②} & p_3^{②} \\ p_1^{③} & p_2^{③} & p_3^{③} \end{bmatrix}$

Fix i

$f_i = e_1 p_i^{①} + e_2 p_i^{②} + e_3 p_i^{③}$

$f_i = \sum_j e_j p_i^j$ ← row ← col

since $\{f^*\}$ is dual basis to $\{f\}$, $f_k^*(f_i) = \delta_{ik}$ i still fixed

Expressing f_k^* in basis $\{e^*\}$:

$$\delta_{ik} = [a_k\, b_k\, c_k]\begin{Bmatrix} e_1^*(\cdot) \\ e_2^*(\cdot) \\ e_3^*(\cdot) \end{Bmatrix}\left(\sum_j e_j p_i^j\right) = [a_k\ b_k\ c_k]\begin{bmatrix} p_i^{①} \\ p_i^{②} \\ p_i^{③} \end{bmatrix} \overset{!}{=} \delta_{ik} \Rightarrow [a_k\ b_k\ c_k] \text{ must be the } i^{th} \text{ row of } P^{-1}$$

co-ord charts

$$\begin{tikzcd}
 & V \arrow[dl, "\alpha_{\{e\}}"'] \arrow[dr, "\alpha_{\{f\}}"] & \\
\mathbb{R}^n \arrow[rr, "P^{-1}"] & & R^n \\
{[v]_e} \arrow[rr, mapsto, bend right] & & {[v]_f}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V^* \arrow[dl, "\theta_{e^*}"'] \arrow[dr, "\theta_{f^*}"] & \\
(\mathbb{R}^n)^* \arrow[rr, "P^T"] & & (\mathbb{R}^n)^* \\
{[\varphi]_{e^*}} & & {[\varphi]_{f^*}}
\end{tikzcd}$$

$\theta_{e^*} = (\alpha_{\{e\}}^*)^{-1}$

OR since Schuams is writing the basis in a <u>row</u>

$$\{f_1^*\ f_2^*\ f_3^*\} = \{e_1^*\, e_2^*\, e_3^*\}(P^{-1})^T$$

□

▷ 2nd Dual Space V^{**}

V^* is a vs itself so it can also have a dual V^{**}

For each $v \in V$, we can define "pt evaluation fcnal" $\hat{v} : V^* \longrightarrow K$, $\varphi \longmapsto \varphi(v)$

Better to call this $\delta_v : V^* \to K$

δ_v is linear: $\delta_v(a\varphi + b\sigma) = a\varphi(v) + b\sigma(v) = a\delta_v(\varphi) + b\,\delta_v(\sigma)$

Thm 11.4 V fin dim $\Rightarrow$ The map $q : V \longrightarrow V^{**}$, $v \longmapsto \delta_v$ is an iso

For arb $\overset{\infty \text{ dim}}{\text{linear}}$ sp X q is <u>never</u> <u>Onto</u> but always linear and On-to-One
So X^{**} is isomorphically (and isometrically if nls) embedded in X

See books on Functional Analysis
Usually X is nls or Banach
cf Cheney Apl notes, Simmonds ITAMA p.231

▷ Annihilators

Let W be any <u>subset</u> of V
Let $\varphi \in V^*$ and $\varphi(w) = 0\ \forall w \in W$
φ 'annihilates' W

$$\Rightarrow W^0 := \{\overset{\text{all}}{\varphi \in V^*} \mid \varphi(W) = \{0\}\}$$

better $\boxed{W^{*0}}$

all φ that annihilate W

Another notation is $W^\perp$
see →

Note $W \subseteq V$ but $W^0 \subseteq V^*$

<u>Claim</u> W^0 is a subsp of V^* (even though W not nec subsp V)

- $0_{V^*} \in W^0$ because obviously $0_{V^*}(w) = 0_K$
- W^0 is closed because if $\varphi, \sigma \in W^0$, then $(a\varphi + b\sigma) \in W^0$ because $(a\varphi + b\sigma)(w) = a\varphi(w) + b\sigma(w) = 0\ \forall w$

Thm 11.5 $\left.\begin{array}{l} V \text{ fin dim} \\ W \text{ subspace } V \end{array}\right\} \Rightarrow$ • $\dim(W) + \dim(W^{0*}) = \dim V$
• $W^{00*} = W$ should say $(W^{0*})^{0*}$ better $*0$

pf. ① Let $\dim V = n$ and $\dim W = r \leq n$ We want: $\dim W^{0*} = n-r$

Let $\{w_1, \ldots, w_r\}$ be a basis for W; extend to basis for V $\{w_1, \ldots w_r, v_1 \ldots v_{n-r}\}$

Let $\{\varphi_1 \ldots \varphi_r, \sigma_1, \ldots, \sigma_{n-r}\}$ be dual basis (that exists by Thm 11.1)

Then $\sigma_i(w_j) = 0$ by design, so each σ_i annihilates $W \Rightarrow \{\sigma_1, \ldots \sigma_{n-r}\} \subset W^{0*}$

claim $\{\sigma_1 \ldots \sigma_{n-r}\}$ is, in fact, a basis for W^{0*}

(a) LI because $\{\sigma_i\}$ is part of basis $\{w_1 \ldots w_r, v_1 \ldots v_{n-r}\}$

(b) Span: choose arb $\theta \in W^{0*}$

$$\theta(\cdot) = \sum^{r} \underbrace{\theta(w_i)}_{0} \varphi_i(\cdot) + \sum^{n-r} \theta(v_j)\sigma_j(\cdot) = \sum^{n-r} \theta(v_j)\sigma_j(\cdot)$$

Thus θ is an LC of only $\{\sigma_j\}$ which is a basis of W^{0*}

$\Rightarrow \dim W^{0*} = n-r = \dim V - \dim W$

② $(W^{*0})^{*0} = W$

we know $\dim V = n$ so $\dim V^* = n$

W^{*0} is subsp of V^* v.s. so by part ① $\underset{n}{\dim V^*} - \underset{(n-r)}{\dim W^{*0}} = \dim (W^{*0})^{*0}$

so $\dim W = r = n-(n-r) = \dim (W^{*0})^{*0}$

Now we must show $W \subseteq (W^{*0})^{*0}$ because that will show equality

Let $w \in W$. For any $\varphi \in W^{*0}$, $\varphi(w) = 0$

But this says the pt eval fcnal δ_w has $\delta_w(\varphi) = \varphi(w) = 0$ This holds $\forall w \in W$

$\{\overset{\text{all}}{\delta_w} \mid w \in W\} \subseteq (W^{*0})^{*0}$ And by the iso $V \longleftrightarrow V^{**}$ $W \subseteq (W^{*0})^{*0}$

▷ Now let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be an $n \times m$ matrix and lets give an interpretation of the homogeneous eq $Ax = 0$ using $\dim W + \dim W^{*0} = \dim V$ recover Rank + Nullity Thm.

cf. ch13 Sheet⑥

So take $V = \mathbb{R}^n$ and $W := \ker(A)$ (solns to $Ax=0$)

What is $[\ker(A)]^{*0}$? $\varphi \in [\ker(A)]^{*0}$ iff $\varphi(w) = 0$ $\forall w \in W = \ker(A)$

Transpose homog eq: $x^T A^T = 0^T$ The n-dim row vector $x^T = \varphi \in (\mathbb{R}^n)^*$

Thus for any $y \in \mathbb{R}^m$ $x^T A^T y = 0^T y = 0_{\mathbb{R}}$ so $x \in \ker(A) \Rightarrow x^T$ annihilates $Im(A^T)$

But $\{\overset{\text{all } u \in \mathbb{R}^n}{u = A^T y} \mid y \in \mathbb{R}^m\} = Image(A^T)$

Let $\dim(W) = \dim(\ker(A)) = r$

$\Rightarrow \dim W + \dim W^{*0} = n$

$r + \dim [Im(A^T)]^{*0} = n \Rightarrow \dim [Im(A^T)]^{*0} = n-r$

ASIDE: Many books, such as B&S p351 and Cheney Apl M write the annihilator as $W^\perp$

I want to show we can identify $W^{*0} \overset{is}{\longleftrightarrow} W^\perp$ even though they live in different spaces.

Schuams defines $\underset{\subseteq V^*}{W^{*0}} := \{ \overset{all}{\varphi \in V^*} \mid \varphi(w) = 0 \ \forall w \in W \}$

and $\underset{\subseteq V}{W^\perp} := \{ \overset{all}{v \in V} \mid \langle w, v \rangle = 0 \ \forall w \in W \}$

But by the "Reisz Rep Thm": For any $\varphi \in V^*$ there is a uniq vector $v_\varphi \in V \ni \varphi(\cdot) = \langle \cdot, v_\varphi \rangle$

Let me write $R : \varphi \longmapsto v_\varphi$ $\quad R: V^* \longrightarrow V$ $\varphi \longmapsto v_\varphi$

$W^{*0} \subseteq R^{-1}(W^\perp)$: $\varphi \in W^{*0} \Rightarrow \langle w, v_\varphi \rangle = 0 \ \forall w \in W \Rightarrow v_\varphi \in W^\perp$

$R(W^{*0}) \supseteq W^\perp$: $\quad q \in W^\perp \Rightarrow \langle w, q \rangle = 0 \ \forall w \in W \Rightarrow$ $\varphi_q(\cdot) = \langle \cdot, q \rangle \in W^{*0}$

But in fact we already know this from the existence of dual basis

$\Theta : V \longrightarrow V^*$

$v = \sum a_i e_i \longmapsto \sum a_i \varphi_i = \varphi$

We know an iso exists — not nec same one!

ASIDE 2 Strang LAAIA has $\boxed{\text{Fund Thm of LA II}}$

$A : \mathbb{R}^n \to \mathbb{R}^m$ matrix

$\ker(A) = [\mathrm{Im}(A^T)]^\perp$

$\ker(A^T) = [\mathrm{Im}(A)]^\perp$

$\mathbb{R}^n = \ker(A) \oplus \underbrace{\ker(A)^\perp}_{\mathrm{Im}(A^T)}$

$\mathbb{R}^m = \ker(A^T) \oplus \mathrm{Im}(A)$

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p252

▷ Transpose of a Linear Map (pullback!)

Let E be a vs with {e} the fixed basis we are working with. Same for F and {f}.

Let $T : E \to F$ be a linear map and $\beta : F \longrightarrow K$ be linear fcnal $\beta \in F^*$

$$\begin{tikzcd}
 & K & \\
E \arrow[ur, "T^*\beta"] \arrow[rr, "T"'] \arrow[d, dotted, no head] & & F \arrow[ul, "\beta"'] \arrow[d, dotted, no head] \\
E^* & & F^* \arrow[ll, "T^*"']
\end{tikzcd}$$

We define <u>pullback</u> $T^*\beta := \beta \circ T$

Thus the existence of T induces the existence of T^*

$\boxed{\text{Thm 11.6} \quad T^* \text{ is linear}}$

Pf. Let $\varphi, \sigma \in F^*$ Then
$$T^*(a\varphi + b\sigma) = (a\varphi + b\sigma) \circ T = a(\varphi \circ T) + b(\sigma \circ T) = a T^*\varphi + b T^*\sigma \quad \square$$

$\boxed{\text{Thm (prob 11.15)} \quad \ker(T^*) = [\mathrm{Im}(T)]^{*0} \text{ a.k.a. } [\mathrm{Im}(T)]^\perp}$

Pf. show $\ker(T^*) \subseteq [\mathrm{Im}(T)]^{*0}$: Let $\varphi \in \ker(T^*) \Rightarrow T^*\varphi = \varphi \circ T = 0$

If $u \in \mathrm{Im}(T)$ then $u = T(v)$ for some v. Then $\varphi(u) = (\varphi \circ T)(v) = 0(v) = 0$

So we have $\varphi(u) = 0 \ \forall u \in \mathrm{Im}(T) \Rightarrow \varphi \in [\mathrm{Im}(T)]^{*0}$

show $\ker(T^*) \supseteq [\mathrm{Im}(T)]^{*0}$: Let $\sigma \in [\mathrm{Im}(T)]^{*0} \Rightarrow \sigma(\mathrm{Im}(T)) = \{0\}$

Then $\forall v \in V \ (T^*\sigma)(v) = (\sigma \circ T)(v) = \sigma(T(v)) = 0$ since $T(v) \in \mathrm{Im}(T)$

This holds $\forall v$ so $T^*\sigma = 0 \Rightarrow \sigma \in \ker(T^*)$ $\square$

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Thm (prob 11.38)

$$\left.\begin{array}{l} V \text{ fin dim} \\ T: V \to U \\ V^* \xleftarrow{T^*} U^* \end{array}\right\} \Rightarrow \operatorname{Im}(T^*) = [\ker(T)]^{*0}$$

This corresponds to Strang's $\operatorname{Im}(A^T) = (\ker(A))^\perp$

Pf. $(\subseteq)$ $\beta = T^*\varphi = \varphi \circ T$ we want $\beta \in [\ker(T)]^{*0}$

i.e. $\beta(w) = 0_K$ if $w \in \ker(T)$

So for any $v \in V$ $\beta(v) = \varphi(Tv)$

if $v = w \in \ker(T)$ then $\beta(w) = \varphi(T(w)) = \varphi(0_U) = 0_K \Rightarrow \beta \in [\ker(T)]^{*0}$

$(\supseteq)$ $\sigma \in (\ker(T))^{*0}$ we want: $\sigma \in \operatorname{Im}(T^*)$

So if $w \in \ker(T)$, $\sigma(w) = 0$

i.e. $\exists \beta \in U^*$ where $T^*\beta = \sigma$

$\beta \circ T = \sigma$

Mayb show dimensions are equal?
like sheet 6 ch 13

Prob 11.16

$$\left.\begin{array}{l} V, U \text{ fin dim} \\ T: V \to U \text{ linear} \end{array}\right\} \Rightarrow \operatorname{rank}(T) = \operatorname{rank}(T^*)$$

where $\operatorname{rank}(T) := \dim(\operatorname{Im}(T))$

Pf. Let $\dim V = n$ $\operatorname{rank}(T) = r$
$\dim U = m$

From <u>Thm 11.5</u> $\dim W + \dim W^{*0} = \dim U$ so let $W := \operatorname{Im}(T)$

$$\Rightarrow \dim(\operatorname{Im}(T))^{*0} = \dim U - \dim(\operatorname{Im}(T)) = m - \operatorname{rank}(T) = m - r$$

By (Prob 11.15) sheet ⑤ $\ker(T^*) = [\operatorname{Im}(T)]^{*0}$

$$\Rightarrow \underbrace{\dim(\ker(T^*))}_{\text{nullity}(T^*)} = \dim[\operatorname{Im}(T)]^{*0} = m \mp r$$

Apply Rank + Nullity Thm to T^*

$$\operatorname{rank}(T^*) + \underbrace{\text{nullity}(T^*)}_{m-r} = \underbrace{\dim U^*}_{m} \Rightarrow \operatorname{rank}(T^*) = m - (m-r) = r = \operatorname{rank}(T)$$

□

2/20/2023

Thm 11.7 Given

$$\begin{tikzcd} \overset{\{e\}}{E} \arrow[r, "T"] \arrow[d, "\varphi"'] & \overset{\{f\}}{F} \arrow[d, "\psi"] \\ K^n \arrow[r, "A"] & K^m \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \overset{\{\tilde{e}\}}{E^*} \arrow[d, "\theta_{\tilde{e}}"] & \overset{\{\tilde{f}\}}{F^*} \arrow[l, "T^*"'] \arrow[d, "\theta_{\tilde{f}}"] \\ K^n & K^m \arrow[l, "A^T"'] \end{tikzcd}$$

That is, if $[T]_e^f = A$

then $\boxed{[T^*]_{\tilde{f}}^{\tilde{e}} = A^T}$

Before giving the pf of this thm, let's warm up and give the construction pf of A, which is the left diagram. This is ch 7 p. 156

Let E have basis $\{e_1 \dots e_n\}$ and F has $\{f_1 \dots f_m\}$. What is downstairs representation of T: $[T]_e^f$ denoted as A. (matrix)

Let $v = \sum v^i e_i$ be arb

Then $T(v) = T(\sum v^i e_i) = \sum v^i T(e_i)$. So we must find $T(e_i)$, $\forall\, i = 1, \dots, n$

$T(e_i)$ must be LC of $\{f\}$: $T(e_i) = \{f_1 \dots f_m\} \begin{bmatrix} a_i^{①} \\ \vdots \\ a_i^{ⓜ} \end{bmatrix} = \{f\} \vec{A}_i$ ith col of matrix A

$$\Rightarrow T(v) = \sum v^i T(e_i) = v^1 \{f\} \vec{A}_1 + \dots + v^n \{f\} \vec{A}_n = \{f_1 \dots f_m\} \underbrace{\begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}}_{\uparrow\ [v]_e}$$

<u>NOW for the pf at hand:</u> what is the induced matrix $[T]_{\tilde{f}}^{\tilde{e}}$ when we have the dual bases $\{\tilde{e}\}$ and $\{\tilde{f}\}$?

Choose any $\beta \in F^*$. Then $\beta = \sum \beta^i \tilde{f}_i$

because it is just a v.s. I'm not doing anything special with indices or "row vectors" here

$(T^*\beta)(v) = T^*(\sum \beta^i \tilde{f}_i(\cdot))(v) = (\sum \beta^i T^* \tilde{f}_i(\cdot))(v)$ and we know this must be $T^*\beta = \sum \lambda^i \tilde{e}_i$ for some λs

$$\begin{tikzcd} & K & \\ E \arrow[ur, "T^*\beta"] \arrow[rr, "T"] & & F \arrow[ul, "\beta"'] \end{tikzcd}$$

Just take
n=3
m=3
here

Consider $(T^* \tilde{f}_i)(v) = \tilde{f}_i(T(v))$

$= \tilde{f}_i(\{f\} A v_e)$ from warm up above

$= \tilde{f}_i \left(v^1 \sum_j f_j A_1^j + v^2 \sum_j f_j A_2^j + v^3 \sum_j f_j A_3^j \right)$

$= v^{①} A_1^{ⓘ} + v^{②} A_2^{ⓘ} + v^{③} A_3^{ⓘ}$ since dual basis, only $j=i$ term survive

$= \tilde{e}_1(v) A_1^i + \tilde{e}_2(v) A_2^i + \tilde{e}_3(v) A_3^i$ $\quad \tilde{e}_j(v) = v^j$ by Lemma 3 sheet ②

$= \left(\{\tilde{e}_1(\cdot)\ \tilde{e}_2(\cdot)\ \tilde{e}_3(\cdot)\} \begin{bmatrix} A_1^i \\ A_2^i \\ A_3^i \end{bmatrix} \right)(v)$

row i of A = col i of A^T

▷ Thus $T^*(\sum \beta_i \tilde{f}_i) =$

$\beta_1 T^* \tilde{f}_1 + \beta_2 T^* \tilde{f}_2 + \beta_3 T^* \tilde{f}_3$

$= \{\tilde{e}_1\, \tilde{e}_2\, \tilde{e}_3\} \begin{bmatrix} A_1^1 \\ A_2^1 \\ A_3^1 \end{bmatrix} \beta_1 + \{\tilde{e}\ \tilde{e}\ \tilde{e}\} \begin{bmatrix} A_1^2 \\ A_2^2 \\ A_3^2 \end{bmatrix} \beta_2 + \{\tilde{e}_1\, \tilde{e}_2\, \tilde{e}_3\} \begin{bmatrix} A_1^3 \\ A_2^3 \\ A_3^3 \end{bmatrix} \beta_3$

$= \{\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3\} A^T \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{bmatrix} = A^T \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ Downstairs co-ords

□