~~Schwams~~ LA 1/15/2020

Ch 12 ①

Bilinear, Quadratic and Hermetian Forms

Let V be a finite dim vs over a field K (think $K = \mathbb{R}$ or $\mathbb{C}$ always)

$f: V \times V \to K$ is <u>bilinear</u> if it is linear in each position separately.

ex A is $n \times n$ matrix $\quad f(u,v) := u^T A v$ is bilinear.

Define $\mathcal{B}(V) := \left\{ \begin{matrix} \text{all bilinear maps} \\ f: V\times V \to K \end{matrix} \right\}$ $\quad \mathcal{B}(V)$ itself is vs with $(f+g)(u,v) := f(u,v) + g(u,v)$

$(\lambda f)(u,v) := \lambda f(u,v)$

Thm 12.1 Let $\{\varphi_i\}_{i=1}^n$ be a basis for dual space V^* $\Rightarrow$ $f_{ij}(u,v) := \varphi_i(u)\varphi_j(v)$ is basis for $\mathcal{B}(V)$

$\dim(\mathcal{B}(V)) = n^2$

Pf. Let $\{e_1 \dots e_n\}$ be basis of V dual to $\{\varphi_i\}$ i.e $\varphi_i(e_j) = \delta_{ij}$

<u>Step 1</u> show $\{f_{ij}\}$ span $\mathcal{B}(V)$. Let $\alpha_{ij} := f(e_i, e_j)$

Let f be arb elt of $\mathcal{B}(V)$.

<u>claim</u>: $f = \sum_i \sum_j \alpha_{ij} f_{ij}$ i.e. for $s,t = 1,\dots,n$ $f(e_s, e_t) = \left(\sum\sum \alpha_{ij} f_{ij}\right)(e_s, e_t)$

$$\left(\sum\sum \alpha_{ij} f_{ij}\right)(e_s,e_t) = \sum\sum \alpha_{ij} f_{ij}(e_s,e_t) = \sum\sum \alpha_{ij} \underbrace{\varphi_i(e_s)}_{\delta_{is}} \underbrace{\varphi_j(e_t)}_{\delta_{jt}} = \alpha_{st} \overset{\text{def}}{\equiv} f(e_s,e_t)$$

Thus $\sum_{ij} \alpha_{ij} f_{ij}(\cdot,\cdot) = f(\cdot,\cdot)$ by linearity.

<u>step 2</u> Show $\{f_{ij}\}$ is LI.

Suppose $\sum\sum \beta_{ij} f_{ij} = 0_{V^*}$

Then for any fixed $s,t \in \{1,\dots n\}$ $\quad 0 = 0_{V^*}(e_s,e_t) = \sum\sum \beta_{ij} \varphi_i(e_s)\varphi_j(e_t) = \beta_{st}$

every coeff is 0 $\quad \square$

If V has basis $\{e\}$ then a bilinear $f: V\times V \to K$ has the matrix representation $f(u,v) = [u]_e^T \left[f(e_i e_j) \right] [v]_e$

To show this: $f(u,v) = f\left(\sum u_i e_i, \sum v_j e_j\right) = \sum\sum u_i v_j f(e_i, e_j) = [u_1 \dots u_n] \left[f(e_i e_j) \right] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

Thm 12.2 Given 2 bases for V $\{e\}, \{f\}$

Change of basis matrix P: $\{f\} = \{e\}P$ $\Rightarrow$ $[A]_f = P^T [A]_e P$

bi-linear f with matrix rep $[A]_e$ $\quad$ 'Congruence' transform

<u>Pf</u> we know $P v_f = v_e$, $P u_f = u_e$ $\quad f(u,v) = u_e^T A_e v_e = (P v_f)^T A_e (P v_f) = v_f^T [P^T A_e P] v_f \quad \square$

<u>Def</u> we say B is <u>Congruent</u> to A if $\exists$ nsing matrix P where $B = P^T A P$.

<u>Def</u> rank of bilinear form f is ordinary rank of any matrix representation A.

f is degenerate if $\operatorname{rank}(f) < \dim(V)$

a bi-linear f is called <u>alternating</u> if $\boxed{f(v,v) = 0 \ \forall\, v \in V}$

This implies: $0 = f(u+v, u+v) = \underbrace{f(u,u)}_{0} + f(u,v) + f(v,u) + \underbrace{f(v,v)}_{0}$

$\Rightarrow \boxed{f(u,v) = -f(v,u)}$ <u>anti-Symm</u> or <u>skew-Symm</u>

<u>Thm 12.3</u> f bi-linear, alternating $\Longrightarrow$ • $\exists$ a basis where $[f] = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$

This is the symplectic 2-form ω of Hamiltonian Mechanics (provided it has full rank)
see remarks after pf.

• The number of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ blocks is uniquely determined by f and equal to $\frac{1}{2}\operatorname{rank}(f)$
• Alternating bi form always has even rank

<u>Pf</u> The idea is that if f is non-degenerate on a vs of dim ≥ 2, then we can carve out a 2-dim subsp where f has this form, and we can keep doing this repeatedly until the dimensions are exhausted or f becomes degen.

<u>Step 1</u> Assume $f \neq 0_{B(V)}$ or nothing to prove.
If $\dim(V) = 1$, then for any 2 vectors u, v, $v \overset{!}{=} \lambda u$ so $f(u,v) = 0$ and again $f = 0_{B(V)}$
So assume $\dim V \geq 2$.
$\exists\, u_1, u_2 \ni f(u_1,u_2) \neq 0$ so rescale wlog so $f(u_1,u_2) = 1$ $\ (f(u_2,u_1) = -1)$
We see $\{u_1, u_2\}$ is LI or else $f(u_1,u_2) = 0$ because $u_2 = \lambda u_1$
Let $U := \operatorname{Span}\{u_1 u_2\}$
Observe $\left[f\big|_U\right]_{\{u\}} = \begin{bmatrix} f(u_1,u_1) & f(u_1,u_2) \\ f(u_2,u_1) & f(u_2 u_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad [f(u_i, u_i) = 0 \text{ Alternating}]$
so the desired form is attained.

<u>Step 2</u> Define $W = \left\{ \begin{smallmatrix}\text{all} \\ w \in V\end{smallmatrix} \;\middle|\; \begin{smallmatrix} f(w,u_1) = 0 \\ f(w,u_2) = 0 \end{smallmatrix} \right\}$ i.e. $\left\{ \begin{smallmatrix}\text{all} \\ w \in V\end{smallmatrix} \;\middle|\; f(w,u) = 0 \ \forall u \in U \right\}$ "$U^\perp$ wrt f"
Show $V = U \oplus W$
obviously $U \cap W = \{0\}$ disjoint
<u>Now show $V = U + W$</u> i.e. for any $v \in V$ $\exists\, u_v, w_v$ so $v = u_v + w_v$
$= (\alpha u_1 + \beta u_2) + w_v$
we must now exhibit $\alpha, \beta, [w_v \overset{!}{\in} W]$ that make this work. $\boxed{w_v = v - \alpha u_1 - \beta u_2}$
we must have $f(w_v, u_1) = 0$ and $f(w_v, u_2) = 0$ and our freedom is coeffs α, β

$$f(w, u_1) = f(v - u_v, u_1) = f(v - \alpha u_1 - \beta u_2, u_1)$$
$$= f(v, u_1) - \alpha_1 \underbrace{f(u_1,u_1)}_{0} - \beta \underbrace{f(u_2,u_1)}_{-1}$$

So we want $f(v, u_1) + \beta \overset{!}{=} 0 \Rightarrow \beta = -f(v, u_1)$

▷ Likewise $\alpha = f(v, u_2)$

Thus for this α, β $\ f(w, u_1) = 0 = f(w, u_2)$ so $w_v \in W$

cont'd →

pf of Thm 12.3 cont'd:

Step 3 So we have shown $V = U \oplus W$. write it as $U_1 \oplus W_1$

W_1 is a VS and $f|_{W_1}$ is still bilinear and alternating.

If $\dim W_1 = 0$ we are done and $[f] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

If $\dim W_1 = 1$ $f|_{W_1} = 0_{B(W_1)}$ and $[f] = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{bmatrix}$

or if f is now degenerate $f|_{W_1} = 0_{B(W_1)}$ and $[f] = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0_{0} \end{bmatrix}$

So if none of those cases, we can repeat the arg of Steps 1,2 and get $V = U_1 \oplus (U_2 \oplus W_2)$

Step 4 Now we keep repeating Step 3's argument and get

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_k + W_k$$

$f|_{U\text{ space}}$ is in the desired form and the rank of f is exhausted

So since each "U-block" has dim 2, $\operatorname{rank}(f) = 2k$

$\boxed{\text{QED}}$

Meyer & Hall p.40-41 have 2n-dim vs V and alternating, non-degen bilinear $\omega \to \mathbb{R}$ or $\mathbb{C}$

For $\mathbb{R}^{2n}$ $\omega(x,y) = x^T J y$ where the symplectic matrix $J = \begin{bmatrix} & I \\ -I & \end{bmatrix}$

How can we convert the matrix form "K" of the thm to standard J?

Full rank, so $K = \begin{bmatrix} \begin{matrix} & 1 \\ -1 & \end{matrix} & & \\ & \begin{matrix} & 1 \\ -1 & \end{matrix} & \\ & & \ddots \\ & & & \begin{matrix} & 1 \\ -1 & \end{matrix} \end{bmatrix}$ Want a perm matrix P so $J = P^T K P$

This is what works $\begin{bmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix} \xrightarrow{P} \begin{bmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{bmatrix}$

Symm bilinear forms, quadratic forms

Bilinear f is Symmetric if $f(u,v) = f(v,u)\ \forall u,v \in V$

Let A be a matrix rep of f. Then $f(x,y) = x^TAy = (x^TAy)^T = y^TA^Tx$ (Scalars)

Symm $\Rightarrow$ $\|$

$f(y,x) = y^TAx$ Since x,y arb $\Rightarrow A = A^T$

Given a bi, symm f there is an associated quadratic form $q: V \longrightarrow K$, $v \longmapsto f(v,v)$

Polar form $f(u,v) = \frac{1}{2}\left[q(u+v) - q(u) - q(v)\right]$ Just algebra. $q(x) = x^TAx$

> **Thm 12.4** f symm, bilinear $\Rightarrow \exists$ a basis $\{v_1, \dots v_n\}$ where f is represented by a diag matrix $[f(v_i,v_j)=0\ i\neq j]$
>
> Alternatively: matrix $A = A^T \Rightarrow \exists$ nsing matrix $P \ni P^TAP = D$ diag
>
> [Note that this would follow directly from Spectral Thm, but we don't have that yet and this matrix P is not the O.N. Q, more like $A = LDL^T$ Strang's "Cholesky"]

Pf assume $f \neq 0_{B(V)}$

Proof by Induction Initialization: For $\dim(V) = 1$ any matrix $[a]$ is diag. Basis is just $\{v_1\}$

Induction: let $\dim(V) = n$

Step 1 Claim: $\exists$ at least one vector v_n where $f(v_n,v_n) \neq 0$

Suppose not. Then $f(v,v) = 0\ \forall v \in V$

Then for any pair u,v $f(u,v) = \frac{1}{2}[\underset{0}{f(u+v,u+v)} - \underset{0}{f(u,u)} - \underset{0}{f(v,v)}]$ polar form

$\Rightarrow f = 0_{B(V)}$ $\Rightarrow\!\!\Leftarrow$

Step 2 Let $U := \operatorname{span}\{v_n\}$ and $W := \{\text{all } v \mid f(v,v_n) = 0\}$ "$\ker(f_{v_n})$"

Show $V = U \oplus W$

Step 2a show $U \cap W = \{0\}$ no vectors in common

Suppose $\exists\ u \in U \cap W$. Because $u \in U$, $u \overset{!}{=} kv_n$ for some $k \neq 0$

Because $u \in W$, $f(u,v_n) = 0 \Rightarrow 0 = f(kv_n,v_n) = kf(v_n,v_n) \neq 0$ by choice of v_n $\Rightarrow\!\!\Leftarrow$

Step 2b show $V = U + W$

Choose any $v \in V$. we want to be able to write $v = u + w$ i.e. $v = kv_n + w$ $\Rightarrow w = v - kv_n$

we must be able to have $k \ni f(w,v_n) = 0$

$$0 \overset{!}{=} f(w,v_n) = f(v - kv_n, v_n) = f(v,v_n) - kf(v_n,v_n) \Rightarrow k = \frac{f(v,v_n)}{f(v_n,v_n)\neq 0}$$

Step 3 $f|_W$ is still a symm, bi form on $(n-1)$ dim VS, so by induction hypoth there is already a basis $\{v_1, \dots, v_{n-1}\}$ satisfying the criteria $f(v_i,v_j) = 0$ for $i \neq j$

Add v_n to this collection $\{v_1, v_2 \dots v_n\}$.

By def of W, $f(v_i,v_n) = 0\ \forall i = 1, \dots, n-1$

$\square$

we can give an alternate pf and exhibit $P^TAP = D$ and show elementary "row" operations on columns, which I had not seen before.

First, here is an example showing the method:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{bmatrix} \text{ symm}$$

Form augmented matrix

$$\begin{array}{cccccc} 3 & 6 & -9 & 3 & & \\ 2 & 4 & -6 & 2 & & \end{array}$$
$$\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

use Gaussian Elim to Clear out 1st col

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + 3R_1$

$$3\ 2 \left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right]$$

Now do exactly the same operations to the cols of A. (The same ops will clear out 1st row since A symm) The augmented part is not affected!

$C_2 \rightsquigarrow C_2 - 2C_1$

$C_3 \rightsquigarrow C_3 + 3C_1$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right]$$

Now zero out col 2 below pivot and then row 2

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right] P^T$$

$$P^TAP = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -5 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -2 & 7 \\ & 1 & -2 \\ & & 1 \end{bmatrix}$$

Note that Strang LAAIA uses Gaussian elim to factor $A = LDU$ and this factorization is unique. Then $A^T = U^TD^TL^T$ but $A = A^T$ so $LDU = U^TD^TL^T$ and $D = D^T$. So by uniqueness we must have $U = L^T \Rightarrow A = LDL^T$ (I think Strang's L is Schuam's P).

Schuam's formalizes this algo as another pf of Thm 12.4 but I'm not writing it up now.

[This pf is very similiar to Stang's ; See my LAAIA ch 6 writeup sheet ⑧] 1/19/2020

Thm 12.5 <u>Sylvester's Law of Inertia</u>

f symm, bilinear $\Rightarrow$ Every matrix rep $[f]$ that is diagonal has the same number of <u>pos</u>, <u>neg</u> and <u>0</u> entries, respectively.

<u>Pf.</u> By Thm 12.4, we have a basis $\{u_1 \dots u_n\}$ where $[f]_u$ is diag (i.e. $f(u_i, u_j) = 0$ $i \neq j$)
This diag matrix has p pos and m neg entries (then $n-(p+m)$ zeroes).

Suppose $\{w_1 \dots w_n\}$ is some other basis where $[f]_w$ is diagonal; it has q pos entries and l neg entries.

<u>Lemma</u> If a matrix A is diagonalized $D = K^T A K$ then the number of nonzero diag entries in D is the rank of A [K is n sing]
i.e. number of 0's on main diag = $\dim(\ker(A))$. rank + nullity = n

pf after this thm.

From the Lemma, we know $\text{rank}(f) = p + m = q + l$
W.l.o.g we can order the bases so $u_1 \dots u_p$ are assoc with pos diag elts $f(u_i, u_i) > 0$
and likewise for $w_1 \dots w_q$; then $w_{q+1}, \dots, w_n$ have $f(w_j, w_j) \leq 0$.
Let $U := \text{Span}\{u_1 \dots u_p\}$ $\quad W := \text{Span}\{w_{q+1}, \dots, w_n\}$ and obviously $U \cap W = \{0\}$ since diff signs for f

<u>Show $p \leq q$</u> a priori $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$
$\qquad = p \quad n-q \quad 0$

and since $U + W$ is a set of vectors in V, $\dim(U+W) \leq \dim V$
$\Rightarrow p + n - q \leq n \Rightarrow p \leq q$

<u>Show $q \leq p$</u> Repeat the same arg with $W := \text{Span}\{w_1 \dots w_q\}$
$\qquad U := \text{Span}\{u_{p+1} \dots u_n\}$

$\dim(W+U) = \dim W + \dim U - \dim(W+U)$
$\qquad = q \quad n-p \quad 0$ $\qquad \Rightarrow q + n - p \leq n$ $\quad q \leq p$

Thus $p = q$ number of pos elts is the same
From the Lemma, num of 0 elts is the same $\Rightarrow$ number neg elts is same $\square$

pf of Lemma

Let $x_1 \dots x_r$ be a basis for $\ker(A)$. Extend this to $\{x_1, \dots, x_r \dots x_n\}$ basis for V
Since K is nsing, $\exists$ basis $\{s_1 \dots s_n\} \ni Ks_i = x_i$. Then $Ds_i = K^T A K s_i = K^T A x_i = K^T 0 = \vec{0}$
$= 0 s_i \quad i \leq r$

$\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} s \end{bmatrix}$ So 0 is an EW of $D \Rightarrow$ diag elt must be 0
Could any other elt of D be 0? Let q satisfy $Dq = 0$ for $q \notin \text{Span}\{s_1 \dots s_r\}$
$Dq = 0 \Rightarrow K^T A K q = 0 \Rightarrow K^T A z = 0$ where $z = Kq \Rightarrow z \in \ker(A)$ but that means
$z \in \text{Span}\{x_1 \dots x_r\} \Rightarrow q \in \text{Span}\{s_1 \dots s_r\} \Rightarrow\Leftarrow$ so no other diag elts of D are 0 $\square$

COR For a quadratic form $q(x) = f(x,x) = x^T A x$, $A = A^T$ $\Rightarrow$ $\exists$ a special $D_1 = \begin{array}{c} \\ s \\ r \\ \\ \end{array}\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & 0 \end{bmatrix}$

Such that $q(x) = \underbrace{x_1^2 + x_2^2 + \dots + x_s^2}_{pos} \underbrace{- x_{s+1}^2 - \dots - x_r^2}_{neg} + 0$

This representation is unique for q

pf. From Thm 12.4 we know $\exists$ $\{e_1 \dots e_n\}$ such that $f(e_i, e_j) = 0$ $i \neq j$ but $f(e_i, e_i) = \alpha_i$ and we want $\tilde{\alpha}_i = \pm 1$ provided $\alpha_i \neq 0$

Redefine $\tilde{e}_i = \frac{1}{\sqrt{|\alpha_i|}} e_i$ when $\alpha_i \neq 0$, then $f\left(\frac{1}{\sqrt{|\alpha_i|}} e_i, \frac{1}{\sqrt{|\alpha_i|}} e_i\right) = \frac{1}{|\alpha_i|} f(e_i, e_i) = \frac{1}{|\alpha_i|} \alpha_i = \pm 1$

[and $f\left(\frac{1}{\sqrt{|\alpha_i|}} e_i, \frac{1}{\sqrt{|\alpha_j|}} e_j\right) = \frac{1}{\sqrt{|\alpha_i|}} \frac{1}{\sqrt{|\alpha_j|}} f(e_i, e_j) = 0$ still] $\square$

What is this in matrix terms? If I have $D = K^T A K$ and I want to convert this to the special D_1 form, we just have to mult on left and right by another diag matrix:

$$\begin{bmatrix} \alpha & & \\ & \beta & \\ & & 0 \end{bmatrix} = K^T A K$$

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{|\alpha|}} & & \\ & \frac{1}{\sqrt{|\beta|}} & \\ & & 1 \end{bmatrix}}_{C^T} \begin{bmatrix} \alpha & & \\ & \beta & \\ & & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{|\alpha|}} & & \\ & \frac{1}{\sqrt{|\beta|}} & \\ & & 1 \end{bmatrix}}_{C} = C^T K^T A (KC)$$

thus (KC) converts to D_1.

Strang LAAIA ch6 views Sylvester a little differently. He considers all possible COVs $x = Cy$ (for nsing C) in quadratic form $q(x) = x^T A x$ and thus is led to consider the set of all transforms of A: $A_c := C^T A C$ [remember A was symm and so is this]

Strang has introduced Spectral Thm and thus $\exists$ O.N. $Q \ni \Lambda = Q^T A Q$ diag matrix of EWs so Strang phrases Sylvester as 'Every A_c has same number of pos, neg and 0 EWs'. Then since "Strang's Cholesky" $A = LDL^T$ has same form, it is a Corollary that D has same # of pos, neg, 0 diag elts even though they are not EWs.