11/5/2017

Ross ch 3.4 <u>Independent Events</u>

Events E, F are <u>indep</u> if $\underbrace{P(E|F)}_{\frac{P(E\cap F)}{P(F)}} = P(E)$

The knowledge that F occured has no effect on P(E).

$$\Rightarrow \boxed{P(E\cap F) = P(E)\,P(F)}$$

They take this as the <u>def</u> of <u>2</u> indep events.

(4a) Pick a card at random.

E := event card is ace $\quad P(E) = \frac{4}{52} = \frac{1}{13}$

F := event card is a spade $\quad P(F) = \frac{13}{52} = \frac{1}{4}$

$P(E\cap F) = P\left(\begin{matrix}\text{card is ace}\\ \text{\& card is spade}\end{matrix}\right) = P(\text{ace of spades}) = \frac{1}{52}$

whereas $P(E)\cdot P(F) = \frac{1}{13}\cdot\frac{1}{4} = \frac{1}{52}$ ← same → So events qualify as indep.

(4c) Toss 2 dice $\quad E = \text{sum is } 6$

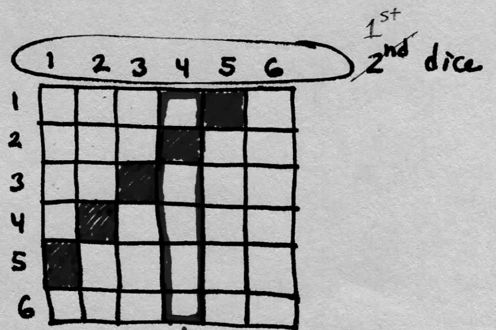
$F = 1^{st}$ dice is a 4

There is only one way to sum to 6 if 1^{st} dice is a 4. That is $\langle 4,2\rangle$.

$P(E\cap F) = P(\langle 4,2\rangle) = \frac{1}{36}$

But $P(E) = \frac{5}{36}$

$P(F) = \frac{6}{36} = \frac{1}{6} \quad P(E)\cdot P(F) = \frac{5}{36}\cdot\frac{1}{6} = \frac{5}{216}$



F restricts us to this col.

$$\boxed{\underline{\text{Prop 4.1}} \quad E \,\&\, F \text{ indep} \Longrightarrow E \,\&\, F^c \text{ are indep}}$$

pf. $E = (E\cap F) \cup (E\cap F^c)$

$P(E) = P(E\cap F) + P(E\cap F^c) = P(E)P(F) + P(E\cap F^c)$ since E&F indep

$P(E) - P(E)P(F) = P(E\cap F^c) \Rightarrow P(E)\underbrace{[1-P(F)]}_{P(F^c)} = P(E\cap F^c)$ $\square$

Now consider 3 Events. E indep of F, F indep of G.

Is E indep of $F\cap G$? NOT NECESSARILY.

(4e) Toss 2 dice again

E := event the sum is 7

F := set of outcomes $\{(4,x) \mid x \text{ is any number on } 2^{nd}\}$ $\quad F\cap G = \langle 4,3\rangle$

G := $\{(y,3) \mid y \text{ any number on } 1^{st}\}$ $\quad$ This forces E to be true

E <u>not</u> indep of $F\cap G$ since $P(E \mid F\cap G) = 1$

<u>Def</u> E, F, G are Indep if:

- $P(E\cap F) = P(E)\cdot P(F)$
- $P(E\cap G) = P(E)\cdot P(G)$
- $P(F\cap G) = P(F)\cdot P(G)$
- $P(E\cap F\cap G) = P(E)\cdot P(F)\cdot P(G)$

and analogously for n events.

Ross ch 5 p.164

This is well described in Schaum's Stats ch 10

11/3/2017

example 3g

Trial of 100 people
Medicine considered to work if 65% of people show improvement.

X = # of people with improved cholesterol.

This is not a good example.
Many better ones in Schaums.

H_0 : Medicine has no effect. $p = 1/2$

Prob of false rejection of H_0

Given that $p = 1/2$

$$P(X \ge 65) = \sum_{i=65}^{100} \binom{100}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{100-i} = \left(\frac{1}{2}\right)^{100} \sum_{i=65}^{100} \binom{100}{i}$$

$\mu = Np = 100 \cdot \frac{1}{2} = 50$ > 10 so we can apply $\mathcal{N}$ approx.

$\sigma := \sqrt{Np(1-p)} = \sqrt{100 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{25} = 5$

$65 \le X$

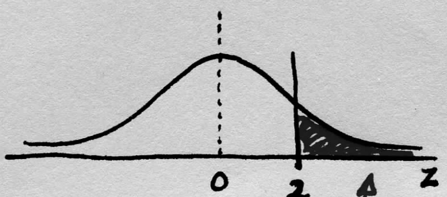
Normalize

$$\frac{\cancel{60}\,65 - 50}{5} \le \frac{X - \mu}{\sigma}$$

$$\frac{15}{5} \le Z$$

$$3 \le Z$$

look in table p.158



area = 0.0013

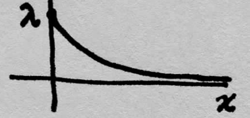
$P(3 \le Z) =$
$= 1 - P(Z \le 3)$
$= 1 - .9987$
$= 0.0013$

This is the prob "α" of a false rejection of H_0:
the prob that ~~H_0~~ the test results indicate we should reject H_0 when in fact H_0 is true.

Exponential RVs

For $\lambda > 0$ define

pdf $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x \text{ neg} \end{cases}$



$u = -\lambda x$
$du = -\lambda dx$

cdf $F(a) = P(X \leq a) = \int_{-\infty}^{a} \lambda e^{-\lambda x} dx = \int_{0}^{a} \lambda e^{-\lambda x} dx = -\int_{u=0}^{-\lambda a} e^{u} du$

$= e^{u}\Big|_{-\lambda a}^{0}$

$= \boxed{1 - e^{-\lambda a}}$

ex 4a Exponential RV usually represents the amount of time until some event occurs – with the 'memoryless' property that just because I waited a time t without it happening has no effect on the probability that it will happen in the next time interval (of length s, say). More on memoryless later.

Let X = time in minutes until phone rings and say $\lambda = 1/10$

Find $P(X > 10) = \int_{10}^{\infty} \frac{1}{10} e^{-1/10 x} dx = -\int_{u=-10\lambda}^{-\infty} e^{u} du = \int_{-\infty}^{-10\lambda} e^{u} du = e^{-10\lambda} = e^{-1}$

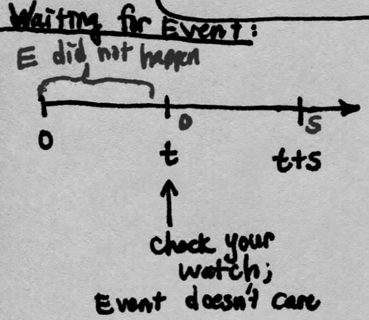
$\lambda = 1/10$

$P(10 < X < 20) = -\int_{-10\lambda}^{-20\lambda} e^{u} du = e^{-1} - e^{-2}$ □

Memoryless cond

$\boxed{\text{Want: } P(X > t+s \mid X > t) = P(X > s)}$

Geometric RVs are the only discrete ones with memoryless prop



$P(A|B) = \frac{P(A \cap B)}{P(B)}$ so

$X > t+s \Rightarrow X > t$

$P(X > t+s \mid X > t) = \frac{P(X > t+s \;\&\&\; X > t)}{P(X > t)} = \frac{P(X > t+s)}{P(X > t)}$

$= \frac{1 - F(t+s)}{1 - F(t)} = \frac{1 - (1 - e^{-\lambda(t+s)})}{1 - (1 - e^{-\lambda t})}$

$= \frac{+e^{-\lambda(t+s)}}{e^{-\lambda t}}$

$= e^{-\lambda s} = P(X > s)$ □

Thus $\frac{P(X > t+s)}{P(X > t)} = P(X > s)$

$P(X > t+s) = P(X > t) P(X > s)$

This works: $e^{-\lambda(t+s)} = e^{-\lambda t} e^{-\lambda s}$

What other fcns could satisfy this eq? See next page →

We just saw the memoryless cond was equivalent to:

$$P(X > t+s) = P(X>t)\cdot P(X>S) \qquad \text{Let } G(u) := P(X>u)$$

$$\Rightarrow G(t+s) = G(t)\cdot G(s)$$

What continuous fcns could work here? [Ross p.166 says 'Right continuous']

Observe $G(\frac{2}{n}) = G(\frac{1}{n}+\frac{1}{n}) = G(1/n)\cdot G(1/n) = G^2(1/n)$

then $G(1) = G(n/n) = G^n(1/n) \Rightarrow G(1/n) = G(1)^{1/n}$

then $G(m/n) = [G(1)]^{m/n}$

Any $x \in \mathbb{R}$ can be approximated by a seq of rationals $(\frac{m_i}{n_i}) \to x$

$$G(x) = G(\lim_i \frac{m_i}{n_i}) \underset{\substack{\uparrow \\ G \text{ cont}}}{=} \lim_i G(\frac{m_i}{n_i}) = \lim_i [G(1)]^{\frac{m_i}{n_i}} = G(1)^x$$

Thus $G(x) = G(1)^x \Rightarrow G(x) = a^x$ where $a = G(1)$ and $a \geq 0$ because $G(1) = G(1/2)^2 \geq 0$

we know
$$a = e^{\ln(a)}$$
$$a^x = e^{\ln(a)x} \qquad \text{Define } \lambda = -\ln(a) \qquad \lambda > 0 \text{ if } a \in (0,1)$$
$$a^x = e^{-\lambda x}$$

$\Rightarrow G(x) = e^{-\lambda x}$ and since $G(x) = P(X>x)$

the cdf is $F(x) = P(X \leq x) = 1 - P(X>x) = 1 - G(x)$
$= 1 - e^{-\lambda x}$ □

p.169 Hazard Rate fcns

RV X = lifetime of a component, let cdf be F and f is pdf.

time of failure

Failure rate $\lambda(t) := \frac{f(t)}{1-F(t)}$

Say component has survived for t hours. what is prob it will fail in the next time increment dt?

$$P(X \in (t, t+dt) \mid X > t) = \frac{P(X \in (t, t+dt))}{P(X>t)} = \frac{f(t)\,dt}{1-F(t)} = \lambda(t)\,dt$$

Now if $X \sim \exp(\lambda)$, previous survival time should make no difference

$$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda e^{-\lambda t}}{1-[1-e^{-\lambda t}]} = \lambda\frac{\cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}} = \lambda \quad \text{const.} \checkmark$$

<u>Thm</u> a Failure rate fcn $\lambda(t)$ uniquely determines cdf F

pf we know $\lambda(t) = \frac{\frac{d}{dt}F(t)}{1-F(t)} \Rightarrow \frac{1}{1-F}\,dF = \lambda\,dt \quad \left[\begin{array}{l} u = 1-F \\ du = -dF \end{array}\right]$

$$\ln(1-F) = -\int_0^t \lambda(t)\,dt + k \qquad \text{plug in } t=0,\ F(0)=0$$

$$\ln(1) = \int_0^0 + k \Rightarrow k = 0$$

$$1 - F(t) = e^{-\int_0^t \lambda(t)\,dt}$$

$$\Rightarrow \boxed{F(t) = 1 - e^{-\int_0^t \lambda(t)\,dt}}$$ □

p.170 **Gamma Distribution**

$X \sim \Gamma(t,\lambda)$ $\quad \lambda > 0$, $t > 0$

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{t-1}}{\Gamma(t)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\Gamma(t) = \int_0^\infty e^{-y} y^{t-1}\,dy$ Gamma fcn

Assuming technical conds of p. 129, the amount of time one has to wait for n events is given by $\Gamma(n,\lambda)$.

Using integration by parts, we can show $\Gamma(t) = (t-1)\Gamma(t-1)$ $\quad \Gamma(1) = \int_0^\infty e^{-x}dx = 1$

thus $\Gamma(n) = (n-1)!$

$\Gamma(n/2, 1/2)$ is $\chi^2_{(n)}$ distrib

▷ **Weibull distrib** failure time of complex system with weakest link

$X \sim W(\nu, \alpha, \beta)$

$$F(x) = \begin{cases} 0 & x \leq \nu \\ 1 - e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} & x > \nu \end{cases}$$

$$f(x) = \begin{cases} 0 & x \leq \nu \\ \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} & x > \nu \end{cases}$$

▷ Cauchy distrib

$$f(x) = \frac{1}{\pi}\frac{1}{[1+(x-\theta)^2]}$$

▷ Beta distrib

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \end{cases}$$

where $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$

Ross P.
Ch 5.6

Functions of a RV

10/20/2018

Also based on Rice MSADA Ch 2.3 and online book 'Intro to Prob, Stats and Random Processes' §4.1.3

$$\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$Y = g(X)$

The basic steps are: (1) Find cdf of transformed var.
(2) Differentiate to get pdf
(3) specify region where result holds.

▷ Let $\boxed{Y = aX + b}$ where $a > 0$

$F_Y(y) := P(Y \le y) = P(aX + b \le y) = P(X \le \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

A priori, $f_Y(y) = \frac{d}{dy} F_Y(y)$ and we have $F_X(s) = \int_{-\infty}^{s} f_X(t)\,dt$ Let $s = s(y) = \frac{y-b}{a}$

$$f_Y(y) = \frac{d}{dy}\int_{-\infty}^{s(y)} f_X(t)\,dt \overset{\text{Leibniz}}{\underset{\text{Integration rule}}{=\!=}} f_X(s(y))\,s'(y) = \boxed{\frac{1}{a} f_X\left(\frac{y-b}{a}\right)}$$

Holds for any X.

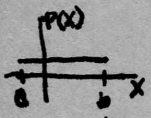
In Rice's example, take $X \sim \mathcal{N}(\mu, \sigma) \Rightarrow f_Y(y) = \frac{1}{a}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(\frac{y-b}{a} - \mu\right)^2}{\sigma^2}}$

$\frac{y - (\mu a + b)}{a\sigma}$

For $a < 0$, we'd get
$P(X \ge \frac{y-b}{a}) = 1 - P(X < \frac{y-b}{a})$

so we see $Y \sim \mathcal{N}(\mu a + b, a\sigma)$ just like on Sheet 14.

▷ $X \sim Unif(a,b)$ $\boxed{Y = X^2}$



Fix y

$F_Y(y) = P(X^2 \le y)$ If $y < 0$, no values of $X \in \mathbb{R}$ satisfy $X^2 \le y$ so $P(X^2 \le y) = 0$

If $y \ge 0$ $X^2 \le y \Leftrightarrow |X| \le \sqrt{y} \Leftrightarrow -\sqrt{y} \le X \le \sqrt{y}$
(X may be neg, not X^2)

$|a| < b$

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_a^{\sqrt{y}} \frac{1}{b-a}\,dx - \int_a^{-\sqrt{y}} \frac{1}{b-a}\,dx \quad (\text{provided } \sqrt{y} < b)$$

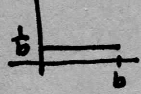
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) =$$

$$= \frac{1}{b-a}\left(\sqrt{y} - \cancel{a} - (-\sqrt{y} - \cancel{a})\right) = \frac{2}{b-a}\sqrt{y}$$

Thus

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{2}{b-a}\sqrt{y} & y < a^2 < b^2 \\ \frac{1}{b-a}\sqrt{y} - a & a^2 < y < b^2 \\ 1 & y > b \end{cases} \Rightarrow f_Y(y) = \begin{cases} \sim \frac{1}{2} y^{-1/2} \\ \sim \\ 0 \end{cases}$$

▷ $X \sim Unif(0,b)$ $\boxed{Y = e^X}$



$f_X(x) = \begin{cases} 1/b & x \in (0,b) \\ 0 & \text{otherwise} \end{cases}$

$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = F_X(\ln y) = \int_{-\infty}^{\ln y} \frac{1}{b}\,dx = \frac{1}{b}\ln y$ provided $\ln(y) \le b$

observe e^X never neg, so $y \overset{!}{\ge} 0$. Also domain $[0,b] \xrightarrow{e^x} [1, e^b]$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 1 \\ \frac{1}{b}\ln(y) & 1 < y < e^b \\ 1 & e^b < y \end{cases} \overset{\frac{d}{dy}}{\Longrightarrow} f_Y(y) = \begin{cases} b/y & 1 \le y \le e^b \\ 0 & \text{otherwise} \end{cases}$$

10/20/2018

▷ Let $X \sim \mathcal{N}(0,1)$ $\begin{matrix}\mu=0\\ \sigma=1\end{matrix}$ $\quad Y = X^2$

$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$

Then $\frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\infty}^{\sqrt{y}} \varphi(t)\,dt - \frac{d}{dy} \int_{-\infty}^{-\sqrt{y}} \varphi(t)\,dt$ where $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$

Leibniz Integration rule $\quad = \varphi(y^{1/2}) \frac{1}{2} y^{-1/2} - \varphi(-y^{1/2})(-\frac{1}{2} y^{-1/2}) = \boxed{\frac{1}{\sqrt{y}} \varphi(\sqrt{y})}$

Fisz p. 60-61 $\quad Z = XY \quad$ This is now 2-dim

$z = f(x,y)$ should be in next chapter writeup