P.138 12 **Maximum liklihood for Poisson RV:**

$$f(k) = P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

Given k, what value of λ maximizes this quantity?

$$g(\lambda) := e^{-\lambda}\frac{\lambda^k}{k!} \qquad g'(\lambda) = \frac{1}{k!}\left[-e^{-\lambda}\lambda^k + e^{-\lambda}k\lambda^{k-1}\right]$$

$$= \frac{e^{-\lambda}\lambda^{k-1}}{k!}\left[k-\lambda\right] \overset{!}{=} 0 \Rightarrow \lambda = k$$

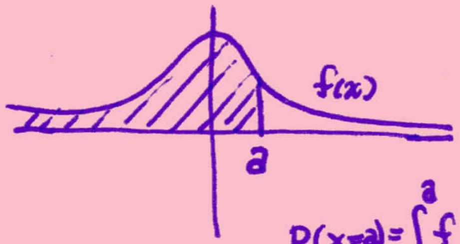
observe that for Poisson to approximate Binomial, we said $\lambda = np$.
Thus for this maximizer $\lambda = k$, we'd have $k = np$ or $p = \frac{k}{n}$ which we found previously. □

Ch 5 Continuum RVs

Here we have a pdf $f_X : \mathbb{R} \to [0,\infty)$ which is a measure or "density" weight fcn for the probability of the pts in the domain (Note: any single pt has m 0!)

$$P(a \le X \le b) := \int_a^b f(x)\,dx$$

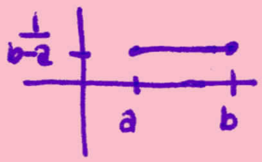
cdf $P(X \le a) =: F(a) = \int_{-\infty}^{a} f(x)\,dx$



$$P(X=a) = \int_a^a f = 0$$

Uniform RV

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} f = 1$$

P.153 example: Bus stop
p. 181 #9 I arrive at bus stop at 10:00 AM. The time the bus actually arrives is a unif RV over interval [10:00, 10:30] or in minutes [0, 30]

Let $X :=$ time I have to wait

$$P(X > 10) = P\left(\begin{smallmatrix}\text{bus arrives}\\ \text{in } [10,30]\end{smallmatrix}\right) = \int_{10}^{30} \frac{1}{30}\,dx = \frac{20}{30} = \frac{2}{3}$$

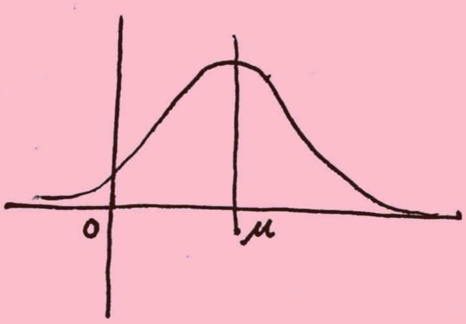
But the real question would be: The bus is scheduled at 10:15 and randomly arrives between [10:00, 10:30]. I randomly arrive between [10, 10:30] what is prob I will catch the bus? (ie bus arrives after I do)

To solve this we need joint pdfs see p. 196

Normal RVs

$X \sim \mathcal{N}(\mu,\sigma)$ pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

P.155



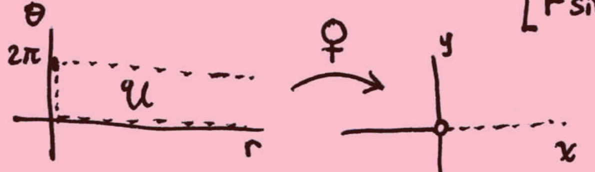
Lets show $\int_{-\infty}^{\infty} f(x)\,dx = 1$:

$$I := \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \quad \text{COV } y := \frac{x-\mu}{\sigma} \quad dy = \frac{1}{\sigma}dx$$

square both sides:

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt = \frac{1}{2\pi} \iint e^{-\frac{1}{2}(y^2+t^2)} dy\,dt$$

Make polar COV $\varphi(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} y \\ t \end{bmatrix}$ $\quad D\varphi_{r\theta} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ $\quad \det D\varphi_{r\theta} = r$



$$\int_{\varphi(\mathcal{U})} f(x)\,d^2x = \int_{\mathcal{U}} f(\varphi(u))\,|\det D\varphi_{r\theta}|\,dr\,d\theta$$

$$\Rightarrow I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2+t^2)} dy\,dt \overset{\text{COV}}{=} \frac{1}{2\pi} \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta = \frac{1}{2\pi}\left[2\pi \int_{r=0}^{\infty} e^{-\frac{1}{2}r^2} r\,dr\right]$$

Now $u := -\frac{r^2}{2}$ $\quad du = -r\,dr$

$r=0 \Rightarrow u=0$

$r=\infty \Rightarrow u = -\frac{1}{2}\infty^2 = -\infty$

$$= -\int_{u=0}^{-\infty} e^u\,du = \int_{u=-\infty}^{0} e^u\,du = e^u\Big]_{-\infty}^{0} = 1-0 = 1$$

$I^2 = 1$

$I = \pm 1$ but it's $+1$ because curve is above x axis, so pos area. □

Thm a linear combination of Normal RVs is a Normal RV:

P.156

$$\left.\begin{array}{l} X \sim \mathcal{N}(\mu,\sigma) \\ Y := aX+b,\ a>0 \end{array}\right\} \Rightarrow Y \sim \mathcal{N}(a\mu+b,\ a\sigma)$$

pf. $F_Y(r) = P(Y \le r) = P(aX+b \le r) = P\left(X \le \frac{r-b}{a}\right)$ because $aX+b \le r \Leftrightarrow aX \le r-b$

$$= F_X\left(\frac{r-b}{a}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{r-b}{a}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Make COV $y = ax+b$ $\quad dy = a\,dx$

$\Rightarrow \frac{y-b}{a} = x$ $\quad x = \frac{r-b}{a} \Rightarrow y = r$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-\frac{1}{2}\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right)^2} \frac{1}{a}\,dy$$

$$= \frac{1}{a\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-\frac{1}{2}\left(\frac{y-\boxed{(a\mu+b)}}{a\sigma}\right)^2} dy$$

This is the form for $\mathcal{N}(a\mu+b,\ a\sigma)$ □

Normalizing a Normal RV

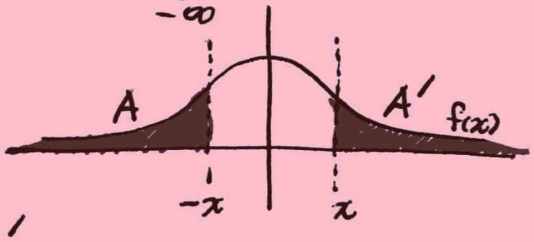
$a > 0$

Given $X \sim \mathcal{N}(\mu, \sigma)$ we want to define $Z := aX + b$ so $Z \sim \mathcal{N}(0,1)$

we just showed $aX + b \sim \mathcal{N}(a\mu + b, a\sigma)$ so take $a := \frac{1}{\sigma}$ $b := \frac{-\mu}{\sigma}$

$\Rightarrow Z := \frac{1}{\sigma}X - \frac{\mu}{\sigma} = \boxed{\frac{X - \mu}{\sigma}}$ is distributed $\mathcal{N}(0,1)$

The CDF for the $\underbrace{\text{std Normal distrib}}_{\mathcal{N}(0,1)}$ is $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$



Lemma $\boxed{\text{Symmetry of } \Phi : \quad \Phi(-x) = 1 - \Phi(x)}$

Pf Since f has reflection symm about y axis, the shaded area under the curve, A, reflects to A'

$\Rightarrow \; m(A) = m(A')$

$\Phi(-x) = m(A)$, $m(A') = 1 - \Phi(x)$ because $\Phi(x) + m(A') \overset{!}{=} 1$ $\square$

p.159 example 3a $\quad X \sim \mathcal{N}(\overset{\mu}{3}, \overset{\sigma}{3})$

(i) Find $P(2 < X < 5)$

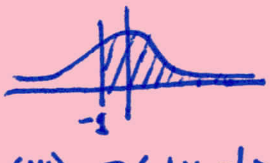
Normalize to $\mathcal{N}(0,1)$ so we can use table on p.158 $\quad Z := \frac{X - \mu}{\sigma}$

$$2 < X < 5 \Rightarrow \frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3} \Rightarrow -\frac{1}{3} < Z < \frac{2}{3}$$

$$P(-\tfrac{1}{3} < Z < \tfrac{2}{3}) = \Phi(\tfrac{2}{3}) - \Phi(-\tfrac{1}{3})$$
$$= \Phi(\tfrac{2}{3}) - [1 - \Phi(\tfrac{1}{3})] \quad \text{Symm Lemma}$$
$$= \underbrace{\Phi(0.667)}_{0.7486} - [1 - \underbrace{\Phi(0.333)}_{0.6293}] = 0.3779 \checkmark$$

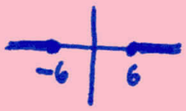
(ii) $P(X > 0)$

$$0 < X \Rightarrow \frac{0-3}{3} < \frac{X-3}{3} \Rightarrow -1 < Z$$



$$P(-1 < Z) = 1 - P(Z \leq -1) = 1 - \Phi(-1) \overset{\text{Symm lemma}}{=} \Phi(1) = 0.8413$$

(iii) $P(|X-3| > 6)$ $\quad 6 < |x-3|$, $\quad 6 < |u|$



$u < -6$ OR $6 < u$

$(-\infty, -6) \cup (6, \infty)$

So here we get $X - 3 < -6$ and $6 < X - 3$

$\frac{X-3}{3} < -\frac{6}{3} = -2$ and $2 < \frac{X-3}{3}$

$Z < -2$ and $2 < Z$

$$P(|X-3| > 6) = P(Z < -2) + P(Z > 2)$$

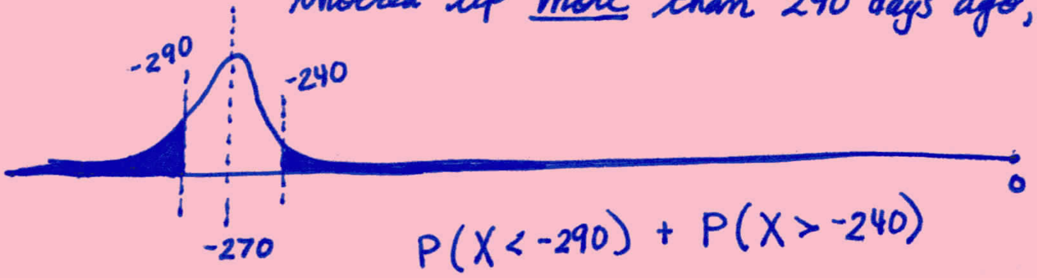
$\Phi(-2) \qquad 1 - P(Z \leq 2)$

$1 - \Phi(2) + \qquad 1 - \Phi(2)$

$2(1 - \Phi(2))$

P.160 example 3c The length of a pregnancy is a Normal RV with $\mu = 270$ days and $\sigma = 10$

a woman gives birth today. what is the prob she was knocked up more than 290 days ago, or less than 240 days ago?



$$P(X < -290) + P(X > -240)$$

$$\frac{X - -270}{10} < \frac{-290 + 270}{10} \qquad \frac{X - -270}{10} > \frac{-240+270}{10}$$

$$Z < -2 \qquad\qquad Z > 3$$

$$P(Z<-2) + P(Z>3)$$
$$= \Phi(-2) + 1 - \Phi(3) = 1 - \Phi(2) + 1 - \Phi(3) = 0.0241$$

P.161 Here are some bounds for Φ that the author said were important:

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)e^{-\frac{1}{2}x^2} < 1 - \Phi(x) < \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{1}{2}x^2} \quad \forall x > 0$$

▷ Special Case of Central Lim Thm — DeMoivre-Laplace limit thm

$S_n :=$ # of wins in n trials
Prob p

$$P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \xrightarrow{n\to\infty} \Phi(b) - \Phi(a)$$

we can approximate Binomial $Bin(n,p)$:

- with Poisson when n large, np moderate
- with Normal $np(1-p)$ large (rule of thumb: $np(1-p) \ge 10$)

Here is a statement of CLT:

Seq of RVs (X_i) i.i.d.
(each has mean μ and var σ^2)

$$P\left(\frac{\sum^n X_i - n\mu}{\sigma\sqrt{n}} \le a\right) \xrightarrow{n\to\infty} \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{1}{2}x^2}dx}_{\Phi(a)} \quad \text{p.157}$$

example 3e Flip coin $n = 40$ times
$p = \frac{1}{2}$ $X =$ # of heads i.e. wins
we take $\mu = np = 40 \cdot \frac{1}{2} = 20$ $\quad \sigma := \sqrt{n(p)(1-p)} = \sqrt{40 \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{10}$

Normal approx
$$P(X=20) = P(19.5 < X < 20.5)$$
$$= P\left(\frac{19.5-20}{\sqrt{10}} < Z < \frac{20.5-20}{\sqrt{10}}\right)$$
$$= P(-0.16 < Z < 0.16)$$
$$= \Phi(0.16) - \Phi(-0.16)$$
$$= \Phi(0.16) - [1 - \Phi(0.16)] = 0.1272$$

□