AOM §38 REWRITE – see sheets (12)-(14) 3/8/2026

(p.315) COR 38.4 to Generalized Cross Prod lemma (11)

M

Let S be a $(n-1)$ dim $\mathcal{O}^R$ mfd in $\mathbb{R}^n$ (maybe $S=\partial M$, maybe not)
Let α be a chart in the pos $\mathcal{O}^R$ on S
Let $\vec{V}_i = D\alpha_x(e_i) =: \vec{\partial}_i \quad \forall i=1,\dots n-1$

$\Rightarrow$

- we now have an explicit expression for OPN $\vec{n}$ on S: $\vec{n} = \chi(\partial_1 \dots \partial_{n-1})$
- The Gauss map (doCarmo) $n: p \mapsto \hat{n}_p$ is C^∞ smooth because it is composition of det and α

[we know $\vec{n}$ points outward because $\det[n, \partial_1 \dots \partial_{n-1}] > 0$ by Cond (4) cf Lemma 38.7]

(p.318) Big Lemma 38.7

M n-mfd in $\mathbb{R}^n$
M has Nat $\mathcal{O}^R$

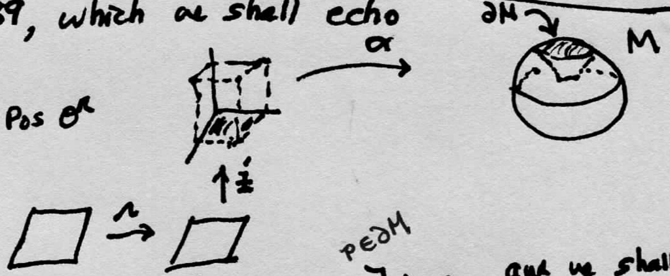
$\Rightarrow$ we know an $\mathcal{O}^R$ is induced on ∂M
This $\mathcal{O}^R_i(\partial M)$ is given by (1) OPN $\hat{n}$
(2) $\hat{n}$ is obtained from the GCP formula

Pf. Most of this was established in (ex 4) p.289, which we shall echo

Let $p \in \partial M$ $\alpha(x) = p$

$\alpha: \mathcal{U}_H \to V$ $\quad \det[D\alpha_x] > 0 \ \forall x$ since $\alpha \in$ Pos $\mathcal{O}^R$

$\det[\partial_1 \dots \partial_n] > 0$



Key Relation: For such an α, we decree n is OPN if it satisfies $\det[n, \partial_1 \dots \partial_{n-1}]_p > 0$, $p \in \partial M$, and we shall justify why this is outward below

To avoid Munkres' $(-1)^n \vec{n}$, I shall consider 2 cases

dim M = n even – no reversing

$\det[n, \partial_1 \dots \partial_{n-1}]_p \overset{!}{>} 0$

$= \det[A_n] = \langle n, \chi(\partial_1 \dots \partial_{n-1}) \rangle \overset{!}{>} 0$

Set $\boxed{n = \chi}$ this guarantees IP pos $\langle n, \chi \rangle = \langle \chi, \chi \rangle$

How do we know n is OPN?

We know $\det[\overbrace{\partial_1 \dots}^{n-1} \partial_n] > 0$ since α pos $\mathcal{O}^R$

$\qquad \| $

$(-1)^{n-1} \det[\partial_n, \partial_1 \dots \partial_{n-1}] > 0$

$= (-1)^{n-1} \det[A_{\partial_n}]$

$n-1$ odd

$= -\langle \partial_n, \chi \rangle = \langle \partial_n, -\chi \rangle > 0$

$\langle \partial_n, -n \rangle > 0$

This says $-n$ has a component in the $\vec{\partial}_n$ direction, and ∂_n points IN, so $-n$ points IN

$\Rightarrow +\vec{n}$ points OUT

$n = \chi$ is OPN

det M = n odd so we reverse $\mathcal{O}^R$ on $\hat{\alpha}$

$\det[n, \ominus\partial_1, \partial_2 \dots \partial_{n-1}]_p \overset{!}{>} 0$

$= \det[-n, \partial_1 \dots \partial_{n-1}] > 0$

$= \det[A_{-n}] = \langle -n, \chi \rangle \overset{!}{>} 0$

Set $-n := \chi$ that is $\boxed{n = -\chi}$

$(-1)^{n-1} \det[\partial_n, \partial_1 \dots \partial_{n-1}] > 0$

$n-1$ even

$\Rightarrow + \det[\partial_n, \partial_1 \dots \partial_{n-1}] > 0$

$\langle \partial_n \chi \rangle > 0$

$\langle \partial_n, -n \rangle > 0$

so again $-n$ has a compoint pointing along $\vec{\partial}_n$ so that component points IN

$\Rightarrow +\vec{n}$ points OUT

$n = -\chi$ is OPN here

3/14/2026

Let's recap the essential ideas, and then test them out on sphere example

If (1) α is a chart in the pos $\mathcal{O}^R$
(2) $\vec{n}$ OPN
(3) $p \in \partial M$
$\Rightarrow$ Key Relation holds

$$\underbrace{\det\left[n_p, \partial_1(p) \ldots \partial_{n-1}(p)\right]}_{= \langle n, q\rangle_p = \langle n, X(a_1 \ldots a_{n-1})\rangle_p} > 0$$

If we know n is OPN, we can use this to see if α is in pos $\mathcal{O}^R$

If we know α has pos $\mathcal{O}^R$, we can use key relation to determine OPN $\vec{n}$

To make $\vec{n} \perp$ to all $\vec{\partial}_i(p)$ and $\langle n, q\rangle_p > 0$ we take $\vec{n} := \vec{q} = X_p(a_1 \ldots a_{n-1}) = \text{GCP}$

Does the dimension n matter? NOT to GCP, but what matters is if α is a pos chart for a particular n dimension.

Let's continue with Munkres' running sphere param example

p. 280 #3
p. 292 #8
p. 296 #3
p. 321 #4
p. 309

Consider $S^{n-1} \subset \mathbb{R}^n$

$S^{n-1} = \partial B^n$ so if B^n has Nat $\mathcal{O}^R$ then S^{n-1} is given induced $\partial\, \mathcal{O}^R$ by OPN

a priori, for the sphere we know $\vec{n}(x) = \vec{x} = \vec{r}$ radial vector

p. 292 #8 Consider our familiar param φ of the top half, and its partner Ψ for bot half:

$$\varphi : \underset{\mathbb{R}^{n-1}}{\dot{B}^{n-1}} \longrightarrow \underset{\mathbb{R}^n}{S^{n-1}}$$

$$u \longmapsto \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ +\sqrt{1-\|u\|^2} \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ +\rho^{1/2} \end{bmatrix} \quad \text{where } \rho := 1-u_1^2 \ldots - u_{n-1}^2$$

$$\Psi : B^{n-1} \longrightarrow S^{n-1}$$

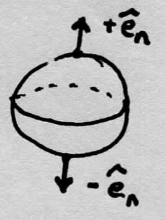
$$u \longmapsto \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ -\rho^{1/2} \end{bmatrix}$$

Here φ is NOT the restriction of a map α for an n-dim M

We will again find that $\mathcal{O}^R$ is fiendishly tricky — φ has pos $\mathcal{O}^R$ only if n is odd
Ψ " " " " " n is even

showing $n=5$

$$D\varphi_u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-u_1}{\rho^{1/2}} & \frac{-u_2}{\rho^{1/2}} & \frac{-u_3}{\rho^{1/2}} & \frac{-u_4}{\rho^{1/2}} \end{bmatrix} \quad \text{and} \quad D\Psi_u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{+u_1}{\rho^{1/2}} & \frac{+u_2}{\rho^{1/2}} & \frac{+u_3}{\rho^{1/2}} & \frac{+u_4}{\rho^{1/2}} \end{bmatrix}$$



- We choose a particularly easy pt to evaluate at: North Pole for φ (co-ord $\vec{u} = \vec{0}$) and South Pole for Ψ (again $\vec{u} = \vec{0}$)
- We know *a priori* OPN $\vec{n} = +\hat{e}_n$ at North P and $\hat{n} = -\hat{e}_n$ at South P

$D\varphi_0 = [e_1 \ldots e_{n-1}]$ $\quad \rho^{1/2}(0) = 1$ $\quad$ and same for $D\Psi_0 = [e_1 \ldots e_{n-1}]$

cont'd →

$D\varphi_x(e_1)$

Observe Identity: $\det\left[\vec{s}, \varphi_{x_1}, \cdots \varphi_{x_{n-1}}\right] \equiv \langle s, \chi(\varphi_{x_1} \cdots \varphi_{x_{n-1}})\rangle$

Just $\det[a\ b\ c] = a\cdot(b\times c)$

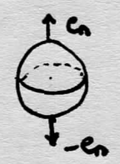
If $\varphi \in Pos\ \mathcal{O}^R$, take $\vec{s} = \chi(\varphi_{x_1} \cdots \varphi_{x_{n-1}})$

then $s \perp Span\{\varphi_{x_1}, \cdots \varphi_{x_{n-1}}\}$ and $\langle \chi, \chi\rangle > 0$

$\Rightarrow \vec{s}$ is OPN.

Continuing from last page:

$$D\Phi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-u_1}{\rho^{1/2}} & \frac{-u_2}{\rho^{1/2}} & \frac{-u_3}{\rho^{1/2}} & \frac{-u_4}{\rho^{1/2}} \end{bmatrix} \underset{\substack{\text{plug}\\\text{in}\\0}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D\Psi_0$$



$$= {}_n\big[\overbrace{e_1\ e_2 \cdots e_{n-1}}^{n-1}\big]_{p=0} \quad \text{and} \quad \rho^{1/2}(0) = 1$$

For top hemisphere w/ $\boxed{\varphi}$, we know $\hat{n} = \hat{e}_n$ at North Pole $(u=0)$

Then the Key Relation becomes $\det\big[e_n, \overbrace{e_1, \dots, e_{n-1}}^{n-1}\big] = (-1)^{n-1}\det I$ $= (-1)^{n-1}$ Pos iff n odd

For bot hemisphere w/ $\boxed{\Psi}$, $\hat{n} = -\hat{e}_n$ at $u=0$

Key Relation $\det\left[-e_n, e_1 \dots e_n\right] = (-1)(-1)^{n-1}\det I$ $= (-1)^n$ Pos iff n even

$\Rightarrow$ φ and Ψ are never <u>both</u> pos $\mathcal{O}^R$ for any n — always we must reverse one or the other.

▷ Lets do some explicit Calculations to validate $\vec{n}_{out} = \chi(\varphi_{x_1} \cdots \varphi_{x_{n-1}})$ when φ pos

<u>Case $n=3$</u> $\quad D\Phi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = D\Psi_0 \qquad n \overset{?}{=} \chi(e_1, e_2) = \det\begin{bmatrix} i & 1 & 0 \\ j & 0 & 1 \\ k & 0 & 0 \end{bmatrix} \overset{\text{transp}}{=} \det\begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{e}_3$ Correct for φ, it is in pos $\mathcal{O}^R$

<u>NOT</u> correct for Ψ but Ψ not in pos $\mathcal{O}^R$ (Correct value $\hat{n} = -\hat{e}_3$)

<u>Case $n=4$</u> $\quad D\Phi_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = D\Psi_0$ but <u>only Ψ pos $\mathcal{O}^R$ for $n=4$</u>

Zero row

$$\chi(e_1 e_2 e_3) = \begin{bmatrix} + & 1 & 0 & 0 \\ - & 0 & 1 & 0 \\ + & 0 & 0 & 1 \\ - & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} +\det\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \\ -\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \\ +\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \\ -\det\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -\hat{e}_4$$

This is the wrong (opposite) for φ but right for Ψ

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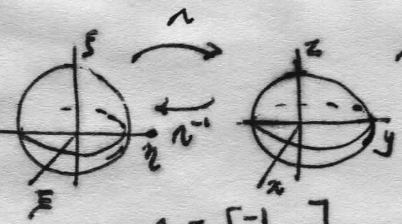
14

So if we reversed the $\mathcal{O}^R$ on φ:

cf §33 sheet 5



$\xrightarrow{\varphi} S^3 \subset \mathbb{R}^4$

$\rho(n(\xi)) = \rho(x)$

$n = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $\quad n(\xi,\eta,\zeta) = \begin{bmatrix} -\xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\tilde{\varphi}$

$\varphi \circ n(\xi,\eta,\zeta) = \varphi(-x,y,z) = \begin{bmatrix} -x \\ y \\ z \\ \rho^{1/2} \end{bmatrix}$

$D\tilde{\varphi}_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\chi(\tilde{\varphi}_{x_1}\ \tilde{\varphi}_{x_2}\ \tilde{\varphi}_{x_3}) = \chi(-e_1, e_2, e_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = +e_4$$

and this is correct for top hemisphere

$n=5$

For $\chi(e_1\, e_2\, e_3\, e_4)$ here we must compute the det of

$$\det\begin{bmatrix} + & 1 & 0 & 0 & 0 \\ - & 0 & 1 & 0 & 0 \\ + & 0 & 0 & 1 & 0 \\ - & 0 & 0 & 0 & 1 \\ + & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \chi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ +\det\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{bmatrix} = \hat{e}_5$$

The pattern is clear, the zero row appears in all terms (except the last one) and kills them. The sign is determined by the checkerboard pattern.

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p. 296 #3 §35 With these maps φ and ψ we shall continue Continuing Prob #3 §33 p. 280 for $S^{n-1} \subset \mathbb{R}^n$

Let $(n-1)$ form $\eta = \sum_{i=1}^{n} (-1)^{i-1} f_i\, dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n$ $\longleftarrow$ This corresponds to vf $\vec{F} = \frac{1}{r^m}\vec{r}$

where $f_i = \frac{1}{\|x\|^m} x_i$ (defined on $\mathbb{R}^n - \{0\}$)

we want to show $\int_{S^{n-1}} \eta \neq 0$ with the following steps:

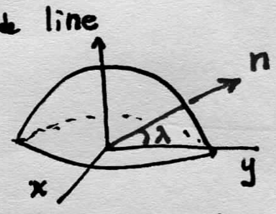
(a) Define $g = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}$ and show $\psi = g \cdot \varphi$ (trivial, I'll skip)

Show ψ has opp $\mathcal{O}^R$ from φ: [we already showed this on sheets 13 – 14.]

Here is one idea though, related to sheets 3 – 4 of §33

$D\varphi_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\rho^{1/2}} & \frac{-y}{\rho^{1/2}} \end{bmatrix}$ $\quad \vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & -x/\rho^{1/2} \\ 0 & 1 & -y/\rho^{1/2} \end{vmatrix} = \begin{bmatrix} x/\rho^{1/2} \\ y/\rho^{1/2} \\ 1 \end{bmatrix}$

Slide this to the equator along a latitude line



we can't directly evaluate at $x=0$ $y=1$ because $\frac{1}{0}$

But we can param the latitude curve $\begin{matrix} x=0 \\ y=\cos\lambda \end{matrix}$ $\quad \rho^{1/2} = \sqrt{1-\cos^2\lambda} = \sin(\lambda)$

$\vec{n}(\lambda) = \begin{bmatrix} 0 \\ \frac{\cos\lambda}{\sin\lambda} \\ 1 \end{bmatrix} = \frac{1}{\sin\lambda}\begin{bmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{bmatrix} \xrightarrow{\lambda\to 0} \infty \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ but if first we normalized $\hat{n}(\lambda) = \begin{bmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So using this idea we could show the normal from ψ comes up from the South to $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ and they are incompatible $\square$

(b) *Show* $\psi^*\eta = -\varphi^*\eta$ and conclude $\int_{S^{n-1}} \eta = 2\int_{B} \varphi^*\eta$

we know $(f^*\omega)_x(v_1, v_2) = \omega_{f(x)}(Df_x(v_1), Df_x(v_2))$

Now $\psi^*\eta = (g\circ\varphi)^*\eta = \varphi^*(g^*\eta)$. Lets look at $g^*\eta$ and just take 2 vectors for args

$$(g^*\eta)_z(v_1, v_2) = \eta_{g(z)}(\underbrace{Dg_z(v_1)}_{g(v_1)}, \underbrace{Dg_z(v_2)}_{g(v_2)}) \text{ since } g \text{ linear}$$

G&P Det Thm p.160
Spivak COM p.82
Related to Munkres p.269, 273

$$= (\det g)\,\eta(v_1, v_2) = (-1)\,\eta(v_1, v_2) \Rightarrow g^*\eta = -\eta$$

We know $\int_{S^{n-1}} \eta = \int_{B} \varphi^*\eta + -\underbrace{\int_{B} \psi^*\eta}$ we put the neg sign because ψ has neg or.

$$-\int_{B} (g\circ\varphi)^*\eta \longrightarrow \varphi^*(g^*\eta) = \varphi^*(-\eta) = -\varphi^*\eta$$

$$= \int_{B} \varphi^*\eta + {}^{+}\int_{B} \varphi^*\eta = 2\int_{B} \varphi^*\eta \qquad \square$$

sphere $S^3_{(0,R)}$
$S^{n-1} = S^3_{(0,R)}$

(c) *Show* $\int_{B} \varphi^*\eta = \pm \int_{B(0,R)} \frac{1}{\sqrt{R^2 - \|u\|^2}} \neq 0$ ← integrand is pos on a set of pos (m)

Depending on n, φ has pos or neg or

See Sheets §33 prob #3

$$\eta = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{\|x\|_2^m}\, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$$

we did in for n=3 →
Lets do it here for n=4

$$\varphi = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \rho^{1/2} \end{bmatrix} \qquad \rho = R^2 - u_1^2 - u_2^2 - u_3^2$$

$$\eta = \frac{x_1}{\|x\|^m} dx_2 \wedge dx_3 \wedge dx_4 - \frac{x_2}{\|x\|^m} dx_1 \wedge dx_3 \wedge dx_4 + \frac{x_3}{\|x\|^m} dx_1 \wedge dx_2 \wedge dx_4 - \frac{x_4}{\|x\|^m} dx_1 \wedge dx_2 \wedge dx_3$$

$$\varphi^*\eta = \varphi^*\left(\frac{x_1}{\|x\|^m} dx_2 \wedge dx_3 \wedge dx_4\right) - \varphi^*(\quad) + \varphi^*(\quad) - \varphi^*(\quad)$$

$$= \frac{x_1(\varphi)}{[\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2]^{m/2}} \frac{\partial(\varphi_2, \varphi_3, \varphi_4)}{\partial(u_1, u_2, u_3)}$$

$$D\varphi_u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-u_1}{\rho^{1/2}} & \frac{-u_2}{\rho^{1/2}} & \frac{-u_3}{\rho^{1/2}} \end{bmatrix} \quad \frac{-u_4}{\rho^{1/2}}$$

$$[u_1^2 + u_2^2 + u_3^2 + (R^2 - u_1^2 - u_2^2 - u_3^2)]^{m/2}$$

$= R^m$

$$\frac{\varphi_1}{R^m} \frac{\partial(\varphi_2, \varphi_3, \varphi_4)}{\partial(u_1 u_2 u_3)}$$

cont'd →

Ⓒ cont'd

$$\varphi^*\eta = \varphi^*\left(\frac{x_1}{\|x\|^m} dx_2\wedge dx_3\wedge dx_4\right) - \varphi^*\left(\frac{x_2}{\|\ \|} dx_1\wedge dx_3\wedge dx_4\right) + \varphi^*\left(\frac{x_3}{\|\ \|} dx_1 dx_2 dx_4\right) - \varphi^*\left(\frac{x_4}{\|\ \|} dx_1 dx_2 dx_3\right)$$

and $$D\varphi_u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-u_1}{\rho^{1/2}} & \frac{-u_2}{\rho^{1/2}} & \frac{-u_3}{\rho^{1/2}} \end{bmatrix}$$

$$= \frac{u_1}{R^m}\frac{\partial(\varphi_2\varphi_3\varphi_4)}{\partial(u_1 u_2 u_3)} - \frac{u_2}{R^m}\frac{\partial(\varphi_1\varphi_3\varphi_4)}{\partial(123)} + \frac{u_3}{R^m}\frac{\partial(\varphi_1\varphi_2\varphi_4)}{\partial(u_{123})} - \frac{\rho^{1/2}}{R^m}\frac{\partial(\varphi_1\varphi_2\varphi_3)}{\partial(u_{123})}$$

$$\frac{1}{R^m}\left(u_1 \det\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-u_1}{\rho^{1/2}} & \frac{-u_2}{\rho^{1/2}} & \frac{-u_3}{\rho^{1/2}} \end{vmatrix} - u_2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{-u_1}{\rho} & \frac{-u_2}{\rho} & \frac{-u_3}{\rho} \end{vmatrix} + u_3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-u_1}{\rho} & \frac{-u_2}{\rho} & \frac{-u_3}{\rho} \end{vmatrix} - \rho^{1/2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right)$$

$$= \frac{1}{R^m}\left(u_1\left(\frac{-u_1}{\rho^{1/2}}\right) - u_2\left(\frac{u_2}{\rho^{1/2}}\right) + u_3\left(\frac{-u_3}{\rho^{1/2}}\right) - \rho^{1/2}\cdot\frac{\rho^{1/2}}{\rho^{1/2}}\right)$$

$$= \frac{1}{R^m}\left(\frac{-u_1^2 - u_2^2 - u_3^2 - (R^2 - u_1^2 - u_2^2 - u_3^2)}{\rho^{1/2}}\right) = \frac{-1}{R^{m-2}}\frac{1}{\rho^{1/2}}$$

$$\Rightarrow \int_B \varphi^*\eta = \frac{-1}{R^{m-2}}\int_{B(0,R)}\frac{1}{\rho^{1/2}}$$ QED

Finally the last problem §38 #4 p.321: Given η and the ball $B^n_{(0,R)} \subset \mathbb{R}^n$ with Nat O^R, let $S^{n-1} = \partial B^n$ with $O_i^R(\partial B^n)$. We want to show $\boxed{\int_{S_R^{n-1}} \eta = v(S_R^{n-1})}$ (n-1) dim surf area.

cf. Spivak COM p.91

write $\eta = \frac{1}{\|x\|^m}\omega$ where $\omega = \sum(-1)^{i-1} x_i\, dx_1\wedge\cdots\widehat{dx_i}\cdots\wedge dx_n$

$$\int_{S_R^{n-1}}\eta = \int_{S_R^{n-1}}\frac{1}{R^m}\omega = \frac{1}{R^m}\int_{S_R^{n-1}}\omega \overset{\text{stokes}}{=} \int_{B_R} d\omega$$

$$= \int_{B_R} n\,\overbrace{dx_1\wedge\cdots\wedge dx_n}^{\text{vol form}} = n\,\mathrm{vol}(B^n_{(0,R)})$$

NOTE

Let $f = a\,dx + b\,dy$

$df = (a_x dx + a_y dy)\wedge dx + (b_x dx + b_y dy)\wedge dy$

$= -a_y\, dx\wedge dy + b_x\, dx\wedge dy$

So for us $d(x_1\, dy\wedge dz - x_2\, dx\wedge dz + x_3\, dx\wedge dy)$

$= dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy$

$= 3\, dx\wedge dy\wedge dz$

$d\omega = n(dx_1\wedge\cdots\wedge dx_n)$

The vol of an n-ball of radius R is $\alpha(n)R^n$ where $\alpha(n) := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

NOTE: $n\alpha(n)$ is exactly the surf area ("volume" in (n-1)dim) of unit sphere S_1^{n-1}, often denoted "$\mathrm{vol}(S^{n-1})$"

Thus $\int_{S_R^{n-1}}\eta = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}R^{n-m}$ (For n=3 this is $\int_{S_R^2}\eta = \frac{4\pi}{R^{m-3}}$) □

Soln by google Gemini 3.0 Pro

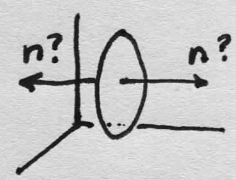
Now finally we are back to §34 p. 289-290 example 5

Let S be a 2-mfd w/ ∂ in $\mathbb{R}^3$

M is $\mathcal{O}^R$ so let ∂S have Induced $\mathcal{O}^R$

(For S, there is not Nat $\mathcal{O}^R$ in general, consider a disc

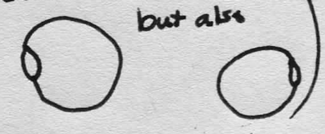
No natural dir for Outward



Since S is 2-dim in $\mathbb{R}^3$, there is $\hat{n}$ vf.

How do we know what is the induced $\mathcal{O}^R$?

(we could deform the disc but also

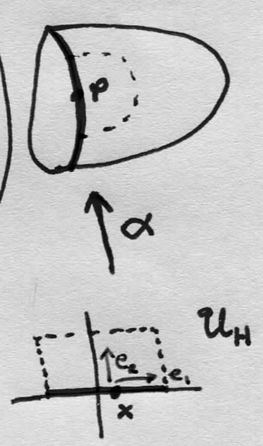


)

Thm The 1-dim mfd $\Gamma = \partial S$ is $\mathcal{O}^R$ by $\hat{t}$, where t is part of the ordered ON basis $\{n, t, w\}_p$

At pt $p \in \partial S$, $\hat{n}$ is from $\mathcal{O}^R$ of S

t and w lie in T_pS, but $w \perp T_p(\partial S)$ and w points into M



pf As usual we param the ∂ with special charts

$\alpha : \mathcal{U}_H \longrightarrow V$

Then $\alpha \circ \dot{\underline{t}} : \overset{\mathbb{R}}{I} \longrightarrow \mathbb{R}^3$ params a sector of ∂S (I is an interval

ie $\hat{\alpha} = \alpha\Big|_{\mathbb{R}\times\{0\}\cap\,\mathcal{U}_H}$ Then $D\alpha_x(e_1)$ is a basis for $T_p(\partial S)$

set $\vec{t} := D\alpha_x(e_1)$

We also know $D\alpha_x(e_2)$ points on the side of $T_p(\partial S)$ where S lies, i.e. it points INWARD (see the arg on sheet 5)

Set $\hat{w} := D\alpha_x(e_2)$ [This must be modified to ensure $\vec{w} \perp \vec{t}$]

We can complete the triad by $\vec{n} = \underbrace{D\alpha_x(e_1)}_{\vec{\alpha}_u} \times \underbrace{D\alpha_x(e_2)}_{\vec{\alpha}_v}$ we know this exists and is the proper $\vec{n}$ by the hypoth S is $\mathcal{O}^R$able.

~~we know n~~

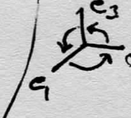
we can form the ordered triad $\{n, t, \hat{w}\}$ $\det[n, t, \hat{w}] > 0$

because $(\alpha_u \times \alpha_v)\cdot(\alpha_u \times \alpha_v) = \|\alpha_u \times \alpha_v\| > 0$.

~~But~~ we know $n \perp \operatorname{Span}\{\vec{\alpha}_u, \vec{\alpha}_v\}$ but how can we ~~find~~ modify $\hat{w}$ to w where $w \perp n$, $w \perp t$? Basically we just need $w \perp t$.

$$\det\begin{bmatrix} | & | & | \\ n & t & \hat{w} \\ | & | & | \end{bmatrix} = \det\begin{bmatrix} - & n & - \\ - & t & - \\ - & \hat{w} & - \end{bmatrix} = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\langle t, \hat{w}\rangle & 1 \end{bmatrix}\begin{bmatrix} n \\ t \\ \tilde{w} \end{bmatrix}\right) = \det\begin{bmatrix} n \\ t \\ w \end{bmatrix}$$

For a RH triad

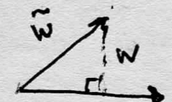


$e_1 \times e_2 = e_3$

$e_2 \times e_3 = e_1$

$e_3 \times e_1 = e_2$

Same for $\{n, t, w\}$



$\tilde{w} = P_w(t)\hat{t} + w$ Gram Schmidt

$\vec{w} = \tilde{w} - \langle \hat{t}, \tilde{w}\rangle \hat{t}$

$\hat{n} = \hat{t} \times \hat{w} \Rightarrow w \times n = w \times (t \times w) = \hat{t}(\hat{w}\cdot\hat{w}) - \hat{w}(\underbrace{\cancel{\hat{w}\cdot\hat{t}}}_{0}) = \hat{t}$

□

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But in general, we don't have a half-space chart α, but rather a more general one $\varphi: D \to S$. We can't orient the boundary by $\vec{t} = D\alpha_x(e_1)$ but we can still form the RH triad $\{\hat{n}, \hat{t}, \hat{w}\}$

but we can still form the RH triad and geometrically we know the line $T_p(\partial S)$ (not the direction)

we know $\vec{n} = \vec{\varphi}_u \times \vec{\varphi}_v$ and geometrically we know the line $T_p(\partial S)$ (not the direction)

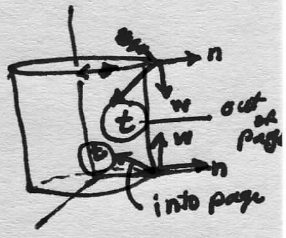
and we know which side of $T_p(\partial S)$ that $\hat{w}$ is on.

p.291

#4 Consider the cylinder given by $\varphi: (0,1)\times(0,1) \longrightarrow \mathbb{R}^3$

$$(u,v) \longrightarrow \begin{bmatrix} \cos(2\pi u) \\ \sin(2\pi u) \\ v \end{bmatrix}$$

We say this the pos or



Describe the vf $\hat{n}$ and $\hat{t}$.

$s \to c$
$c \to -s$

$$\varphi_u = 2\pi\begin{bmatrix} -\sin(2\pi u) \\ \cos(2\pi u) \\ 0 \end{bmatrix} \qquad \varphi_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \varphi_u \times \varphi_v = 2\pi\begin{vmatrix} i & j & k \\ -s & c & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\pi\begin{bmatrix} c \\ s \\ 0 \end{bmatrix} = 2\pi\hat{e}_r$$

We can see what $\hat{w}$ is in the picture, then we know $t = w \times n$

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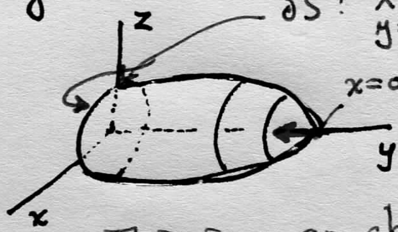
p.308

#4 Consider the following surf S w/ ∂: $4x^2 + y^2 + 4z^2 = 4$, $y \geq 0$

$y = \pm 2\sqrt{1 - x^2 - y^2}$

We will cover $S - \partial S$ with a single chart (which we shall call pos or)



∂S: $x^2 + z^2 = 1$, $y = 0$

$x = 0, z = 0$ $y^2 = 4$, $y = +2$

This is a graph over the xz plane $y = f(x,z)$

$$\varphi: \overset{\circ}{\Delta}(0,1) \longrightarrow \mathbb{R}^3 \quad (\mathbb{R}^2)$$

$$\langle u, v\rangle \longmapsto \begin{bmatrix} u \\ 2\sqrt{1-u^2-v^2} \\ v \end{bmatrix} \qquad \text{let } \rho := 1 - u^2 - v^2$$

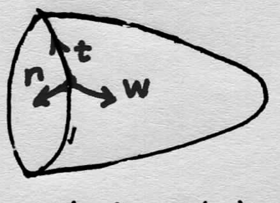
(a) What is the $\vec{n}$ vf on S? What is the induced $\vec{t}$ vf on ∂S?

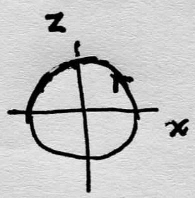
$$D\varphi_u = \begin{bmatrix} 1 & 0 \\ \frac{-2u}{\rho^{1/2}} & \frac{-2v}{\rho^{1/2}} \\ 0 & 1 \end{bmatrix} \qquad \vec{n} = \varphi_u \times \varphi_v = -\begin{bmatrix} \frac{2u}{\rho^{1/2}} \\ 1 \\ \frac{2v}{\rho^{1/2}} \end{bmatrix}$$

Consider the tip pt $u = 0$ $v = 0$

$\vec{n} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ points inward

$t = w \times n$

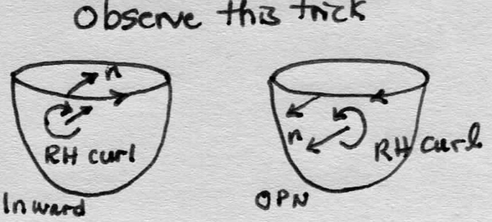




Geometrically we see this is the direction on ∂S

we know $\vec{n}$ points inward in the interior, so it still points inward on ∂S by continuity

Observe this trick



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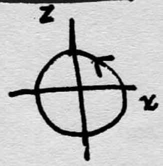
Now we are going to separately evaluate both sides of Stokes' Thm $\int_{\partial S} \omega = \int_S d\omega$ where $\omega = y\,dx + 3x\,dz$

$\oint_{\partial S} F \cdot ds = \int_S (\nabla \times F) \cdot n\,dS$

(b) compute $\int_{\partial S} \omega$



This direction is param by $\sigma(t) = \begin{bmatrix} \cos t \\ 0 \\ \sin t \end{bmatrix}$ $0 \le t \le 2\pi$

Do it the vector calculus way!

$\vec{F} = \begin{bmatrix} y \\ 0 \\ 3x \end{bmatrix}$ $\int_0^{2\pi} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{bmatrix} 0 \\ 0 \\ 3\cos t \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ 0 \\ \cos t \end{bmatrix} dt$

$= \int_0^{2\pi} 3\cos^2 t\, dt = 3\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$

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$= 3\left[\pi + \underbrace{\frac{1}{4}\sin 4\pi}_{0} - 0 - 0\right] = \boxed{3\pi}$

Same

Diff forms way!

$\int_{\partial S} \omega = \int_{(0,2\pi)} \sigma^* \omega = \int_0^{2\pi} \sigma^*(y\,dx + 3x\,dz)$

$\pi_2(\cdot)$ $3\pi_1(\cdot)$

$= \int_0^{2\pi} \underbrace{\pi_2(\sigma(t))}_{0} \sigma^* dx + 3\pi(\sigma(t))\, \sigma^* dz$ $\begin{bmatrix} \cos \\ 0 \\ \sin \end{bmatrix}$

one way to "see" this

d commutes with *

$\sigma^* dz = d(\sigma^* z)$
$\pi_3(\sigma)$
$= d(\sin t)$
$= \cos t$

$= \int_0^{2\pi} 0 + 3\cos(t)\, \underbrace{\dot{\sigma}^3}_{\cos t}\, dt$

$= \int_0^{2\pi} 3\cos^2(t)\, dt$

(c) Compute $\int_S d\omega$

$\omega = y\,dx + 3x\,dz \Rightarrow d\omega = dy \wedge dx + 3\,dx \wedge dz$
$= -dx \wedge dy + 3\,dx \wedge dz$

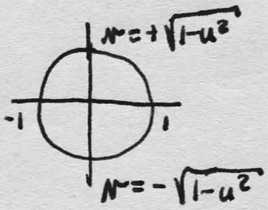
We don't have to integrate over ∂S, since it is m 0

$\int_S d\omega = \int_{\varphi(\dot{\Delta})} d\omega = \int_\Delta \varphi^* d\omega = \int_\Delta \varphi^*(-1\,dx \wedge dy + 3\,dx \wedge dz) = \int_\Delta -\frac{\partial(\varphi^1, \varphi^2)}{\partial(u, v)} + 3\frac{\partial(\varphi^1, \varphi^3)}{\partial(u,v)}\, du\,dv$

$\frac{\partial(\varphi^1, \varphi^2)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ \frac{-2u}{\rho^{1/2}} & \frac{-2v}{\rho^{1/2}} \end{vmatrix} = \frac{-2v}{\rho^{1/2}}$

$\frac{\partial(\varphi^1, \varphi^3)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$= \int_{\Delta(0,1)} \left(\frac{2v}{\sqrt{1-u^2-v^2}} + 3\right) du\,dv = \int_{u=-1}^{1} \int_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \underbrace{\frac{2v}{\sqrt{1-u^2-v^2}}}_{f(v)}\, dv\,du + \iint 3\, dv\,du$



$f(-v) = -f(v)$ odd fcn of v
By sym, pos half of disc cancels neg half
$= 0$

convert to polar
area of disc
$3\int_0^{2\pi} \int_{r=0}^{1} r\,dr\,d\theta = 3\pi(1)^2 = \boxed{3\pi}$ ✓

cont'd →

Let's also do (c) by vector calculus to illustrate something

Munkres gave $\omega = y\,dx + 3x\,dz$ which I translated to $\vec{F} = \begin{bmatrix} y \\ 0 \\ 3x \end{bmatrix}$

Now we need $\displaystyle\int_S (\nabla\times F)\cdot n\,dS$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ y & 0 & 3x \end{vmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$$

?

whereas $d\omega = \underbrace{-dx\wedge dy}_{\text{"}z\text{"}} + \underbrace{3\,\underbrace{dx\wedge dz}_{\text{"}y\text{"}}}_{\text{really } -3\,dz\wedge dx} \rightsquigarrow \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$

$dz\wedge dx$ is the cannonical form

$x \to y \to z \to x$

$$\int_\Delta \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}\cdot(\vec{\varphi}_u\times\vec{\varphi}_v)\,du\,dv$$

$$= \int_\Delta \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} \cancel{-2u/\rho^{1/2}} \\ -1 \\ \cancel{-2v/\rho^{1/2}} \end{bmatrix} = \int_\Delta \left(3 + \frac{-2v}{\sqrt{1-u^2-v^2}}\right)du\,dv$$

↑ from (a)

Now follow the same steps as before

□