an $\mathcal{O}^R$ is a atlas of charts with pos overlap

2/13/2026

①

Summary

Pos $\mathcal{O}^R$ on $\mathbb{R}^n$ is an ordered basis $\{b_1 \dots b_n\} \ni \det [b_1 \cdots b_n] > 0$

M is $\mathcal{O}^R$-able if it can be <u>covered</u> with charts $(\alpha, \mathcal{U})$ with pos overlap

(P. 284) M (n-1)-mfd in $\mathbb{R}^n$ normal $\hat{n}_p \in T_p(\mathbb{R}^n)$ but $\perp$ to $T_p M \Rightarrow \hat{n}_p$ uniq up to sign

Abbrev $\vec{\partial}_i := D\alpha_x(e_i)$ $\alpha: \underset{\mathcal{U}}{\mathbb{R}^{n-1}} \longrightarrow \mathbb{R}^n$ $\alpha(x) = p$

$\{\partial_1 \dots \partial_{n-1}\}$ is a basis for $T_p M$ — we call this basis RH

Later we will give a C^∞ fcn to make $\hat{n}$ from $D\alpha_x$ ch 38

(★) We specify $\hat{n}_p$ by requiring $\det [n\ \partial_1 \dots \partial_{n-1}] > 0$ — OPN

For $M^n \subset \mathbb{R}^n$ Nat $\mathcal{O}^R$ is all charts $\ni D\alpha_x > 0$ ← Here M does not have a $\hat{n}$; There is no dim for it. $\vec{n}$ only makes sense for (k-1) dim mfd in $\mathbb{R}^k$. Half space discussion explains inward or outward pointing.

we need reverse $\mathcal{O}^R$ map $r = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(n-1)}$ neg on 1st comp

$$D(\alpha \circ r)\{e_1 \dots e_{n-1}\} = D\alpha_x \begin{bmatrix} -1 & & \\ & 1 & \\ & & \ddots \ 1 \end{bmatrix} [e_1 \dots e_{n-1}]_{n-1} = D\alpha_x \begin{bmatrix} | & | & & | \\ -e_1 & e_2 & \cdots & e_{n-1} \\ | & | & & | \end{bmatrix}$$

we really apply this at $r^{-1}(x)$

linearity

$$= \begin{bmatrix} | & & | \\ D\alpha_x(-e_1) & \dots & D\alpha_x(e_{n-1}) \\ | & & | \end{bmatrix} = [-\partial_1\ \partial_2 \dots \partial_{n-1}]_n$$

This basis has neg (or reversed) $\mathcal{O}^R$ for $T_p M$

Now if we add $\hat{n}$ (so we have n basis vectors – sorry for double meanings of "n")

Reversing $\mathcal{O}^R$ on α changes n to $-n$

$\det\{n, \partial_1 \dots \partial_{n-1}\} > 0$ by def of n

If now we put the basis from $\alpha \circ r$, we would have to <u>also</u> put a neg on n or else det would be neg

$\det [-n, -\partial \dots \partial_{n-1}] > 0$ only if we ~~add~~ mult (-1) to n

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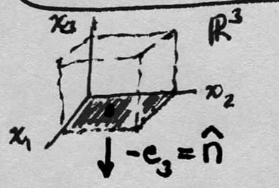
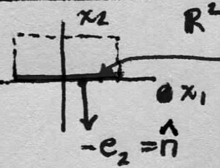
Aside from G&P ch 3.2 #6

Half Space H^k is the prototype of Mfd-w-∂

$$H^k := \{ \text{all } x \in \mathbb{R}^k \mid x^{(k)} \geq 0 \}$$





this portion is what I call "∂_m" mfd boundary

This n is for $\partial_m H^k$

Obviously, for $\partial_m H^k$ we can see $\hat{n} = -\hat{e}_k$ is the OPN

<u>By convention</u>, we put n in slot 1 and evaluate the sign of ordered basis

chart here is $\alpha = id$

$$\det [n, e_1 \dots e_{k-1}] \overset{\det}{=} [-e_k, e_1 \dots e_{k-1}] = (-1)^k \det [e_1 \dots e_k] = (-1)^k \det I = (-1)^k$$

If k even, this is +1 no problem

If k odd, we'd get -1 so we must reverse $\mathcal{O}^R$ $(\alpha \circ r = id \circ r)$ to get $-n = +e_k$ ~~so geometrically we'd have IPN in this case~~?

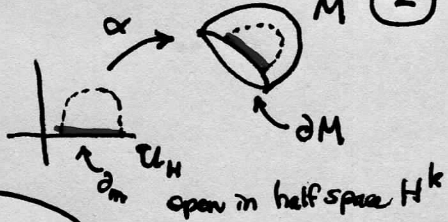
This explains why we must reverse $\mathcal{O}^R$

so for k = odd, we can't say the <u>$Id_{\mathbb{R}^k}$</u> is a <u>pos</u> chart anymore because we know ~~what~~ n <u>must</u> be $-e_k$ and we <u>must</u> keep our precious $\det [n, e_1 \dots e_{k-1}] > 0$ the Key Relation

Induced $\mathcal{O}^R$ on ∂M [I will abbrev as $\mathcal{O}^R_i(\partial M)$]

For mfd-w/-∂, we have ∂ charts $\alpha : \mathcal{U}_H \longrightarrow U$

we can restrict α to $\partial_m \mathcal{U}_H$ and get charts for mfd ∂M

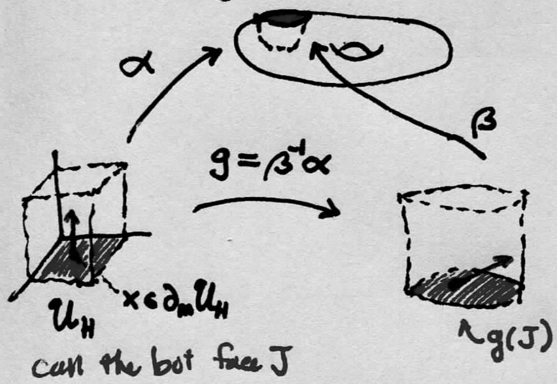


∂_m means mfd ∂, not topological ∂

p.287

Thm 34.1 Let $k > 1$, M k-mfd, $\mathcal{O}^R$able, $\partial M \neq \emptyset$ $\Big\} \Rightarrow \partial M$ is an $\mathcal{O}^R$able mfd

Pf Show if α, β are co-ord patches (a.k.a charts in my lingo) on M with pos overlap,
Then if they are restricted to the ∂ (charts $\hat{\alpha}, \hat{\beta}$), $\hat{\alpha}$ and $\hat{\beta}$ also have pos overlap.
Let us assume we cut down the domains so $\alpha(\mathcal{U}_H) = U = \beta(\mathcal{V}_H)$
Thus $g : \mathcal{U}_H \longrightarrow \mathcal{V}_H$ is a diffeo (and we need $\det Dg_x > 0 \ \forall x \in \mathcal{U}_H$ to be pos overlap i.e. $\mathcal{O}^R$ pres)



We must now restrict to the ∂_m portion
The Key is that $g : \partial H^k \longrightarrow \partial H^k$
since both charts are the same piece of ∂M
Restricting $\hat{g} = \hat{\beta}^{-1} \circ \hat{\alpha}$ $\quad \underset{\text{in}(\mathbb{R}^{k-1})}{\hat{g} : \partial H^k} \longrightarrow \underset{\mathbb{R}^{k-1} \times \{0\}}{\partial H^k \times \{0\}}$
Let's just show it for $k=3$ – then general idea is clear

g maps any curve γ contain in J to a curve contained in $g(J) \subset \mathbb{R}^2 \times \{0\}$

Let $u = \langle x, y, 0 \rangle \in J$ $\quad$ γ could be such that $\gamma'(0) = e_1$, and we could have a $\gamma \ni \gamma'(0) = e_2$

$Dg_u(e_1)$ must lie in xy plane

$$Dg_x = \begin{bmatrix} g^1_x & g^1_y & g^1_z \\ g^2_x & g^2_y & g^2_z \\ g^3_x & g^3_y & g^3_z \end{bmatrix} \qquad Dg_u(e_1) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix} \Rightarrow g^3_x(u) = 0$$

and the same arg using e_2 gives $g^3_y(u) = 0$

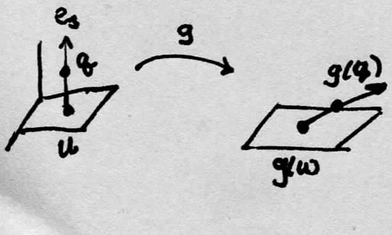
$$\Rightarrow Dg_u = \begin{bmatrix} g^1_x & g^1_y & g^1_z \\ g^2_x & g^2_y & g^2_z \\ 0 & 0 & g^3_z \end{bmatrix}$$

B

This matrix is block upper triang
this block $B = D\hat{g}_u$
If we can show $\det B > 0$ then the restricted transition map $\hat{g}$ has pos overlap and we are done (better to say $\hat{g}$ preserves $\mathcal{O}^R$)

Is $\det B > 0$? we know $\det Dg_x > 0 \ \forall x \in \mathcal{U}_H$, so in particular, true for $x = u$
So $\det Dg_u > 0$ and by block matrix dets (Strang LAAIA ch 4 sheets) $\underline{\det Dg_u = \det B \cdot [g^3_z(u)]}$

<u>so we are done if $g^3_z(u) > 0$</u>



Let q be a pt on curve $\gamma(t) = u + t\hat{e}_3$
$g(q)$ must be in the interior of $\mathcal{V}_H$ <u>above $g(J)$</u>
The ray from $g(u)$ to $g(q)$ converges to $Dg_u(e_3)$ as $q \to u$
we know $Dg_u(e_3) \notin \mathbb{R}^2 \times \{0\}$ or else g not a diffeo
$\Rightarrow \dfrac{\partial g^3}{\partial z}(u) > 0$

$\square$

M has dim k $\Rightarrow$ ∂M has dim k-1

we consider the charts that contain points of ∂M : $\alpha : \mathcal{U}_H \longrightarrow V$

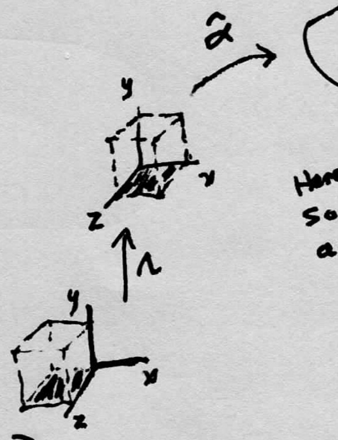
and we restrict their domain $\hat{\alpha} : \partial_m \mathcal{U}_H \longrightarrow \partial_m V$

How do we do this?

Define inclusion map $\mathbf{i} : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}^k$

$$\langle x^1, \dots, x^{k-1} \rangle \longmapsto \begin{bmatrix} x^1 \\ \vdots \\ x^{k-1} \\ 0 \end{bmatrix}$$

Define $\hat{\alpha} = \alpha \circ \mathbf{i}$ image is $\mathbb{R}^{k-1} \times \{0\}$



Here dim k=3 so its odd and we reverse

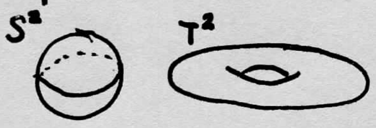
The Induced $\mathcal{O}^R$ is then the restricted charts

$$\begin{cases} \left\{ \overset{\text{all}}{\hat{\alpha}} = \alpha \Big|_{(\mathbb{R}^{k-1} \times \{0\}) \cap \mathcal{U}_H} \;\Big|\; (\alpha, \mathcal{U}_H) \text{ is pos chart on } M \right\} & k \text{ is even} \\ \left\{ \overset{\text{all}}{\hat{\alpha}} = \alpha \circ r \Big|_{\mathbb{R}^{k-1} \times \{0\} \cap r^{-1}(\mathcal{U}_H)} \;\Big|\; (\alpha, \mathcal{U}_H) \text{ pos} \right\} & k \text{ is odd} \end{cases}$$

Now we shall go on a lengthy digression and out of sequence with Munkres

- A long discussion of the ∂ $\mathcal{O}^R$ induced on n-mfds M in $\mathbb{R}^n$. This starts with (ex 4) p.289 when n=3 but follows this thread thru the remaining sections of ch 7. This continues until sheet _____
- A long discussion following (ex 5) p.289-291 where we have a 2-dim Surf in $\mathbb{R}^3$ that is not the ∂_m of a 3-mfd and again we include and group together the examples of this in §34, 35, 36, 37, 38
- We are grouping all the $\mathcal{O}^R$ material here. Then on separate sheets we shall write up the other material that Munkres includes in these sections, like Stokes Thm.

p.289 (ex 4)



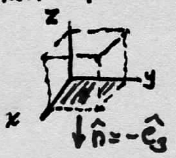
Each of these is the ∂_m of a solid 3-mfd in $\mathbb{R}^3$

If M is a 3-mfd in $\mathbb{R}^3$ $\mathcal{O}^R$ Naturally [det $D\alpha > 0$ $\forall$ charts $\alpha \in \mathcal{O}^R$]

What can we say about Induced $\mathcal{O}^R$ on ∂M? $\mathcal{O}^R_i(\partial M)$

Let's consider the protitype case again to see how it all fits together

Half space H^3 a priori, we know what Outward pointing normal is: $-\hat{e}_3$



(*) we also REQUIRE $\det[\vec{n}, D\alpha_x(e_1), D\alpha_x(e_2)] > 0$ for $n \perp \mathrm{Span}\{D\alpha_x(e_i)\}$

The obvious chart α would be the identity I

$$\hat{\alpha} = \alpha\circ\dot{\imath} : \langle x,y\rangle \longmapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad D\hat{\alpha}_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = [e_1\ e_2]$$

So we have $\det[-e_3\ e_1\ e_2] \overset{?}{>} 0$ but no, $\det[-e_3\ e_1\ e_2] = (-1)(-1)^2 \det I = -1$ by transposing cols.

BUT another way to see it is $\det[\vec{a}\ \vec{b}\ \vec{c}] \equiv \vec{a}\cdot(\vec{b}\times\vec{c})$

So we are REQUIRING $\det[a,b,c] = a\cdot(b\times c) > 0$ (*)

In our case $-e_3\cdot(e_1\times e_2) > 0$ $e_1\times e_2 = +e_3$: $\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(*) is so important that we will reverse the identity $(I\circ r)$ and $D\alpha = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = [-e_1\ e_2]$

Then $(-e_1\times e_2) = -e_3$ and $-e_3\cdot e_3 = +1$

Thus, at least for the case $n=3$, we see why we needed to reverse:

$\boxed{\text{TO MAKE SURE } b\times c = n_{OPN} \text{ That is what (*) is doing}}$ $(n\cdot n > 0)$

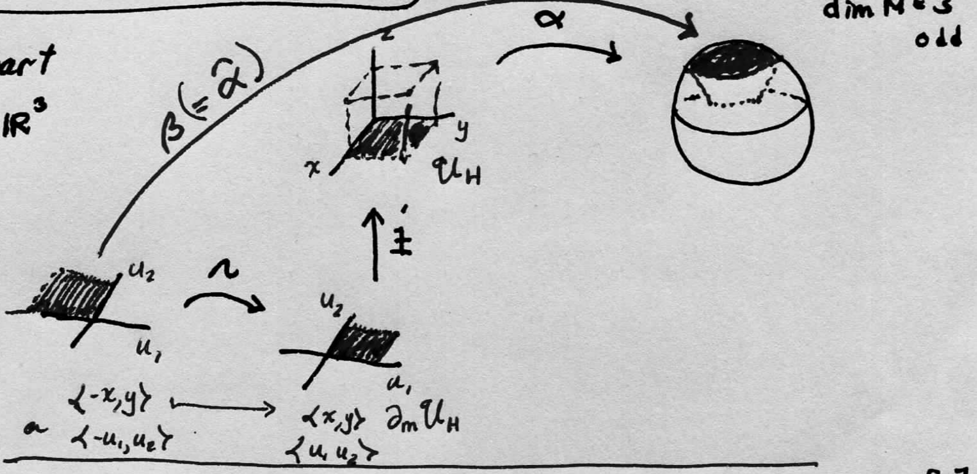
Thm $\mathcal{O}^{\partial}_{i}(\partial M)$ is the one given by OPN vf
$\det(D\alpha) > 0$ Nat $\mathcal{O}^{\partial}$ on M

dim $N = 3$ odd

Pf. Step 1 Set up boundary chart

a.k.a. $(\hat{\alpha} :=)\ \beta := \alpha\circ\dot{\imath}\circ r$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$D\beta_u = D\alpha\cdot D\dot{\imath}\cdot Dr = D\alpha\cdot\dot{\imath}\cdot r$$



$$\begin{array}{cc} & 2 \\ 3 & \begin{bmatrix} D\beta_u \end{bmatrix} \end{array} = \begin{array}{cc} & 3 \\ 3 & \begin{bmatrix} D\alpha_x \end{bmatrix} \end{array} \underbrace{\overset{2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{2}{\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}}}_{\begin{array}{c}\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ -e_1\ \ e_2\end{array}}$$

I will write the std basis for $\mathbb{R}^2$ with font $\mathfrak{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathfrak{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ whereas for $\mathbb{R}^3$ I write $e_1\ e_2\ e_3$

$$D\beta_u(\{\mathfrak{e}_1, \mathfrak{e}_2\}) = \begin{bmatrix} D\beta_u \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D\alpha_x \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} D\beta_u(\mathfrak{e}_1) & D\beta_u(\mathfrak{e}_2) \end{bmatrix} = \begin{bmatrix} D\alpha_x\left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) & D\alpha_x\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} -D\alpha_x(e_1) & D\alpha_x(e_2) \end{bmatrix}$$

$$-\vec{a}_1 \qquad \vec{a}_2$$

cont'd →

3/21/2026

Step 2

Now $\vec{n}$ is specified by (1) $n \perp T_p(\partial M) = D\beta_u(\mathbb{R}^2)$ — can be written $n \perp \operatorname{span}\{a_1, a_2\}$

(2) $\underbrace{\det[n, D\beta_u(e_1), D\beta_u(e_2)]}_{\det[n,\, -D\alpha_x(e_1),\, D\alpha_x(e_2)] \overset{!}{>} 0} \overset{!}{>} 0$ ★

i.e. $\det[n, -a_1, a_2] \overset{!}{>} 0$

But we know $\det[a\ b\ c] = a\cdot(b \times c)$ — a b c

Thus we have an explicit formula for $\vec{n}$: $\boxed{\text{set } \vec{n} := (-\vec{a}_1)\times(\vec{a}_2)}$ The neg sign is because $k=3$ odd

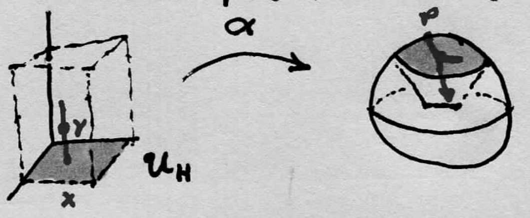
Then both conds are satisfied: $-a_1 \times a_2 \perp \operatorname{Span}\{-a_1, a_2\}$ and $(-a_1\times a_2)\cdot(-a_1 \times a_2) = \|a_1 \times a_2\|^2 > 0$

Step 3

Now we must show $\vec{n}$ is <u>Outward</u> pointing.

First we show $a_3 = D\alpha_x(e_3)$ points <u>Inward</u>, then we compare it to $\vec{n}$.

$\alpha(x) = p \in \partial M$



Consider the (straight) curve $\gamma(t) = x + t\hat{e}_3$ for $t \geq 0$

Consider a pt y on γ and near x

Obviously $\alpha(y)$ is in the interior of M

The ray from p to $\alpha(y)$ points into M $\forall y$ near x and as $y \to x$ this ray converges to lie on vector $D\alpha_x(e_3)$.

$D\alpha_x(e_3)$ cannot be in $T_p(\partial M)$ or else $D\alpha_x$ would map 3 dims into 2 and not be a diffeo

Therefore $\underbrace{D\alpha_x(e_3)}_{a_3}$ points INTO M. [This same arg would hold for $\dim M = n$ and $D\alpha_x(e_n)$]

Step 4

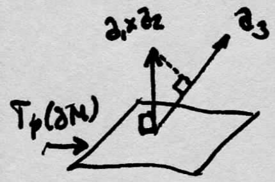
What is the relation between $\vec{n}$ and $\vec{a}_3$?

α has pos $\mathcal{O}^R$ $\Rightarrow$ $\underbrace{\det[a_1\ a_2\ a_3]}> 0$

$= (-1)^2 \det[a_3\ a_1\ a_2] > 0$

$= a_3 \cdot (a_1 \times a_2) > 0$

$= \langle a_3, a_1 \times a_2 \rangle > 0$



$a_1 \times a_2$ has a component in the pos $\vec{a}_3$ dir (into M)

$\Rightarrow -a_1 \times a_2$ has a comp opposite to $\vec{a}_3$ (outward)

$\Rightarrow \vec{n} = -a_1 \times a_3$ so $\vec{n}$ has a comp pointing <u>Out</u> from M

$\vec{n} \perp T_p(\partial M)$

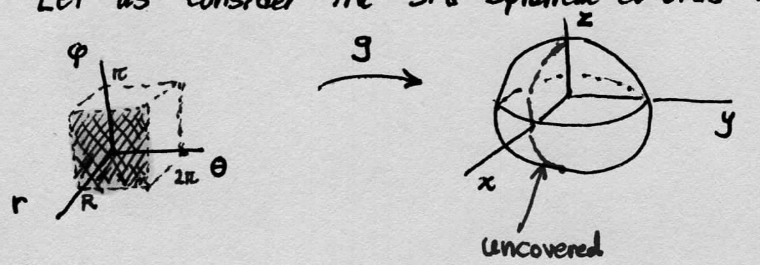
$\vec{n}$ is Outward Pointing

□

ASIDE:
Let's carry this out in a concrete case: 2/22/2026

6

Let us consider the std spherical co-ords of M&T: ch 6 sheet 14



$$g: [0,R]\times(0,2\pi)\times(0,\pi) \longrightarrow \mathbb{R}^3$$
$$(r,\ \theta,\ \varphi) \longmapsto \begin{bmatrix} r c_\theta s_\varphi \\ r s_\theta s_\varphi \\ r c_\varphi \end{bmatrix}$$

$$Dg_p = \begin{bmatrix} c_\theta s_\varphi & -r s_\theta s_\varphi & r c_\theta c_\varphi \\ s_\theta s_\varphi & r c_\theta s_\varphi & r s_\theta c_\varphi \\ c_\varphi & 0 & -r s_\varphi \end{bmatrix}$$

$$\det\left[Dg_p\right] = c_\theta s_\varphi\left(-r c_\theta s_\varphi r s_\varphi - 0\right) + r s_\theta s_\varphi\left(-s_\theta s_\varphi r s_\varphi - r s_\theta c_\varphi^2\right) + r c_\theta c_\varphi\left(0 - r c_\theta s_\varphi c_\varphi\right)$$

$$= -r^2 s_\varphi\Big[\underbrace{s_\varphi^2 c_\theta^2 + s_\varphi^2 s_\theta^2}_{s_\varphi^2(c_\theta^2+s_\theta^2)} + \underbrace{s_\theta^2 c_\varphi^2 + c_\theta^2 c_\varphi^2}_{c_\varphi^2(s_\theta^2+c_\theta^2)}\Big]$$

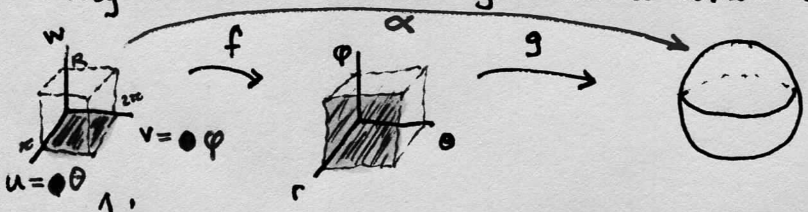
$$= -r^2 \sin(\varphi) \quad \text{neg always since } \varphi\in(0,\pi)$$

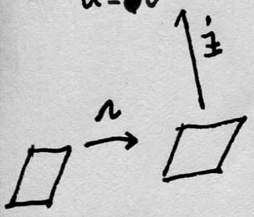
So there are 2 problems with g being a chart that fits into Munkres' framework:

(1) $\det Dg_p < 0$ so not the pos $\mathcal{O}^R$ on $\mathbb{R}^3$

(2) The domain is not a proper $\mathcal{U}_H$ in the half-space – the clsd face must be on the "bottom" for 3rd co-ord

To remedy item (2) we can begin from a correct domain, and this also fixes item (1):



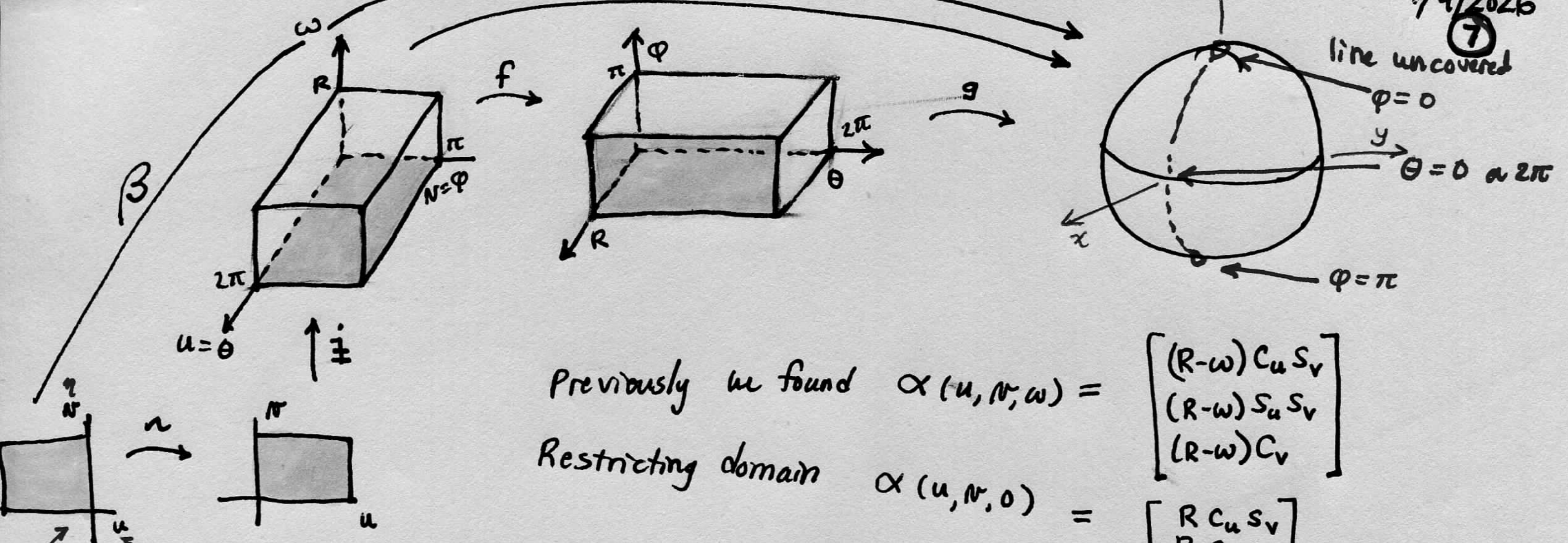


$$f(u,v,w) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R-w \\ u \\ v \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ \varphi \end{bmatrix}$$

Define $\alpha(u,v,w) = \begin{bmatrix} (R-w)c_u s_v \\ (R-w)s_u s_v \\ (R-w)c_v \end{bmatrix}$

$$D\alpha_u = \begin{array}{c} \begin{bmatrix} -(R-w)s_u s_v & (R-w)c_u c_v & -c_u s_v \\ (R-w)c_u s_v & (R-w)s_u c_v & -s_u s_v \\ 0 & -(R-w)s_v & -c_v \end{bmatrix} \\ \begin{array}{ccc} \vec{\alpha}_u & \qquad\qquad \vec{\alpha}_v & \qquad\qquad \vec{\alpha}_w \end{array} \end{array}$$

$$\det\left[D\alpha_u\right] = (R-w)^2 s_v > 0$$
since $v\in(0,\pi)$



Maybe should call these co-ords ξ, η like sheet 5 of §33

Previously we found $\alpha(u,v,\omega) = \begin{bmatrix} (R-\omega)C_u S_v \\ (R-\omega)S_u S_v \\ (R-\omega)C_v \end{bmatrix}$

Restricting domain $\alpha(u,v,0) = \begin{bmatrix} R C_u S_v \\ R S_u S_v \\ R C_v \end{bmatrix}$

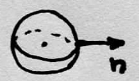
$$D\beta_u = \left[D\alpha_{(u,v,0)} \right] \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -R S_u S_v & R C_u C_v & -C_u S_v \\ R C_u S_v & R S_u C_v & -S_u S_v \\ 0 & -R S_v & -C_v \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R S_u S_v & R C_u C_v \\ -R C_u S_v & R S_u C_v \\ 0 & -R S_v \end{bmatrix}$$

$$\vec{n} = \beta_\xi \times \beta_\eta = (-\partial) \times \partial_2 = R^2 \begin{vmatrix} i & j & k \\ S_u S_v & -C_u S_v & 0 \\ C_u C_v & S_u C_v & -S_v \end{vmatrix} = R^2 \begin{bmatrix} + C_u S_v^2 \\ + S_u S_v^2 \\ \underbrace{S_u^2 S_v C_v + C_u^2 S_v C_v}_{S_v C_v} \end{bmatrix} = R^2 \begin{bmatrix} C_\theta S_\varphi^2 \\ S_\theta S_\varphi^2 \\ S_\varphi C_\varphi \end{bmatrix}$$

In terms of familiar spherical angles

Unfortunately, this param breaks down and we can't evaluate the pts we'd like (on the red line)

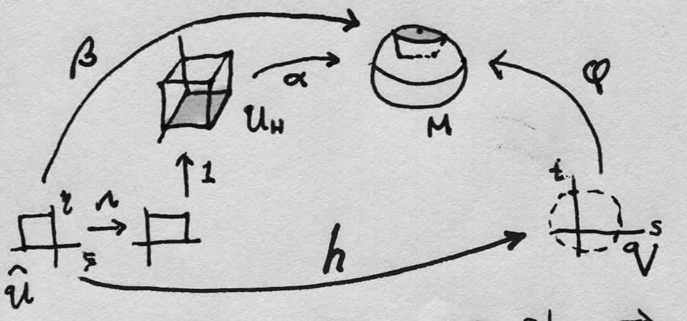
But we can plug in $\theta = \pi/2$ and $\varphi = \pi/2$ $\quad \cos(\pi/2) = 0$, $\sin(\pi/2) = 1$ $\quad \vec{n} = R^2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ This does point outward!



So everything is validated!

▷ BUT Munkres set up a framework with special charts $\alpha : \mathcal{U}_H \longrightarrow U_\partial$ ($\mathbb{R}^{n-1}$, $\mathbb{R}^n$) ($\alpha \in \text{Pos } \mathcal{O}^R$) and these could be restricted to ∂ charts.

What if we parameterized a patch of the surf with some arb map φ?



Since β and φ are both charts on $S = \partial M$ and they overlap, there is a transition fcn h that is a diffeo $\left[\det Dh_{a_\xi} \neq 0 \ \forall u \in \hat{\mathcal{U}} \right]$

$h = \varphi^{-1} \circ \beta \qquad Dh_{a_\xi} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ M&T shows ch 6

$\varphi \circ h = \beta$

$$\beta = \varphi \circ h \qquad D\beta_\xi = D\varphi_s \cdot Dh_\xi \Rightarrow \begin{bmatrix} \beta_\xi & \beta_\eta \end{bmatrix} = \begin{bmatrix} \varphi_s & \varphi_t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{matrix} \vec{\beta}_\xi = a\vec{\varphi}_s + c\vec{\varphi}_t \\ \vec{\beta}_\eta = b\vec{\varphi}_s + d\vec{\varphi}_t \end{matrix}$$

We previously found $\vec{n} = \vec{\beta}_\xi \times \vec{\beta}_\eta$ so here

$$\begin{aligned} \vec{n} &= (a\varphi_s + c\varphi_t) \times (b\varphi_s + d\varphi_t) \\ &= ab\overset{0}{\cancel{(\varphi_s \times \varphi_s)}} + ad(\varphi_s \times \varphi_t) + cb(\varphi_t \times \varphi_s) + cd\overset{0}{\cancel{(\varphi_t \times \varphi_t)}} \\ &= (ad - bc)(\vec{\varphi}_s \times \vec{\varphi}_t) = (\det Dh_\xi)\, \vec{\varphi}_s \times \vec{\varphi}_t \end{aligned}$$

Thus $n = \vec{\beta}_\xi \times \vec{\beta}_\eta$ is a scalar mult of $\vec{\varphi}_s \times \vec{\varphi}_t$

If $\det(Dh_\xi) > 0$ both vectors point in same dir and, when normalized, both give the same $\hat{n}$

This is generalized in Thm on sheet 9 for p.312

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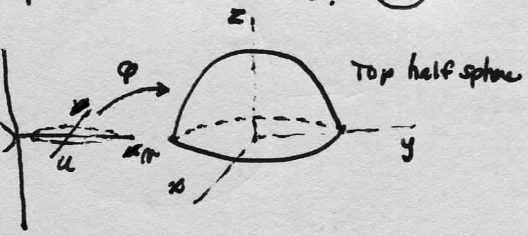
So if $\det Dh_{\xi} > 0$ h is $\mathcal{O}^R$ pres, and β and φ are part of the same $\mathcal{O}^R$.

▷ Let's see this in the example from §33 #3 sheet ③

we have $\varphi : \underset{\mathbb{R}^2}{B^2_{(0,R)}} \longrightarrow \underset{\mathbb{R}^3}{S^2_{(0,R)}}$ where $\rho := R^2 - (u^2+v^2)$

$\langle u, v \rangle \longmapsto \begin{bmatrix} u \\ v \\ \rho^{1/2} \end{bmatrix}$



Then $\varphi_u \times \varphi_v = \begin{bmatrix} u/\rho^{1/2} \\ v/\rho^{1/2} \\ 1 \end{bmatrix}$ If we plug in $\langle u,v \rangle = \langle 0,0 \rangle$ we get the North Pole and we see $\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ OPN

OR if we plug in $\langle 0, R \rangle$ we get $\begin{bmatrix} 0 \\ R/0 \\ 0 \end{bmatrix}$ so we can't evaluate this

But anyway, it is Outward pointing $\vec{n}$ everywhere it is defined.

Summary: So even though φ is not the restriction of a half-space map, we can still find $\vec{n}$ by $\vec{\varphi}_u \times \vec{\varphi}_v$.

The present bundle of sheets is focused only on $\mathcal{O}^R$, so here we shall skip over

§35 Integrating Forms over $\mathcal{O}^R$ Mfds

§36 Geo Interpretation of forms and Integrals

§37 Generalized Stokes' Thm

We do include here $\mathcal{O}^R$ problems from these sections

and select the $\mathcal{O}^R$ parts from §38

▷ Now we focus on the n-dim case

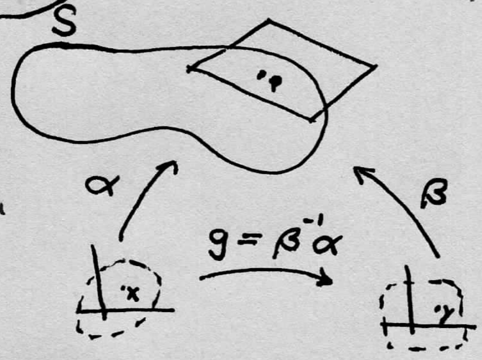
Let S be $\mathcal{O}^r$ $(n-1)$-mfd in $\mathbb{R}^n$ (maybe $S = \partial M^n$, maybe not)

$\hat{n}_p \in T_p(\mathbb{R}^n)$ but $\hat{n}_p \perp T_p(S)$

We define $\hat{n}$ (specifically $\hat{n}_{out}$) by $\det\left[n_{out}, D\alpha_x(e_1) \dots D\alpha_x(e_{n-1})\right] > 0$ ⊛

Important thm

$\boxed{\textbf{Thm } n \text{ is well defined (i.e. indep of particular chart } \alpha \text{) with pos } \mathcal{O}^r}$



pf. α and β are overlapping charts, both in the pos $\mathcal{O}^r$ on S.

Thus the transition fcn g has $\det Dg_x > 0$ $\forall x$

$$\alpha = \beta \circ g \qquad D\alpha_x = D\beta_y \cdot Dg_x \qquad \begin{array}{c} \\ n \end{array}\!\!\underbrace{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_{n-1} \\ | & & | \end{bmatrix}}_{D\alpha_x}^{\,n-1} = n\begin{bmatrix} D\beta_y \end{bmatrix}^{n-1}\begin{bmatrix} Dg_x \end{bmatrix}^{n-1}_{\,n-1}$$

Now, for an arb vector $v \in \mathbb{R}^n$, form the augmented matrix:

$$n\begin{bmatrix} | & | & & | \\ v & a_1 & \cdots & a_{n-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & \\ v & \left[D\beta_y\, Dg_x\right] \\ | & \end{bmatrix} = \begin{array}{c} 1 \\ (n-1) \end{array}\!\!\left[\begin{array}{c:c} v^1 & | \;\; | \\ w & D\beta_y \end{array}\right]\left[\begin{array}{c:c} 1 & 0\,0\,0 \\ \hline 0 & \\ 0 & Dg_x \\ 0 & \end{array}\right]\begin{array}{c} 1 \\ (n-1) \end{array} \quad ⊛⊛$$

Justified

Here is the general formula for block multiplying matrices

$$\left[\begin{array}{c:c} \alpha & b^T \\ \hdashline c & D \end{array}\right]\left[\begin{array}{c:c} \beta & e^T \\ \hdashline f & G \end{array}\right] = \left[\begin{array}{c:c} \alpha\beta + b^T f & \alpha e^T + b^T G \\ \hdashline \beta c + Df & c e^T + DG \end{array}\right]$$

So in our case, we will break $D\beta_y$ into $\left[\begin{array}{c} b^T \\ \hdashline B \end{array}\right]$ ← 1^{st} row, ← last $(n-1)$ rows

$$\left[\begin{array}{c:c} v^1 & b^T \\ \hdashline w & B \end{array}\right]\left[\begin{array}{c:c} 1 & 0^T \\ \hdashline 0 & G \end{array}\right] \overset{\text{Confirming by multiply}}{=} \left[\begin{array}{c:c} v^1 & b^T G \\ \hdashline w & BG \end{array}\right] = \begin{bmatrix} v & D\beta_y\, Dg_x \end{bmatrix}$$

Key Thm for α vs. φ and $n = \chi(\varphi_{u_1} \cdots \varphi_{u_m})$

Apply det to ⊛⊛ and using facts about det of a block matrix (Strang LAAIA ch 4 sheets)

$$\det\begin{bmatrix} v & a_1 \dots a_{n-1} \end{bmatrix} = \det\left[\begin{array}{c:c} v & D\beta_y \end{array}\right] \cdot \underbrace{\det\left[\begin{array}{c:c} 1 & 0^T \\ \hdashline 0 & Dg_x \end{array}\right]}_{\det[1]\cdot\det[Dg_x]}$$

$$= \det\left[\begin{array}{c:c} v & D\beta_y \end{array}\right] \det Dg_x \quad \text{This is pos}$$

This means $\det\overset{D\alpha_x}{\begin{bmatrix} v & a_1 \dots a_{n-1} \end{bmatrix}} > 0 \iff \det\begin{bmatrix} v & D\beta_y \end{bmatrix} > 0$

So if, in addition, $v \perp T_p(S)$ then $v = n$ O.P.N.

NOTE: Munkres gives the theory in terms of special charts $\alpha: U_H \to V$ that can be restricted to ∂M. This thm lets us work with any chart φ on ∂M

□

Preliminaries: Given LI set of vectors $\{v_1 \dots v_{n-1}\}$ in $\mathbb{R}^n$, we want to exhibit a vector $\vec{q}$ that is $\perp$ to $\operatorname{Span}\{v_1 \dots v_{n-1}\}$.

chose arb vector $\vec{a} \in \mathbb{R}^n$ and form matrix $A_a := \begin{bmatrix} | & | & & | \\ a & v_1 & \dots & v_{n-1} \\ | & | & & | \end{bmatrix}$

Expand $\det(A_a)$ down col a by cofactors – each signed cofactor will become a component of an n-vector $\vec{q}$:

$$\circledast \quad \det(A_a) = \sum_{i=1}^{n} a_i \underbrace{(-1)^{i+1} \det C_{i1}}_{q^i}$$

where C_{i1} is formed by deleting row i and col 1 from A_a (or just row i from $V = \begin{bmatrix} | & & | \\ v_1 & \dots & v_{n-1} \\ | & & | \end{bmatrix}$)

Thus we can write $\vec{q} =: \chi(v_1, \dots v_{n-1})$

Lemma 38.3 (Generalized Cross-Prod of $\{v_1 \dots v_{n-1}\}$) [Mentioned in Spivak COM p.84]

$\vec{q}$ defined as in $\circledast$ $\Longrightarrow$
(1) $\det A_a = \langle a, q \rangle$ Generalizing $\det[a\, b\, c] = \vec{a} \cdot (\vec{b} \times \vec{c})$
(2) $q \perp v_i \ \forall i$ (thus $q \perp \operatorname{Span}\{v_1 \dots v_{n-1}\}$)
(3) $q \neq 0$
(4) ordered basis $[q, v_1, \dots, v_{n-1}]$ is RH i.e. $\det[q, v_1 \dots v_{n-1}] > 0$
[observe this is the Key Relation for OPN when $\alpha \in \operatorname{Pos} \mathcal{O}^{\theta}$ and $v_i = D\alpha_x(e_i)$]
(5) $\|q\| = \sqrt{\det V^T V} = \operatorname{Vol}(\mathcal{P}(v_1, \dots v_{n-1}))$ parallelipiped from p.182

Pf. (1) From $\circledast$ $\det(A_a) = \sum a^i q^i = \langle a, q \rangle$

(2) For any $i \leq n-1$, let $a = v_i$ and plug it in $\underbrace{\det[v_i\, v_1 \dots v_{n-1}]}_{0 \text{ since dup col}} = \langle v_i, q \rangle$

(3) choose any $a \notin \operatorname{Span}\{v_1 \dots v_{n-1}\}$
Then A_a is nsing $\Rightarrow \det A_a \neq 0 \Rightarrow \langle a, q \rangle \neq 0 \Rightarrow q \neq 0$

(4) Plug in $a = q$: $\det[q, v_1 \dots v_{n-1}] = \langle q, q \rangle = \|q\|^2 > 0$

(5) $$A_q^T A_q = \begin{bmatrix} - & q & - \\ - & v_1 & - \\ & \vdots & \\ - & v_{n-1} & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ q, & v_1 & \dots & v_{n-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} |q|^2 & 0 \\ 0 & V^T V \end{bmatrix}$$

$$\det(A_q^T A_q) = \underbrace{\det A_q}_{\|q\|^2} \cdot \underbrace{\det A_q}_{\cancel{\|q\|^2}} = \cancel{\|q\|^2} \det(V^T V) \Rightarrow \|q\| = \sqrt{\det V^T V} \qquad \square$$

$\chi(a, b) = q$

Observe $\vec{a} \times \vec{b} = \vec{q}$ with vectors in $\mathbb{R}^3$ is a special case: $n = 3$, so $n-1 = 2$ and we have ordinary cross prod of 2 vectors.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \overset{\text{transp}}{=} \begin{vmatrix} i & a_1 & b_1 \\ j & a_2 & b_2 \\ k & a_3 & b_3 \end{vmatrix} = \hat{\imath} \overbrace{(a_2 b_3 - a_3 b_2)}^{(-1)^{1+1} C_{11}} - \hat{\jmath} \overbrace{(a_1 b_3 - a_3 b_1)}^{(-1)^{2+1} C_{21}} + \hat{k} \overbrace{(a_1 b_2 - a_2 b_1)}^{(-1)^{3+1} C_{31}}$$

$$= \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix} = \vec{q}$$