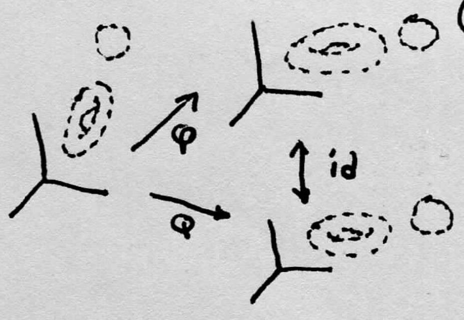
Munkres AOM §33 12/26/2025

p.275–280 ①



Let $\mathcal{a}$ be an open set in $\mathbb{R}^k$ (thus a k-mfd in $\mathbb{R}^k$)

[Thus, given the std basis, the Id map is a co-ord chart. Any other co-ord chart would just be a COB]

Let η be a k-form on $\mathcal{a}$. Thus in the Id chart,

$$\eta_{(x)} = f(x)\, dx^1 \wedge \dots \wedge dx^k$$

Define $\displaystyle\int_{\mathcal{a}} \eta := \int_{\mathcal{a}} f$ ordinary multivariable calculus integral

$\int \varphi^* \eta$ pull back the form

Does this depend on the choice of the co-ord sys? Pnb #4 p.280

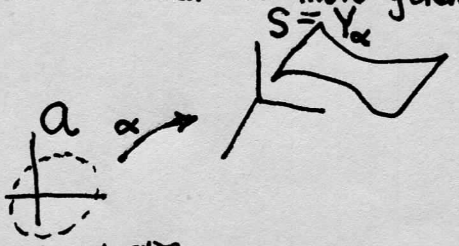
we will get the same result if we restrict to O.N. coord changes Q with $\det(Q) = +1$

G&P det thm p.165 $\quad Q^* f\, dx^1 \wedge \dots dx^k = \underbrace{\det(dQ_x)}_{1} f(Q(x))\, dx^1 \wedge \dots dx^k$ The wedge prod is designed to make this happen

OR we can see this by ordinary COV formula

$$\int_{Q(\mathcal{a})} f(u)\, d^k u = \int_{\mathcal{a}} f(x) \, |\det DQ_x|\, d^k x$$ (well, how abs val rules out or changes)

▷ Now consider the more general case of integrating over a parameterized mfd



open?

closed sets for integration

$$\int_S \omega := \int_{\mathcal{a}} \alpha^* \omega$$

Thm 33.1 Invariance under diffeo

Given smooth diffeo $g : \mathcal{a} \to \mathcal{B}$

$\alpha(\mathcal{a}) = Y = \beta(\mathcal{B})$

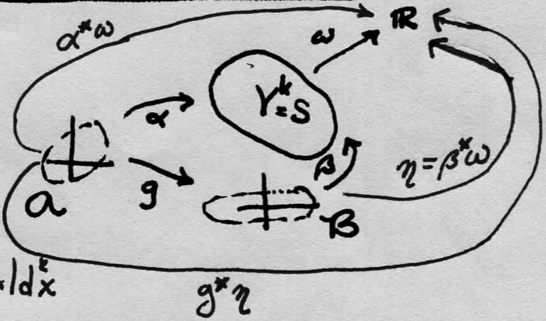
ω k-form on Y

ω integrable on $Y_\alpha = \alpha(\mathcal{a})$

det (Dg_x) does not change sign for any $x \in \mathcal{a}$

$\Rightarrow$ ω integrable over $Y_\beta = \beta(\mathcal{B})$

$$\int_{Y_\beta} \omega = \pm \int_{Y_\alpha} \omega \quad \text{where sign} = \text{sign}(\det Dg_x)$$



pf. $\displaystyle\int_{\alpha(\mathcal{a})} \omega \overset{?}{=} \int_{\beta(\mathcal{B})} \omega$

$$= \int_{\mathcal{B}} \beta^* \omega = \int_{\mathcal{B}} \eta = \int_{\mathcal{B}} f\, d^k y = \int_{g(\mathcal{a})} f\, d^k y \overset{\text{COV Thm}}{=} \int_{\mathcal{a}} f \circ g \, |\det Dg_x|\, d^k x$$

$\alpha = \beta \circ g$

$\alpha^* = g^* \beta^*$

Let $\eta = \beta^* \omega$

$\eta = f\, dx^1 \wedge \dots \wedge dx^k$

Same up to sign ($\det Dg_x$)

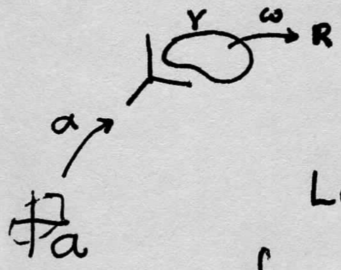
$$\int_{\mathcal{a}} \alpha^* \omega = \int_{\mathcal{a}} g^* \eta = \int_{\mathcal{a}} g^*(f\, dx^I) \overset{\text{p.269}}{=} \int_{\mathcal{a}} f \circ g \det Dg_x \, d^k x$$

□

Thm 33.2 Pulling down a k-form from $\mathbb{R}^n$

Let $\omega = f \underbrace{dx^{i_1} \wedge \cdots \wedge dx^{i_k}}_{dx^I}$ $\Rightarrow$ $\displaystyle\int_{\alpha(a)} \omega = \int_a (f \circ \alpha) \det\left[\frac{\partial \alpha^I}{\partial x^I}\right]$

Pf. ω is k-form in $\mathbb{R}^n$, it is built from $dx^1, dx^2 \ldots dx^n$ but only k size subsets.



we know $\displaystyle\int_Y \omega = \int_a \alpha^* \omega = \int_a \alpha^*(f\, dx^I) \overset{\substack{\text{Thm 32.2}\\ \text{p.269}}}{=} \int_a (f \circ \alpha) \det\left[\frac{\partial \alpha^I}{\partial x_\alpha}\right]$

Let's illustrate for $\mathbb{R}^3$ $\quad \omega = a\, dx \wedge dy + b\, dy \wedge dz + c\, dz \wedge dx$

$$\int_S \omega = \int_a \alpha^*(a\, dx \wedge dy) + \int_a \alpha^*(b\, dy \wedge dz) + \int_a \alpha^*(c\, dz \wedge dx)$$

$$\overset{\text{def}}{=} \int a(\alpha) \frac{\partial(\alpha^1, \alpha^2)}{\partial(u,v)}\, du\, dv + \int_a \overset{b(\alpha)}{\cancel{b(\alpha)}} \frac{\partial(\alpha^2, \alpha^3)}{\partial(u,v)}\, du\, dv + \int_a c(v) \frac{\partial(\alpha^3, \alpha^1)}{\partial(u,v)}\, du\, dv \qquad \square$$

Remark In ordinary calculus, we can write $\int f(x,y)\, dx\, dy$ but Munkres doesn't use this notation because it doesn't reflect $dx \wedge dy = -dy \wedge dx$ $\quad$ I will often use it, however.

p.280

Prob 1 $\quad$ Let $a = (0,1) \times (0,1)$

Let $\alpha : a \longrightarrow \mathbb{R}^3$ $\qquad Y_\alpha := \alpha(a)$

$(u,v) \longmapsto \begin{bmatrix} u \\ v \\ u^2+v^2+1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\qquad$ Evaluate $\displaystyle\int_{Y_\alpha} y\, dy \wedge dz + xz\, dx \wedge dz$

This is $\displaystyle\int_a v \frac{\partial(y,z)}{\partial(u,v)} + u(u^2+v^2+1)\frac{\partial(x,z)}{\partial(u,v)}\, du\, dv$

$$\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2u & 2v \end{vmatrix} = -2u \qquad \begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2u & 2v \end{vmatrix} = 2v$$

$$\Rightarrow \int_a \omega = \int_0^1 \int_0^1 -v(2u) + 2uv(u^2+v^2+1)\, du\, dv$$

$$= 2\int_0^1 \int_0^1 (u^3 v + u v^3)\, du\, dv$$

Good enough - no need to do it

$\square$

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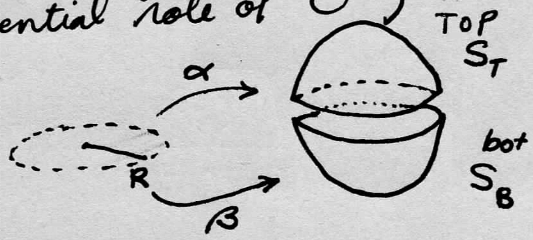
p. 280 ③

(#3) This problem is the tip of an iceberg of problems in this chapter involving the sphere S^{n-1} and this certain param by top half / bot half;

p. 280 #3 p. 292 #8 p. 296 #3 p. 321 #4

Here we will focus on $S^2 \subset \mathbb{R}^3$. We want to integrate over S^2 using the following charts (and show the essential role of $\mathcal{O}^R$)



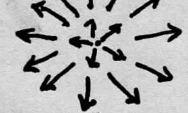
Let $\rho := R^2 - (u^2 + v^2)$ $S^2 = S^2(0,R)$ here

$$\alpha : \underset{\mathbb{R}^2}{B^2} \longrightarrow \underset{\mathbb{R}^3}{S^2} \qquad \langle u,v \rangle \longmapsto \begin{bmatrix} u \\ v \\ +\rho^{1/2} \end{bmatrix}$$

$$\beta : B^2 \longrightarrow S^2 \qquad \langle u,v \rangle \longmapsto \begin{bmatrix} u \\ v \\ -\rho^{1/2} \end{bmatrix}$$

We want to integrate $\displaystyle\int_{S^2} \omega$ where $\omega := \frac{1}{\|x\|^m}(x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy)$

We can write ω as the vf $\vec{F} = \frac{1}{r^m}\vec{r} = \frac{1}{r^m}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ radial flux



For $m=3$ we get Inv Squared Law

$$\int_{S^T} \omega = \int_{S_T} \vec{F} \cdot \hat{n}\, dS$$

▷ First we will do the calculations using vector calculus. We will also see how the "obvious" α and β give incompatible $\mathcal{O}^R$s. Then we will repeat the first calculation in the language of diff forms.

$$\alpha(u,v) = \begin{bmatrix} u \\ v \\ \rho^{1/2} \end{bmatrix} \qquad \rho := R^2 - u^2 - v^2$$

We need $\displaystyle\int_{B^2} \vec{F}(\alpha) \cdot (\vec{\alpha}_u \times \vec{\alpha}_v)\, du\, dv$

$$\vec{\alpha}_u = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2}\rho^{-1/2}(-2u) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{-u}{\rho^{1/2}} \end{bmatrix}$$

$$\vec{\alpha}_v = \begin{bmatrix} 0 \\ 1 \\ -\frac{v}{\rho^{1/2}} \end{bmatrix}$$

$$\vec{\alpha}_u \times \vec{\alpha}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -u/\rho^{1/2} \\ 0 & 1 & -v/\rho^{1/2} \end{vmatrix} = \begin{bmatrix} u/\rho^{1/2} \\ v/\rho^{1/2} \\ 1 \end{bmatrix}$$

Eval this at $\langle 0,0 \rangle$ (North Pole) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{e}_3$ Points UP Outward to ball

$$\beta(u,v) = \begin{bmatrix} u \\ v \\ -\rho^{1/2} \end{bmatrix}$$

$$\vec{\beta}_u = \begin{bmatrix} 1 \\ 0 \\ \frac{+u}{\rho^{1/2}} \end{bmatrix} \qquad \vec{\beta}_v = \begin{bmatrix} 0 \\ 1 \\ \frac{+v}{\rho^{1/2}} \end{bmatrix}$$

$$\beta_u \times \beta_v = \begin{vmatrix} i & j & k \\ 1 & 0 & +u/\rho^{1/2} \\ 0 & 1 & +v/\rho^{1/2} \end{vmatrix} = \begin{bmatrix} -u/\rho^{1/2} \\ -v/\rho^{1/2} \\ 1 \end{bmatrix}$$

Evaluate this at $\langle 0,0 \rangle$ (South Pole)

Again $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{e}_3$ Points UP but INWARD to ball this time

Now we must compute $\displaystyle\int_{S_T=\alpha(B)} F\cdot(\alpha_u\times\alpha_v)\,du\,dv$ and $\displaystyle\int_{S_B=\beta(B)} F\cdot(\beta_u\times\beta_v)$

$$F = \frac{1}{r^m}\begin{bmatrix}x\\y\\z\end{bmatrix} \qquad \|r\|^m = \left[u^2+v^2+(R^2-u^2-v^2)\right]^{m/2} = \left[R^2\right]^{m/2} = R^m$$

$$F(\alpha(u,v)) = \frac{1}{R^m}\begin{bmatrix}u\\v\\+\rho^{1/2}\end{bmatrix} \qquad \vec F\cdot(\vec\alpha_u\times\vec\alpha_v) = \frac{1}{R^m}\begin{bmatrix}u\\v\\\rho^{1/2}\end{bmatrix}\cdot\begin{bmatrix}u/\rho^{1/2}\\v/\rho^{1/2}\\1\end{bmatrix} = \frac{1}{R^m}\left(\frac{u^2}{\rho^{1/2}}+\frac{v^2}{\rho^{1/2}}+\frac{\rho^{1/2}\rho^{1/2}}{\rho^{1/2}}\right)$$

$$= \frac{1}{R^{m-2}}\frac{1}{\rho^{1/2}}$$

So we have $\displaystyle\frac{1}{R^{m-2}}\int_{B^2=\Delta(0,R)}\frac{1}{\sqrt{R^2-u^2-v^2}}\,du\,dv$ Convert to polar $u = r\cos\theta$, $v = r\sin\theta$

$$= \frac{1}{R^{m-2}}\int_{\theta=0}^{2\pi}\int_{r=0}^{R}\frac{1}{\sqrt{R^2-r^2}}\,r\,dr\,d\theta = \frac{2\pi}{R^{m-2}}\int_{r=0}^{R} w^{-1/2}\,r\,dr \qquad w := R^2-r^2,\quad dw = -2r\,dr$$

$$= \frac{2\pi}{R^{m-2}}\frac{1}{(-2)}\int_{r=0}^{R} w^{-1/2}\,dw = \frac{2\pi}{R^{m-2}}\frac{1}{(-2)}\left[2w^{1/2}\right]_{R^2}^{0} = \frac{-2\pi}{R^{m-2}}\left[0-R\right] = \boxed{\frac{2\pi}{R^{m-3}}}$$

▷ Now for the bot half:

$$\frac{1}{R^m}\int_{B^2}F(\beta)\cdot(\overrightarrow{\beta_u\times\beta_v})\,du\,dv \qquad \frac{1}{R^m}\begin{bmatrix}u\\v\\-\rho^{1/2}\end{bmatrix}\cdot\begin{bmatrix}-u/\rho^{1/2}\\-v/\rho^{1/2}\\1\end{bmatrix} = \frac{1}{R^m}\left(\frac{-u^2}{\rho^{1/2}}+\frac{-v^2}{\rho^{1/2}}-\frac{\rho^{1/2}\rho^{1/2}}{\rho^{1/2}}\right)$$

$$= \frac{1}{R^m}\left(\frac{-u^2-v^2-(R^2-u^2-v^2)}{\rho^{1/2}}\right)$$

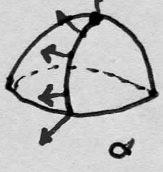
Then again we compute

$$\frac{-1}{R^{m-2}}\int_{B^2}\frac{1}{\sqrt{R^2-u^2-v^2}}\,du\,dv = \boxed{-\frac{2\pi}{R^{m-3}}} \qquad = \frac{1}{R^m}\left(\frac{-R^2}{\rho^{1/2}}\right) = \frac{-1}{R^{m-2}}\frac{1}{\rho^{1/2}}$$

$$\Rightarrow \int_{S^2} = \int_{S_T}+\int_{S_B} = \frac{2\pi}{R^{m-3}}-\frac{2\pi}{R^{m-3}} = 0$$

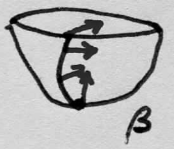
BUT THIS CAN'T BE RIGHT!
FLUX OF $\vec F$ IS CLEARLY GOING OUT SYMMETRICALLY IN TOP AND BOT
$\Rightarrow$ $\boxed{\mathcal{O}^R \text{ IS THE PROBLEM}}$

Later (Munkres p.289) we shall see that the Natural $\mathcal{O}^R$ on $B^3(0,R)$ induces an $\mathcal{O}^R$ on $\partial B^3 = S^2$ that corresponds to outward Pointing Normal.
Thus α belongs to the pos $\mathcal{O}^R$ on S^2, but β does not.
But even without this knowledge, we can see they are incompatible.



α

we can consider all $\hat n$ on a latitude line down to equator

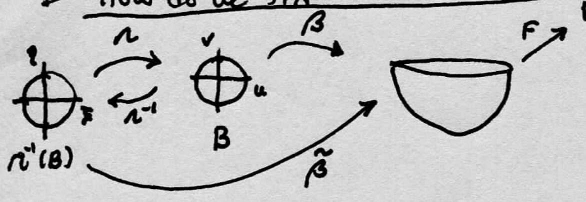


β

we can see the $\hat n$ vectors are opposite when they meet at equator.

▷ How do we fix this? Reverse the $\mathcal{O}^r$ of β

[a simpler version of this problem was done in M&T ch 7.4 Sheet 5]



Observe

$$\rho(r(\xi,\eta)) = R^2 - (-\xi)^2 - \eta^2 = R^2 - \xi^2 - \eta^2 = \rho(\xi,\eta) \quad \text{invar under } r$$

$$\overset{\shortparallel}{\rho(-\xi,\eta)}$$

$$\tilde\beta(\xi,\eta) = \beta(r(\xi,\eta)) = \beta(-\xi,\eta) = \begin{bmatrix} -\xi \\ \eta \\ -\rho^{1/2} \end{bmatrix} \qquad D\beta_\xi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ \frac{\xi}{\rho^{1/2}} & \frac{\eta}{\rho^{1/2}} \end{bmatrix}$$

$$\tilde\beta_\xi \times \tilde\beta_\eta = \begin{vmatrix} i & j & k \\ -1 & 0 & \xi/\rho^{1/2} \\ 0 & 1 & \eta/\rho^{1/2} \end{vmatrix} = \begin{bmatrix} -\xi/\rho^{1/2} \\ \eta/\rho^{1/2} \\ -1 \end{bmatrix}$$

$$F(\tilde\beta(\xi,\eta)) = F(-\xi,\eta,-\rho^{1/2}) = \frac{1}{R^m}\begin{bmatrix} -\xi \\ \eta \\ -\rho^{1/2} \end{bmatrix} \Rightarrow F(\tilde\beta)\cdot\tilde\beta_\xi \times \tilde\beta_\eta = \frac{1}{R^m}\begin{bmatrix} -\xi \\ \eta \\ -\rho^{1/2} \end{bmatrix} \cdot \begin{bmatrix} -\xi/\rho^{1/2} \\ \eta/\rho^{1/2} \\ -1 \end{bmatrix}$$

$$= \frac{1}{R^m}\left(\frac{\xi^2}{\rho^{1/2}} + \frac{\eta^2}{\rho^{1/2}} + \rho^{1/2}\right) \quad \text{Same calc as for } \alpha$$

$$= \frac{+1}{R^{m-2}} \frac{1}{\rho^{1/2}}$$

$$\Rightarrow \int_{r^{-1}(B)} F(\tilde\beta)\cdot\tilde\beta_\xi \times \tilde\beta_\eta \, d\xi\, d\eta = \frac{1}{R^{m-2}} \int_{r^{-1}(B)} \frac{1}{\rho^{1/2}(r(\xi,\eta))} \, d\xi\, d\eta$$

$$\boxed{\begin{array}{l}\text{COV Thm} \\ \displaystyle\int_{g(B)} f(\xi)\, d^2\xi = \int_B f(g(u))\,|\det Dg_u|\, d^2u\end{array}}$$

$$\overset{\text{COV}}{=} \frac{1}{R^{m-2}} \int_B \frac{1}{\rho^{1/2}(r(r^{-1}(u,v)))} |\det Dr^{-1}_u|\, du\, dv = \underbrace{\frac{1}{R^{m-2}} \int_B \frac{1}{\rho^{1/2}_{(u)}} |-1|\, du\, dv}_{\text{Same as prev Calc for } F\cdot\alpha_u\times\alpha_v}$$

$$= \frac{2\pi}{R^{m-3}}$$

Thus now

$$\int_{S^2} = \int_{S_T} + \int_{S_B} = \frac{2\pi}{R^{m-3}} + \frac{2\pi}{R^{m-3}} = \frac{4\pi}{R^{m-3}}$$

If $m=3$ (as for Inv Sq Law) $F = \frac{1}{r^3}\vec{r}$

Flux thru S^2 would be 4π, indep of R
(This is the solid angle for whole sphere M&T ch 7.4 Sheets 16, 17)

▷ Let's make the first α calc again, using diff forms:

$$\omega = \frac{1}{r^m}(x\, dy\wedge dz - y\, dx\wedge dz + z\, dx\wedge dy)$$

compute $\displaystyle\int_{\alpha(B^2)} \omega$ where $\alpha: B^2 \longrightarrow \mathbb{R}^3$, $(u,v) \longmapsto \begin{bmatrix} u \\ v \\ \rho^{1/2} \end{bmatrix}$ $\quad D\alpha_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -u/\rho^{1/2} & -v/\rho^{1/2} \end{bmatrix}$

$$\frac{1}{R^m}\int_{\alpha(B)} \overset{\alpha_1}{x}\, dy\wedge dz - \overset{\alpha_2}{y}\, dx\wedge dz + \overset{\alpha_3}{z}\, dx\wedge dy$$

$$= \int_B \left(u \frac{\partial(y,z)}{\partial(u,v)} - v\frac{\partial(x,z)}{\partial(u,v)} + \rho^{1/2}\frac{\partial(x,y)}{\partial(u,v)}\right) du\, dv$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{-u}{\rho^{1/2}} & \frac{-v}{\rho^{1/2}} \end{vmatrix} = u/\rho^{1/2}$$

$$\frac{\partial(x,z)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{-u}{\rho^{1/2}} & \frac{-v}{\rho^{1/2}} \end{vmatrix} = v/\rho^{1/2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\int_B \left(\frac{u^2}{\rho^{1/2}} + \frac{v^2}{\rho^{1/2}} + \rho^{1/2}\right) du\, dv$$

and from here it is just the same calc we have already done.

□