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Gallilean Transforms

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I really should have written
$G : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3 \times \mathbb{R}$
$(x,t) \longmapsto \begin{bmatrix} g(x,t) \\ t \end{bmatrix}$

What is the most general form of co-ord transform $g_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that preserves the form of Newton's 2nd Law $\vec{F} = m\vec{a}$ $\left[\, m\ddot{\alpha}(t) = F(\alpha(t)) \,\right]$

$\frac{d}{dt}(m(t)\dot{\alpha}(t))$

we aren't using the most general form of Newton's Law,

$= m(t)\ddot{\alpha}(t) + \dot{m}\dot{\alpha}$

① Let us consider first the case of a conservative force $F = -\nabla U$.

we know $U = \underset{\sim}{U} \circ g$
we want $\nabla \underset{\sim}{U} = (?)(\nabla U)$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[rr, "g"] \arrow[dr, "U"'] & & \mathbb{R}^3 \arrow[dl, "\underset{\sim}{U}"] \\ & \mathbb{R} & \end{tikzcd}$$

$x \qquad \underset{\sim}{x}$

$U = \underset{\sim}{U} \circ g$

we are ASSUMING m is const.

ⓐ $\underset{\sim}{U}(\underset{\sim}{x}, t) = U(x,t) = U(g^{-1}(\underset{\sim}{x},t),t)$

Then $\underset{\sim}{F} = -\nabla \underset{\sim}{U} = -T(D_1 \underset{\sim}{U}(\underset{\sim}{x},t))$ where T is the transpose operator.

$$= -T(D_1 U_x \cdot Dg^{-1}_{\underset{\sim}{x}})$$
$$= -(Dg^{-1}_{\underset{\sim}{x}})^T (D_1 U_x)^T$$
$$= (Dg^{-1}_{\underset{\sim}{x}})^T F$$

Later we will show Dg_x is an O.G. matrix.

Then $F(\alpha) = m\ddot{\alpha}$

$\underbrace{D(g^{-1})^T F}_{} = m\, D(g^{-1})^T \ddot{\alpha}$ mult both sides by $(Dg^{-1}_{\underset{\sim}{x}})^T$

$m\ddot{\beta} = \underset{\sim}{F}(\beta)$ ⓐ

$$\Rightarrow \boxed{\cancel{m}\ddot{\beta} = \cancel{m}\, D(g^{-1}_{\underset{\sim}{x}})^T \ddot{\alpha}} \quad (\star)$$

ⓑ Now we compute $\ddot{\beta}$ from $\beta(t) = g(\alpha(t),t)$ and we see necessary conds on g such that $(\star)$ can hold.

This is really the first component of $G : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$, $(x,t) \mapsto \begin{bmatrix} g(x,t) \\ t \end{bmatrix}$ $x = \alpha(t)$

but that does not ruin this argument

$g : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3$ diffeo
$(x,t) \longmapsto$

$\beta(t) = g(\alpha(t),t)$ transform of trajectory curve of particle.

$\dot{\beta}(t) = D_1 g_{(\alpha(t),t)}(\dot{\alpha}) + D_2 g_{(\alpha(t),t)}$.

$$\ddot{\beta}(t) = D_1 D_1 g_{(\alpha(t),t)}(\dot{\alpha},\dot{\alpha}) + \underbrace{D_2 D_1 g_{(\alpha(t),t)}(\dot{\alpha})} + D_1 g_{(\alpha(t),t)}(\ddot{\alpha}) + \underbrace{D_1 D_2 g_{(\alpha,t)}(\dot{\alpha})}$$
$$+ D_2 D_2 g_{(\alpha(t),t)}$$

$g : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3 \times \mathbb{R}$

cont'd →

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Thus $\ddot{\beta} = D_1 D_1 g_{(\alpha(t),t)}(\dot{\alpha},\dot{\alpha}) + 2 D_2 D_1 g_{(\alpha(t),t)}(\dot{\alpha}) + D_1 g_{(\alpha(t),t)}(\ddot{\alpha}) + D_2 D_2 g_{(\alpha,t)}$

cut down to 2 components to make writing it out easier.

$$\underbrace{[\dot{\alpha}_1, \dot{\alpha}_2]\left[\begin{bmatrix} g^{①}_{11} & g^{①}_{12} \\ g^{①}_{21} & g^{①}_{22} \end{bmatrix} \begin{bmatrix} g^{②}_{11} & g^{②}_{12} \\ g^{②}_{21} & g^{②}_{22} \end{bmatrix}\right]\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}}_{\text{must vanish}} + \underbrace{2\begin{bmatrix} \dot{g}^{①}_1 & \dot{g}^{①}_2 \\ \dot{g}^{②}_1 & \dot{g}^{②}_2 \end{bmatrix}\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}}_{\text{vanish}} + \underbrace{\begin{bmatrix} g^{①}_1 & g^{①}_2 \\ g^{②}_1 & g^{②}_2 \end{bmatrix}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \end{bmatrix}}_{\text{This is the only term that can survive}} + \underbrace{\begin{bmatrix} \ddot{g}^{①} \\ \ddot{g}^{②} \end{bmatrix}}_{\text{vanish}}$$

my notation for this object (a "tensor"?)

To make $\ddot{\beta} = (D(g_x^{-1}))^T \ddot{\alpha}$ from p.①, we must have:

(i) $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} g^{(k)} = 0 \quad \forall\, i,j,k \quad \left[\text{i.e. } D_1^2 g = 0\right]$

(ii) $\frac{\partial}{\partial x_j} \dot{g}^{(k)} = 0$

(iii) $\ddot{g} = 0$

(iv) AND $D_1 g_x \overset{!}{=} (D_1(g_x^{-1}))^T$

That is to say $(D_1 g_x)^T \overset{!}{=} D_1(g^{-1})_{g(x)}$
$= (D g_x)^{-1}$

but since $g^{-1} \circ g = Id$
$D(g^{-1})_{g(x)} D g_x = I$
$D(g^{-1})_{g(x)} = (D g_x)^{-1}$

<u>And thus Dg_x is an O.G. matrix.</u>

▷ so if we Taylor expand g:

$$g^t(x) = g^t(0) + D g^t_0(x) + \underbrace{\tfrac{1}{2!} D^2 g_0(x,x)}_{0} + \underbrace{\cdots}_{0} \quad \text{using (i)}$$

$$g^t(0) \to h(t), \qquad D g^t_0(x) \to A_{(t)} X$$

From (ii) $\quad \frac{\partial}{\partial x_j}\dot{g} = 0 \Rightarrow \dot{A}(t) = 0 \Rightarrow A$ does not depend on t

(iii) $\quad \ddot{g} = 0 \Rightarrow \ddot{h} = 0 \Rightarrow h(t) = \vec{a}t + \vec{b}$

$$\Rightarrow \boxed{g^t(x) = \vec{a}t + \vec{b} + Ax}$$

$Dg_x = A$ and from (iv) this is an O.G. matrix.

This is the most general transform that preserves the form of Newton's 2nd Law

"Q" since O.G.

thus $G(x,t) = \begin{bmatrix} g(x,t) \\ t \end{bmatrix} = \begin{bmatrix} Ax + \vec{a}t + \vec{b} \\ t \end{bmatrix}$

cont'd →

If there is no $U \ni F = -\nabla U$, how do we transform F?

(II) Now for a non-conservative F, when we apply a Gallilean transform $g_{A,b,c}(x,t) = Ax + tb + C$, we simply <u>Define</u>

$$\underset{\sim}{F} = D(g^{-1})^T_{g(x)} F = AF$$

and then the preceeding argument goes thru unchanged.

(III) Show the Gallilean transforms form a group:

$$G_{A,b,c}(x,t) = \begin{bmatrix} Ax + tb + c \\ t \end{bmatrix}$$ where A is O.G.

(i) closure

$$H_{P,d,e} \circ G_{Q,b,c}(x,t) = \Big(P(Qx + tb + c) + td + e,\ t\Big)$$
$$= \big(\underbrace{PQ}_{OG}x + t(Pb + d) + Pc + e,\ t\big) \checkmark$$

(ii) associative

$$K_{R,f,g} \circ (H_{P,d,e} \circ G_{Q,b,c}) = \big(RPQx + t\big(R(Pb+d) + Pb + d + g +$$

tedious

(iii) $\exists$ id $e = \begin{bmatrix} Ix \\ t \end{bmatrix} = G_{I,0,0}$

(iv) $\exists$ inverse $G^{-1}_{A,b,c} = G_{A^T, -A^Tb, -A^Tc}$

$H_{P,d,e} \rightsquigarrow$ $P = A^T$, $d = -A^Tb$, $e = -A^Tc$