12/27/2023

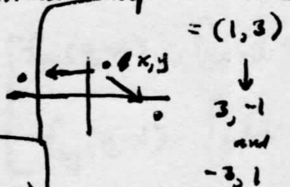
Sym and computing Derivatives

This also has a bearing on Sym of Hamiltonian, see 12/31 sheets

From Rudin POMA Prob 9.27 counterex showing $f_{xy}(0,0) \neq f_{yx}(0,0)$

If $f(x) = f(S(x))$ Let $S: \mathbb{R}^n \to \mathbb{R}^n$ linear map

$$\circledast \quad Df_x = Df_{S(x)} S$$



Consider example $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

observe f is preserved by $(x,y) \longmapsto (y,-x)$ and $(-y,x)$

$S_1 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ and $S_2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ $\quad S_2 = -S_1$ or $S_2 = S_1^T$

Let's only work with $(x,y) \neq (0,0)$ here.

By brute force, compute $Df_x = \left[\dfrac{x^4y + 4x^2y^3 - y^5}{[x^2+y^2]^2} \quad \dfrac{x^5 - 4x^3y^2 - y^4x}{[x^2+y^2]^2} \right]$

Just take $S = S_1$ $\quad (y,-x)$

$$\circledast \Rightarrow [D_1 f_{xy} \;\; D_2 f_{xy}] = [D_1 f_{(y,-x)} \;\; D_2 f_{(y,-x)}] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= [-D_2 f_{(y,-x)} \quad D_1 f_{(y,-x)}]$$

that is to say

$$[f_x(x,y) \;\; f_y(x,y)] = [-f_y(y,-x) \;\; f_x(y,-x)]$$

so given $f_x(x,y)$, subs $x \to y$, $y \to -x$ and mult whole thing by -1:

Find f_y by this: $f_y(x,y) = -f_x(y,-x)$

$$\frac{x^4y + 4x^2y^3 - y^5}{[\;\;]^2}$$

invariant under S $\to [\;\;]^2$

$$\frac{y^4(-x) + 4y^2(-x)^3 - (-x)^5}{[\;\;]^2}$$

$$\frac{-xy^4 - 4y^2x^3 + x^5}{[\;\;]^2}$$

Same!

so we can use the sym to save labor, or to check against errors made while differentiating.

$\triangle$ $D^2 f_x = D^2 f_{S(x)} S S$? $\quad S^T D^2 f_{S(x)} S$?

by a theorem we know $f_{xy} = f_{yx}$ if f is C^2 at that point.

$$D^2 f_x = \begin{bmatrix} \dfrac{12xy^5 - 4x^3y^3}{[\;]^3} & \dfrac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{[\;]^3} \\ \dfrac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{[\;]^3} & \dfrac{-12x^5y + 4x^3y^3}{[\;]^3} \end{bmatrix}$$

I will compute $S^T D^2 f_{S(x)} S$:

$$\underbrace{\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(y,-x)} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}}_{\begin{bmatrix} -f_{xy} & f_{xx} \\ -f_{yy} & f_{yx} \end{bmatrix}}}_{\begin{bmatrix} f_{yy} & -f_{yx} \\ -f_{xy} & f_{xx} \end{bmatrix}}$$

Thus

$$\Rightarrow \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} f_{yy}(y,-x) & -f_{yx}(y,-x) \\ -f_{xy}(y,-x) & f_{xx}(y,-x) \end{bmatrix}$$

so $f_{xx}(x,y) = f_{yy}(y,-x)$

$f_{yy}(x,y) = \boxed{f_{xx}(y,-x)}$

so once I find $f_{xx}(x,y)$ I can plug in $(y,-x)$ and get $f_{yy}(x,y)$

For the off-diag elts:

$f_{xy}(x,y) = -f_{yx}(y,-x)$

$\qquad \| \qquad\qquad\qquad \|$

$f_{yx}(x,y) = \boxed{-f_{xy}(y,-x)}$

so find $f_{xy}(x,y)$, plug in $(y,-x)$, mult by -1 $\Rightarrow$ get $f_{yx}(x,y)$

But we know $f_{xy}(x,y) \overset{!}{=} f_{yx}(x,y)$ [cont of 2nd partials]

That means $f_{xy}(x,y) = -f_{y(x}$

A&M 2nd

12/31/2023

Preview pages xxi

First sheet here is review of Hamiltonian formalism
2nd sheet is showing Sym of H (plus other conds) $\Rightarrow$ Sym of Solns, since Sym of vf X_H

Potential $V(q)$ $\quad q = \langle q^1, q^2, q^3 \rangle \in \mathbb{R}^3$

Particle moves on a curve $\quad m\ddot{q} = -\nabla V(q)$

momentum $p_i = m\dot{q}_i$ $\qquad$ energy $H(q,p) = \frac{1}{2}m\|p\|^2 + V(q)$

Newton equiv to Ham : $\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\partial H / \partial q_i \end{cases}$

$J = \begin{bmatrix} & I \\ -I & \end{bmatrix}$ $\quad$ I 3x3 id

$$\dot{\xi} = J\,\nabla H(\xi) \qquad \xi = (q,p)$$

$$X_H := J\,\nabla H$$

$\xi(t)$ satisfies Ham eqs $\Leftrightarrow$ $\xi(t)$ in integral curve of X_H

ie $\quad \dot{\xi}(t) = X_H(\xi(t))$ $\qquad \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} & I \\ -I & \end{bmatrix} \begin{bmatrix} \partial H/\partial q \\ \partial H/\partial p \end{bmatrix}$

$\boxed{\text{we can show } AJA^T = J \Leftrightarrow A^T J A = J \text{ done on that page xxii}}$

$\omega(v_1, v_2) = v_1 \cdot J \cdot v_2 = v_1^T J v_2$ $\quad$ row vector

$\omega(X_H(\xi), v) = dH_\xi(v)$

$$\begin{tikzcd} \overset{\xi}{\mathbb{R}^3 \times \mathbb{R}^3} \arrow[rr, "f"', "\text{symp}"] \arrow[dr, "H"'] & & \overset{\eta}{\mathbb{R}^3 \times \mathbb{R}^3} \arrow[dl, "K"] \\ & \mathbb{R} & \end{tikzcd}$$

$\xi(t)$ = flow curve defined by $\dot{\xi} = J(dH_\xi)^T$

$\eta(t)$ satisfies H's eqs $\quad$ ie $\quad \dot{\eta} = J(dK_\eta)^T$

$\Leftrightarrow$ f is Symp

$$[Df_\xi]\, J\, [Df_\xi]^T = J \quad \forall \xi$$

Pf. $\quad \dot{\eta} = Df_\xi\, \dot{\xi}$

$= A\dot{\xi} = AJ(dH_\xi)^T$ $\quad$ but $H = K \circ f$

$dH_\xi = dK_{f(\xi)}\, Df_\xi$

$(dH_\xi)^T = [Df_\xi]^T (dK_{f(\xi)})^T$

$= \underbrace{AJA^T}_{\overset{!}{=} J} (dK_\eta)^T$

$\overset{!}{=} J$ if we are to satisfy H eqs $\quad \square$

In my masters thesis p.52, I consider a ~~linear~~ sym

$S: \mathbb{R}^6 \to \mathbb{R}^6$

1. $[DS_\xi]^T J [DS_\xi] = J$ (or $-J$) reversing sym
2. $H \circ S = H$
3. $S^2 = I$

I want to show $\boxed{DS_\xi(X_H(\xi)) = -X_H(S(\xi))}$

Here for linear S $\boxed{S(X_H(\xi)) = -X_H(S(\xi))}$

$\dot{\xi} = \underbrace{J(DH_\xi)^T}_{X_H(\xi)}$

$S J (DH_\xi)^T = -J\underbrace{(DH_{S(\xi)})^T}_{-JS^{-T}DH_\xi^T}$

$-JS^T DH_\xi^T$

~~so I am done if~~

~~$SJ = -JS^{-T}$~~

So I am done if

$$SJ = -JS^T$$
$$SSJ = -SJS^T$$
$$J = -SJS^T$$
$$-J = SJS^T$$

or

$$SJS^T = -J$$

This is not exactly (1) but $S_{246} = S_{246}^T$ since diag matrix

$$\Rightarrow S_{246}^T J S_{246} = -J$$

$(H \circ S)(\xi) = H(\xi)$

$DH_{S(\xi)} DS_\xi = DH_\xi$

$DH_{S(\xi)} S = DH_\xi$

For example, p.55

$S_{246} = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix}$ "$\mathcal{L}$" there

$S = S^T$

$S^T J S = -J$

$JS = -S^{-T}J$

$S^T(DH_{S(\xi)})^T = DH_\xi^T$

$(DH_{S(\xi)})^T = S^{-T} DH_\xi^T$

$DH_{S(\xi)} S = DH_\xi$

$DH_{S(\xi)} \underbrace{SS}_{I} = DH_\xi S$

$(DH_{S(\xi)})^T = S^T(DH_\xi)^T$

$i^2 = -1$

Observe this arg $\quad J^2 = -I$

$$AJA^T = J$$
$$AJA^T(A^T)^{-1} = J(A^T)^{-1}$$
$$JAJ\,I = J^2(A^T)^{-1}$$
$$= -(A^T)^{-1}$$
$$JA\underbrace{J(-J)}_{+I} = -(A^T)^{-1}(-J)$$
$$JA = (A^T)^{-1}J$$
$$A^T J A = J$$

□