Math Symmetry

The Symmetry Principle

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ & Y \\ X \arrow[uu, "S"] \arrow[ur, "f"'] & \end{tikzcd}$$

$f: X \to Y \quad S: X \to X$

$f(S(x)) = f(x) \; \forall x$ [f invariant under S]

$f(x^*) = r \quad x^*$ is soln

$\Sigma = f^{-1}(r)$ set of solutions

$\Rightarrow$ $f(S(x^*)) = r$

so x^* is a soln $\Rightarrow$ Sx^* is a soln

Thus $S(\Sigma) = \Sigma$

(the set of solns is mapped to itself by S)

- If the soln is known to be uniq, then $Sx^* = x^*$ ($x^* \in Fix(S)$). This can enable us to know some properties of x^* by its invariance under S.

NOTE If $f(S(x)) = f(x) \; \forall x$

then $f(S(Sx)) = f(S(x)) = f(x)$

Thus for any $k \geq 0$ $f(S^k(x)) = f(x)$

If S^{-1} exists, let $y = S^{-1}(x)$

then $f(Sy) = f(y)$

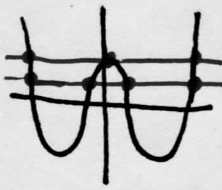
$\underset{f(x)}{\underset{\parallel}{f(SS^{-1}(x))}} = f(S^{-1}x)$ Thus the Sym holds $\forall S^k \in \langle S \rangle$ cyclic group generated by S

$S^0 = I$

[Nothing said about S being a linear map yet although usually we choose such an S]

ex consider an even poly $f: \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto ax^6 + bx^4 + cx^2$



we seek to solve $f(x) = r$ [This looks like a horiz line $y=r$ cutting the graph]

f is invariant under the reflection $S(x) = -x \quad S: \mathbb{R} \to \mathbb{R}$

$f(S(x)) = a(-x)^6 + b(-x)^4 + c(-x)^2 = f(x)$

Then if $f(x^*) = r$ so does $f(Sx^*) = f(-x^*) = r$ $\left[\begin{matrix} \text{soln} & & \text{soln} \\ x^* & \Rightarrow & -x^* \end{matrix}\right]$

Σ is the set of intersection pts for a given $y=r$ line $\quad S(\Sigma) = \Sigma$

A given x^* is not mapped to itself unless $x^* = -x^*$ i.e. $x^* = 0$ as shown on $y = r_1$ line

[odd polys coming up later]

ex $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad x^2 + y^2 = R^2$

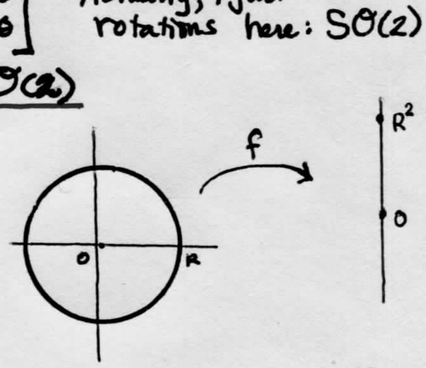
$\langle x, y \rangle \longmapsto x^2 + y^2$

This is invariant under the set of maps $S_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Actually, I just have rotations here: $SO(2)$

$f(S_\theta(x,y)) = f(x,y)$: $\quad$ This is the Lie Group $O(2)$

$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx - sy \\ sx + cy \end{bmatrix}$

$$\begin{aligned} &(cx - sy)^2 + (sx + cy)^2 \\ &= c^2x^2 - \cancel{2csxy} + s^2y^2 + s^2x^2 + \cancel{2csxy} + c^2y^2 \\ &= (c^2 + s^2)x^2 + (c^2 + s^2)y^2 \\ &= x^2 + y^2 \end{aligned}$$



The solution set $\Sigma = f^{-1}(R^2)$ $\quad$ we know $\overset{x^*}{\langle R, 0 \rangle}$ is in Σ

$S_\theta x^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} R\cos\theta \\ R\sin\theta \end{bmatrix}$ for $\theta \in [0, 2\pi]$ This sweeps out the set $S^1_{(0,R)}$

$\boxed{\text{This gives an explicit picture of the solns } f^{-1}(R^2) \text{ based on the Sym of the eq.}}$

Equivariant Sym $f: X \to X$, $S: X \to X$

more restrictive $\quad S^{-1} \circ f \circ S(x) = f(x) \; \forall x \; [\text{i.e } f \circ S = S \circ f]$ $\Rightarrow$ any x^* is a soln

$f(x^*) = r$ where $r \in Fix(S)$ $\quad \Rightarrow Sx^*$ also soln $\quad \Rightarrow S(\Sigma) = \Sigma$

Pf: if $f(x^*) = r$ then $S^{-1} f S(x^*) = f(x^*) = r$

$\Rightarrow f(Sx^*) = S(r) = r$ iff $r \in Fix(S)$

$\Rightarrow f(Sx^*) = r$ so for any soln x^*, Sx^* is also a soln $\square$

Again if $f \circ S = S \circ f$ then $\underline{f \circ S^k = S^k \circ f}$ because $f \circ S \circ (Sx) = SfS(x) = S^2 f(x)$

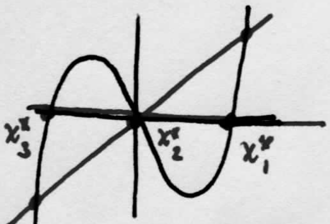
For inverses if $y = S^{-1}x$ $\quad fSy = Sfy$ i.e. $fSS^{-1}x = SfS^{-1}x$

$f(x) = SfS^{-1}(x)$

of course, this holds if $x \in Fix(S)$ $\quad S^{-1} \circ f(x) = f \circ S^{-1}(x)$ $\quad$ all of cyclic group $\langle S \rangle$ commutes with f

ex $\quad f: \mathbb{R} \to \mathbb{R}$ $\quad$ odd degree poly - all terms

$x \mapsto ax^5 + bx^3 + cx$ $\quad S(x) = -x \quad S: \mathbb{R} \to \mathbb{R}$



$f(Sx) = a(-x)^5 + b(-x)^3 + c(-x) = -f(x) = S(f(x))$

The only pt r where $S(r) = r$ $(-r = r)$ is $r = 0$

The example I drew would be more like $x(x-2)(x+2) = x^3 - 4x$

Then $x_1^* = 2$, $x_2^* = 0$, $x_3^* = -2$ $\quad \Sigma = \{-2, 0, 2\}$

$S(x_1^*) = x_3^*$ but x_2^* is FP of S! $\quad S(\Sigma) = \Sigma$

$y = mx$ There is reflection sym thru 0 but we are only using the horizontal line $y = 0$

$\triangleright$ There is a more general form

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[u, "S"] \arrow[r, "f"] & Y \arrow[u, "T"'] \end{tikzcd}$$

$T^{-1} \circ f \circ S = f$

or $f \circ S = T \circ f$

commonly

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, "S"] \arrow[r, "f"] & \mathbb{R} \arrow[u, "T"'] \end{tikzcd}$$

T is multiplication by some const.

We shall discuss this later

Let's look at some Sym problems found online.

(ex) This is a famous problem I saw in "Prob solving using Sym" Michael Penn youtube but also published in "Applications of Sym to Prob solving" Leiken, Berman, Zaslavsky. Probably the main one: Poyla Math. Discovery Part I p.153

No one does my sym methods of soln

Solve this sys of eqs:

$$\begin{aligned} x + 2y + 3z &= 30 \\ 2x + 3y + z &= 30 \\ 3x + y + 2z &= 30 \end{aligned}$$

[Each eq defines a plane in $\mathbb{R}^3$; we seek pts common to all i.e. their intersection]

$Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$$

Yes $A = A^T$ (sym matrix) but here the sym runs deeper

(1) Each row is perm of same elts

(2) special form of b – invariant under perms

Method 1 to Solve

We do note that A is nsing, so Linear Alg tells us $\exists$ only <u>one</u> soln

We can interpret the problem as saying:

There is $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\vec{x} \longmapsto [1\ 2\ 3]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and we seek $f(x^*, y^*, z^*) = 30$

And this f is invariant under 2 perm syms: $f(S_{231}x) = 30 \Rightarrow [1\ 2\ 3]\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [3\ 1\ 2]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 30$ (by eq 3)

and $f(S_{312}x) = 30 \Rightarrow [1\ 2\ 3]\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [2\ 3\ 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 30$ (by eq 2)

NOTE: I am doing the circular perms $1 \to 2 \to 3 \to 1$

and writing S_{231} for $\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$ and S_{312} for $\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$

$S_{312} = S_{231}^T = S_{231}^{-1}$

This is <u>not</u> following the std convention for naming perms and their associated matrices (which would basically be the transpose of what I have) but it works [See my sheets on Abstract Alg]

Since there is only 1 soln x^*, but $S_{231}x^*$ and $S_{312}x^*$ also satisfy the eq.

$\Rightarrow S_{231}x^* \overset{!}{=} x^*$ say $x^* = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Then $\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{!}{=} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{bmatrix} b \\ c \\ a \end{bmatrix} \overset{!}{=} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{aligned} b &\overset{!}{=} a \\ c &\overset{!}{=} b \\ c &= a \end{aligned}$$

$$\Rightarrow x^* = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

Now to determine the value of a:

$$f(a,a,a) \overset{!}{=} 30 \qquad \begin{aligned} a + 2a + 3a &= 30 \\ 6a &= 30 \\ a &= 5 \end{aligned} \qquad x^* = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

Cont'd

$\boxed{\text{Method 2 to Solve}}$

without affecting the soln, we can flip the last 2 eqs

$$\begin{aligned} x + 2y + 3z &= 30 \\ 3x + y + 2z &= 30 \\ 2x + 3y + z &= 30 \end{aligned} \qquad A := \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

This doesn't have $A = A^T$ but it is a <u>circulant</u> matrix

Rename S_{231} as just S. We will show this satisfies the <u>equivariant</u> form $S^{-1}AS = A$

Equivalent to show $AS = SA$

$$\begin{array}{cc} A & S \\ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} & \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \end{array} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} \checkmark$$

$Ax = b \qquad b = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$

$b \in Fix(S)$ so $ASx = Sb = b$ — As before we know $\exists!\, x^*$ so $Sx^* \overset{!}{=} x^*$

by same arg as prev page, $x^* = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ and likewise we solve $a + 2a + 3a = 30$, $a = 5$

▷ This also shows we have an eigenvalue eq: $Ax^* = \lambda x^*$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 6\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

$$Ax^* = \lambda x^* \;\Downarrow\; A(Sx^*) = \lambda(Sx^*)$$

This holds generally

$$\begin{aligned} S^{-1}AS &= A \\ S^{-1}ASx &= Ax = \lambda x \\ A(Sx) &= \lambda(Sx) \end{aligned}$$

▷ When would a general 3×3 matrix satisfy $AS = SA$ for <u>this perm</u> S

[Strang LAMA discusses this more generally ch 5]

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} = \begin{bmatrix} c & a & b \\ f & d & e \\ i & g & h \end{bmatrix}$$

whereas

$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

$$\Rightarrow \begin{aligned} h &\overset{!}{=} d \overset{!}{=} c \\ i &\overset{!}{=} e \overset{!}{=} a \\ g &\overset{!}{=} f \overset{!}{=} b \end{aligned}$$

$$\Rightarrow A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

circulant

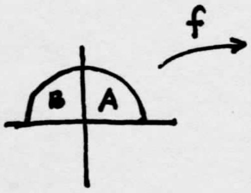
11/26/2022

Using Symm in Integration Problems

want to compute $\int_D f(x)\,d^3x$ where we can recognize $D = A \cup B$

and there is a map S where $S: D \to D$ (invariant set)

1. $S(A) = B$
2. $|\det DS_u| = 1$
3. $f \circ S = f$ on $A \cup B$ (fcn that respects the symm)



D

$$\int_{A\cup B} f(x)\,d^3x = \int_A f(x)\,d^3x + \int_{B=S(A)} f(x)\,d^3x$$

$$= 2\int_A f(x)\,d^3x \qquad \square$$

$$\int_{S(A)} f(x)\,d^3x \overset{\text{COV Thm}}{=} \int_A \underbrace{f(S(u))}_{f(u)}\,\underbrace{|\det DS_u|}_{1}\,d^3u$$

$$= \int_A f(u)\,d^3u$$ and "u" can be renamed "x".

we could combine (2) and (3) by

$$S^*(f\,dx^1 \wedge dx^2 \wedge dx^3) = f \circ S \det DS_u\, du^1 \wedge du^2 \wedge du^3$$

and say <u>this</u> is the fcn that respects the symm (but then no $\det DS_u = -1$) [G&P det thm]

9/16/2025

1

Sym in Integration

First let's discuss COV in integration

$$I(f, dx, (a,b)) := \int_a^b f(x)\,dx$$

Here is the typical case: we are given $\int_a^b F(x)\,dx$ but we recognize $F(x) = f(g(x))\,g'(x)$

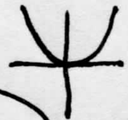
$u := g(x)$
$du = g'(x)\,dx$ thus $x \in (a,b) \Rightarrow u \in g((a,b))$ so $\int_{x=a}^{b} f(\underbrace{g(x)}_{u})\underbrace{g'(x)dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$

$$I(F, dx, (a,b)) \overset{\text{refactor}}{=} I(f\circ g, g'(x)dx, (a,b)) \overset{\text{cov thm}}{=} I(f, du, g((a,b)))$$ 'u' is a dummy name; rename it 'x'

$$= I(f, dx, g(a,b))$$

so if $g((a,b)) = (a,b)$ we have a <u>sym</u>

(★) $\int_a^b F(x)\,dx = \int_a^b f(x)\,dx$ same result from integrating 2 fcns on same interval



▷ Consider $\int_{-a}^{a} f(x)\,dx$ Sym interval $(-a,a)$ and <u>Even fcn</u> $f(-x) = f(x)$

Let $S(x) = -x$ and $f(S(x)) = f(x)$ <u>f invariant under S</u>

$$\int_{-a}^{a} f(x)dx = \underbrace{\int_{-a}^{0} f(x)dx}_{} + \int_0^a f(x)dx$$

since $f(S(x)) = f(x)$ $\quad -\int_{-a}^{0} f(S(x))\,S'(x)dx \overset{\text{COV}}{=} -\int_{S(-a)=+a}^{S(0)=0} f(u)\,du = \int_0^a f(u)\,du$ and rename 'u' as 'x' $\Rightarrow \int_0^a f(x)\,dx$

$$\Rightarrow \boxed{\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx}$$ where we had the ingredients (1) $S((-a,a)) = (-a,a)$ (2) $f(S(x)) = f(x)\ \forall x \in (-a,a)$

▷ Consider $\int_{-a}^{a} f(x)\,dx$: Sym interval $(-a,a)$ and <u>Odd fcn</u> $f(-x) = -f(x)$

$S(x) = -x$ but now $f(S(x)) = S(f(x))\ \forall x \in (-a,a)$ <u>f equivariant under S</u>

$$\int_{-a}^{a} f(x)dx = \underbrace{\int_{-a}^{0} f(x)dx}_{} + \int_0^a f(x)dx$$

$-\int_{-a}^{0} S(f(S(x))(S'(x))dx$ $\quad S'(x)dx = -dx$ but $S(f(Sx)) = -f(S(x))$ so we end up with $\int_{-a}^{0} f(S(x))S'(x)dx$

apply COV Thm $= \int_{S(-a)}^{S(0)} f(u)\,du = \int_a^0 f(u)\,du = -\int_0^a f(x)\,dx$ switching limits and renaming 'u'

$$\Rightarrow \boxed{\int_{-a}^{a} f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx = 0}$$

Here is an example using this (from youtube PrimeNewtons)

ex $\displaystyle\int_{-3}^{3} (x^5 \cos x + 1)\sqrt{9-x^2}\,dx$ $\qquad \underbrace{\underset{\text{odd}}{x^5}\,\underset{\text{even}}{(\cos x)}\,\underset{\text{even}}{\sqrt{9-x^2}}} + \underset{\text{even}}{\sqrt{9-x^2}}$

This whole thing is odd: $h(x) = x^5 \cos x\sqrt{9-x^2}$

$h(-x) = -h(x)$

$\displaystyle\int_{-3}^{3} h(x)\,dx = 0$ and we only need to evaluate $\displaystyle\int_{-3}^{3}\sqrt{9-x^2}\,dx = 2\int_0^3 \sqrt{9-x^2}\,dx$

$\triangleright$ Let's discuss some trickier problems, going back to ⊛: $\displaystyle\int_a^b F(x)\,dx = \int_a^b f(x)\,dx = \mathcal{I}$

This let's us do things like $\displaystyle\int_a^b (F-f)\,dx = 0$ (if that helps)

or $\displaystyle\mathcal{I} = \int_a^b F\,dx$ and $\displaystyle\mathcal{I} = \int_a^b f\,dx$ so $\displaystyle 2\mathcal{I} = \int_a^b (F+f)\,dx$

The later case helps for a certain problem on 1987 Putnam exam

First the prototype:

$$\mathcal{I} = \int_{-a}^{a} \frac{f(x)^{\,p}}{f(x)+f(-x)}\,dx = \int_{u=a}^{-a} \frac{f(-u)}{f(-u)+f(u)}(-du) = \int_{-a}^{a} \frac{f(-x)^{\,q}}{f(-x)+f(x)}\,dx$$

Youtube Center of Math 'Sym in Integrals'

COV $u=-x$, $du=-dx$

renaming 'u' as 'x'. We have domain sym and sym in the denominator

$$\mathcal{I} + \mathcal{I} = \int_{-a}^{a} \frac{p}{p+q} + \int_{-a}^{a}\frac{q}{q+p} = \int_{-a}^{a} \cancel{\frac{p+q}{p+q}} = \int_{-a}^{a} 1 = a - -a \Rightarrow \begin{matrix} 2\mathcal{I} = 2a \\ \mathcal{I} = a\end{matrix}$$

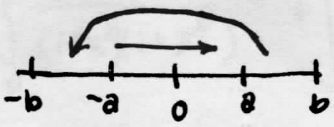
cf. youtube Lets Solve Math Problems

ex $$\int_{-1}^{1} \frac{\sqrt{\ln(6-u)}}{\sqrt{\ln(6-u)} + \sqrt{\ln(6+u)}}\,du = \int_{-1}^{1}\frac{f(6-u)}{f(6-u)+f(6+u)} = \int_{-1}^{1} \frac{F(-u)}{F(-u)+F(u)}$$

where $F(u) = f\circ\tau_6(u) = f(-u+6)$

$\Rightarrow \mathcal{I} = 1$

More Generally



$$\mathcal{I} = \int_a^b \frac{f(x)}{f(x)+f(-x+[a+b])}\,dx$$

Reflect (a,b) to $(-b,-a)$ by $x \mapsto -x$

Translate back to (a,b) by $x \mapsto x + (a+b)$

} This is the sym

COV $u = -x + (a+b) \rightsquigarrow x = -u + (a+b)$ sym!

$x = a \Rightarrow u = b$

$x = b \Rightarrow u = a$ $\qquad dx = -du$

(1) we must preserve (a,b)

(2) we must preserve denom.

$\downarrow$

$$\int_{u=b}^{a} \frac{f(-u+(a+b))}{f(-u+(a+b))+f(u)}(-du) = \int_{u=a}^{b} \frac{f(-u+(a+b))}{f(-u+(a+b))+f(u)}\,du \overset{\text{rename}}{=} \int_a^b \frac{f(-x+(a+b))}{f(-x+(a+b))+f(x)}$$

By the same trick $2\mathcal{I} = b-a$ or $\mathcal{I} = \dfrac{b-a}{2}$

ex $\displaystyle\int_2^4 \frac{\sqrt{\ln(\cancel{9-x})(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}}\,dx$ $\qquad u = -x+6 \quad du = -dx$

$x=2 \Rightarrow u=4$

$x=4 \Rightarrow u=2$

$\displaystyle\int_{u=4}^{2} \frac{\sqrt{\ln(u+3)}}{\sqrt{\ln(u+3)} + \sqrt{\ln(-u+9)}}(-du)$ rename u as x

$$= \int_2^4 \frac{f(x+3)}{f(x+3)+f(-x+9)}\,dx \text{ so } \mathcal{I} = \int_2^4 \frac{f(-x+9)}{f(-x+9)+f(x+3)}\,dx \text{ and } \mathcal{I} = \int_2^4 \frac{f(x+3)}{f(x+3)+f(-x+9)}\,dx$$

Thus $\displaystyle 2\mathcal{I} = \int_2^4 1\,dx = 4-2 = 2$ Thus again $\boxed{\mathcal{I} = 1}$ This is the Putnam problem $\square$

NOT a very general technique

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Here is a related discussion, which is not really sym but a similar transform

[Youtube Center of Math 'Sym in Integrals']

$$\int_0^\infty f(x)\,dx = \int_0^1 f\,dx + \int_1^\infty f\,dx$$

we can transform this 2^{nd} integral to have the same domain as the 1^{st}

$u := 1/x \quad du = -x^{-2}\,dx \Rightarrow dx = \frac{-1}{u^2}\,du$

$$\int_{x=1}^{\infty} f(x)\,dx = \int_{u=1}^{0} f(1/u)\left(\frac{-1}{u^2}\,du\right) = \int_{u=0}^{1} \frac{f(1/u)}{u^2}\,du$$

rename 'u' as 'x'
Same domain as 1^{st} term

$$\Rightarrow \int_0^\infty f(x)\,dx = \int_0^1 \left(f(x) + \frac{f(1/x)}{x^2}\right)dx$$

(ex) $$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_0^1 \frac{\ln x}{1+x^2}\,dx + \underbrace{\int_{u=1}^{0} \frac{\ln(u^{-1})}{1+\left(\frac{1}{u}\right)^2}\left(\frac{-1}{u^2}\,du\right)}_{\int_{u=1}^{0} \frac{-\ln u}{u^2+1}(-du) \;=\; -\int_{x=0}^{1} \frac{\ln x}{x^2+1}\,dx}$$

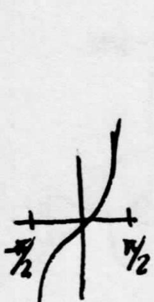
$$\Rightarrow \int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$$

(ex) $$\int_0^\infty \frac{1}{(1+x^2)(1+x^\alpha)}\,dx = \int_0^1 \frac{1}{(1+x^2)(1+x^\alpha)}\,dx + \underbrace{\int_{u=1}^{0} \frac{1}{\left(1+\frac{1}{u^2}\right)\left(1+\frac{1}{u^\alpha}\right)}\frac{-1}{u^2}\,du}$$

$$-\int_{u=1}^{0} \frac{1}{(u^2+1)\left(\frac{u^\alpha+1}{u^\alpha}\right)}\,du = \int_0^1 \frac{x^\alpha}{(x^2+1)(x^\alpha+1)}\,dx$$

renaming

$$= \int_0^1 \frac{1}{(1+x^2)(1+x^\alpha)}\,dx + \int_0^1 \frac{x^\alpha}{(1+x^2)(1+x^\alpha)}\,dx$$



$$= \int_0^1 \frac{\cancel{(1+x^\alpha)}}{(1+x^2)\cancel{(1+x^\alpha)}}\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \mathrm{Tan}^{-1}(1) - \mathrm{Tan}^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$\square$