11/10/2025

15

19-45

$$\mathrm{Curl}(F)(x) = \lim_{|V_\varepsilon| \to 0} \frac{1}{\mathrm{vol}(V_\varepsilon)} \int_{\partial V_\varepsilon} \hat{n} \times \vec{F} \, dS$$

From (19-23) $\int_V (\nabla \times F) \, dV = \int_{S=\partial V} (\hat{n} \times \vec{F}) \, dS$

dot prod both sides by $\hat{e}_i$

$$\int_V (\nabla \times F)^{(i)} \, dV = \int_{\partial V} (n \times F)^{(i)} \, dS$$

$$\Rightarrow \quad (\nabla \times F)^{(i)}(y_\varepsilon) = \frac{1}{\mathrm{vol}(V_\varepsilon)} \int_{\partial V_\varepsilon} (n \times F)^{(i)} \, dS$$

take $\lim\limits_{y_\varepsilon \to x} |V_\varepsilon| \to 0$

$$(\overrightarrow{\nabla \times F})(x) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{vol}(V_\varepsilon)} \int_{\partial V_\varepsilon} (\overrightarrow{n \times F}) \, dS$$

19-46 we can combine all of these

$$\nabla \star (\cdot) = \lim_{|V_\varepsilon| \to 0} \frac{1}{|V_\varepsilon|} \int_{\partial V_\varepsilon} dS \star (\cdot)$$

19-49 is in Feynman ch 4 writeup

REA   12/14/2025

15a

19-48   REAs problem statement and solution are misleading. This problem is really about showing $1/r$ is the Green's fcn for the Laplacian in 3-dim

This is discussed in detail in Griffel AFA ch 3.5

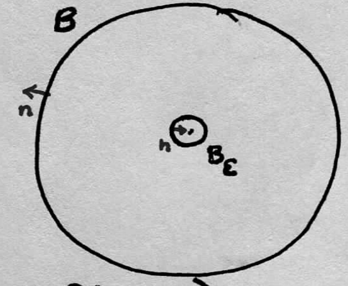
Here we want to show $\boxed{-\nabla^2 \underset{\sim}{\Lambda}_{1/r}(\varphi) = 4\pi \underset{\sim}{\delta}(\varphi)}$ where φ is a C^∞ fcn with cpt supp.

Specifically the aspect we are showing here is evaluating

$$\int_V (\nabla^2\varphi)\frac{1}{r}\,dV$$

we can regard $V = B(0,R)$ where R is large enough that this ball contains the supp(φ).

This integral is well-defined and finite despite the singularity at $r=0$, but we need to reduce it to a more useful form



Since $\nabla^2(1/r) \equiv 0$ for all pts $r \neq 0$ we can write

$$\int_{V=B} (\nabla^2\varphi)\frac{1}{r}\,dV = \int_{V=B}\Big(\overset{f\,\nabla^2 g}{\frac{1}{r}\nabla^2\varphi} - \overset{g\,\nabla^2 f}{\varphi\nabla^2(1/r)}\Big)\,dV$$

origin $\{0\}$ has m0

Now we want to apply Green's Id (which is Int by Parts $\rightsquigarrow$ which is Div Thm)
For Div Thm we can have no sings in the region, so we cut out a ball $B(0,\varepsilon)$ and we call the new region $V_\varepsilon := B \setminus B_\varepsilon$

Green II
REA 19-29

$$\int_{V_\varepsilon}\Big(\frac{1}{r}\nabla^2\varphi - \varphi\nabla^2(1/r)\Big)\,dV = \int_{\partial V_\varepsilon}\Big(\frac{1}{r}\nabla\varphi - \varphi\nabla(1/r)\Big)\cdot n\,dS$$

$$= \underbrace{\int_{\partial B}\Big(\frac{1}{r}\nabla\varphi - \varphi\nabla(1/r)\Big)\cdot n\,dS}_{0} + \int_{S(0,\varepsilon)}\Big(\frac{1}{r}\nabla\varphi - \varphi(\nabla(1/r))\Big)\cdot n\,dS$$

$\nabla\varphi \equiv 0$ and $\varphi \equiv 0$ on ∂B

$$\int_{S(0,\varepsilon)} \frac{1}{r}\nabla\varphi - \varphi\cdot\Big(\frac{-1}{r^3}\vec{r}\Big)\cdot(-\hat{e}_r)\,dS \qquad \vec{r} = \varepsilon\hat{e}_r$$

$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\Big(\frac{1}{\varepsilon}\nabla\varphi - \frac{\varphi}{\varepsilon^{\cancel{3}\,2}}(-\cancel{\varepsilon}\hat{e}_r)\Big)\cdot(-e_r)\,\varepsilon^2\sin\varphi\,d\theta\,d\varphi$$

$$= \iint \cancel{-\varepsilon\nabla\varphi\cdot\hat{e}_r} - \underbrace{\iint \varphi(r,\theta,\varphi)\sin\varphi\,d\varphi\,d\theta}_{\text{MVT for integrals}}$$

0 as $\varepsilon \searrow 0$ since $\nabla\varphi$ bdd

$$\varphi(x_\varepsilon)\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta$$

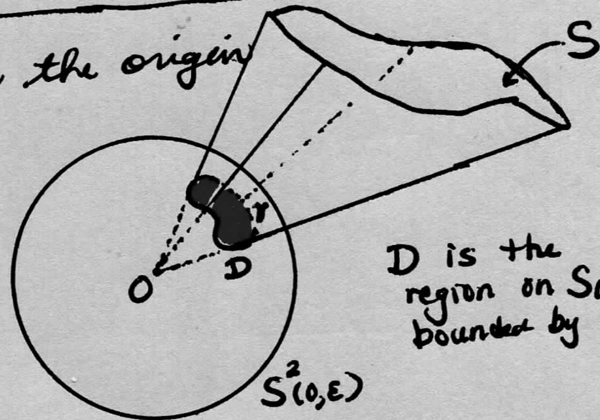
$$\Rightarrow \int_{V_\varepsilon}\nabla^2\varphi\,\frac{1}{r}\,dV = \int_{V_\varepsilon}\frac{1}{r}\nabla^2\varphi - \varphi\nabla^2(1/r)\,dV = -\varphi(x_\varepsilon)\,4\pi$$

$$\lim_{\varepsilon\searrow 0}\int_{V_\varepsilon}\nabla^2\varphi\,\frac{1}{r}\,dV = -\lim_{\varepsilon\searrow 0}\varphi(x_\varepsilon)\,4\pi$$

$$\int_V \nabla^2\varphi\,\frac{1}{r}\,dV = -4\pi\,\varphi(0) = -4\pi\,\underset{\sim}{\delta}(\varphi)$$

□

(19-50) The $\frac{1}{r^2}$ flux thru S is equal to the solid angle Ω covered by the radial projection of S



Consider the vf $\vec{F} = \frac{1}{r^2}\hat{e}_r$ radiating from the origin

Draw a simple clsd curve γ on $S^2(0,\varepsilon)$ and the radial lines thru γ form a cone. at some distance outside $S(0,\varepsilon)$ let the surf S form a single sheet cap (each ray penetrates S only once)

D is the region on $S(0,\varepsilon)$ bounded by γ.

Thm ($\frac{1}{r^2}$ Flux Cone) The solid angle $\Omega = \frac{\text{area}(D)}{\varepsilon^2}$

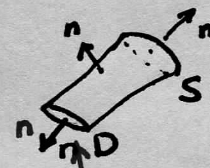
(1) The $F = \frac{1}{r^2}\hat{r}$ flux thru surf region D is the solid angle $\Omega = \frac{\text{area}(D)}{\varepsilon^2}$

(2) The flux thru the arb surf S has the same value Ω

Pf. step 1 Consider the region V of the cone that is between caps D and S. there are no singularities of F so we can use div thm here,

Div thm $\int_V (\nabla\cdot F)\,dV = \int_{\partial V} F\cdot n\,dS$ and $\nabla\cdot F = \nabla\cdot\left(\frac{1}{r^3}\vec{r}\right) = 0$ M&T p.196

Step 2 So $\int_{\partial V} F\cdot n\,dS = 0$



$\partial V = D \cup \{\text{sides of cone}\} \cup S$ oriented as shown

This is the inward pointing normal on $S^2(0,\varepsilon)$ i.e. $-\hat{e}_r$

$$\int_D F\cdot n\,dS + \underbrace{\int_{\text{Sides}} F\cdot n\,dS}_{0} + \int_S F\cdot n\,dS = 0$$

0 because $F\cdot n = \hat{r}\cdot\hat{n} = 0$ since $n \perp r$ here

$$\Rightarrow \int_S F\cdot n\,dS = -\int_D F\cdot n\,dS \quad (\star)$$

Step 3 Evaluate $\int_D F\cdot n\,dS = \int_D \left(\frac{1}{\varepsilon^2}\hat{e}_r\right)\cdot(-\hat{e}_r)\,dS = -\frac{1}{\varepsilon^2}\int_D dS = -\frac{1}{\varepsilon^2}\text{area}(D)$

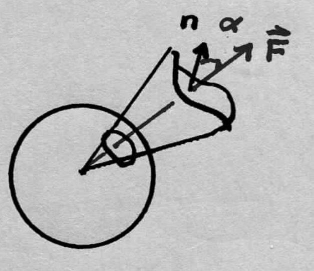
plug into $(\star)$: $\int_S F\cdot n\,dS = +\frac{1}{\varepsilon^2}\text{area}(D)$. and by def of solid angle $\Omega = \frac{A}{r^2}$ Just like $\theta = \frac{s}{r}$

$$\Rightarrow \int_S F\cdot n\,dS = \Omega \qquad \square$$

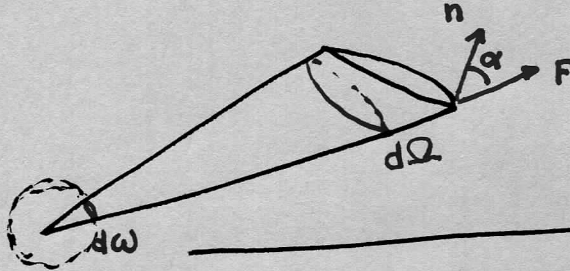
Cont'd →

$$\text{solid angle } \Omega = \int_S \frac{\cos\alpha}{r^2}\, dS$$

$$F\cdot n = \frac{1}{r^2}\hat{r}\cdot\hat{n} = \frac{1}{r^2}\cos\alpha$$



This common picture is misleading



like should be in the center

▷ Now let's write up the discussion in Spivak COM #5.31 p. 131

$$\vec{F} = \frac{1}{r^3}\vec{r} = \frac{1}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \omega := \frac{x}{r^3}\, dy\wedge dz - \frac{y}{r^3}\, dx\wedge dz + \frac{z}{r^3}\, dx\wedge dy$$

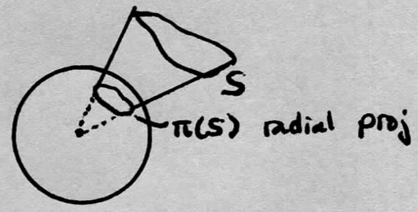
He calls this "$d\Theta$" but not really accurate.

$r = \sqrt{x^2+y^2+z^2}$ on $\mathbb{R}^3 - \{0\}$

(a) $d\omega = 0$ [this is $\nabla\cdot F = 0$]
"ω is closed"

(b) $\omega_{\vec{r}}(v,w) = (\vec{v}\times\vec{w})\cdot\frac{1}{r^3}\vec{r}$ not sure what good this is?

(d) and (c): Solid angle $\Theta = \int_{\pi(S)} dA$



and we want to show $\Theta = \int_S \omega$

Generalized Stokes $\int_M d\omega = \int_{\partial M} \omega$

$$0 = \int_{\pi(S)} \omega + \cancel{\int_{\text{sides}} \omega}^{\,0} + \int_S \omega$$

no flux

From here just evaluate $\int_{\pi(S)} \omega = \int_D \alpha^*\omega$ by calculus methods

No fcn Θ can exist on $S^2(0,r) \ni d\Theta = \omega$ (on all $S^2(0,r)$)

we found $\int_{\pi(S)} \omega = \int_{\pi(S)} dA$ take $\pi(S) = S^2(0,r)$ maximal cone

Then $\int_{S(0,r)} \omega = \text{area}(S^2_{(0,r)}) = 4\pi r^2 \neq 0$

But <u>If</u> $\omega = d\Theta$ $\quad 4\pi r^2 = \int_{S_r} d\Theta \overset{\text{Stokes}}{=} \int_{\partial S_r} \Theta = \int_{\emptyset} \Theta = 0 \Rightarrow\Leftarrow$ □

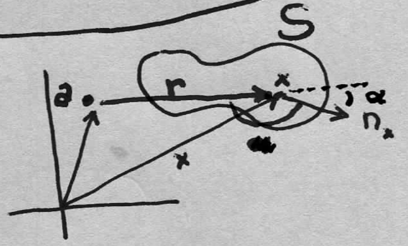
REA

11/13/2025

18

(19-51) Here is another formula to compute vol via a surf integral this time wrt an arb ref pt a, not nec O.

$$\left.\begin{array}{l}\text{Given a region } V \text{ and ref pt } a \notin V \\ \text{Let } x \text{ be a variable pt in } S = \partial V \\ \vec{r} := \vec{x} - \vec{a} \quad \text{and} \quad \hat{n}_x \text{ normal at } x\end{array}\right\} \Rightarrow \mathrm{vol}(V) = \int_S \|r\| \cos\alpha \, dS$$



Pf. Div $\displaystyle\int_V (\nabla \cdot F)\, dV = \int_S F \cdot n \, dS$

Take $\vec{F} = \vec{r} = \vec{x} - \vec{a}$

$r \cdot n = \|r\| \cdot 1 \cos\alpha$

$$\int_V \underbrace{\nabla \cdot (\vec{x} - \vec{a})}_{1+1+1} dV = \int_S \underset{(x-a)\cdot n}{r \cdot n} \, dS$$

$$= \int_S \|r_x\| \cos\alpha \, dS$$

$$3 \,\mathrm{vol}(V) = \int_S \|r_x\| \cos\alpha \, dS$$

$$\mathrm{vol}(V) = \frac{1}{3} \int_S r \cos\alpha \, dS \qquad \square$$

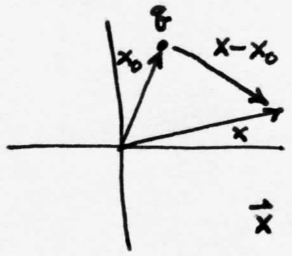
Marsden & Tromba VC 5/26/2003

Ch 7.4

(10) Generalize the book's version of Gauss' Law to (a) charge not at origin (b) charges (more than one, but discrete) and (c) a continuum charged body.

(a) Gauss' Law is given as $\int_S \vec{E}\cdot\hat{n}\,dS = Q$ if S encloses the charge.

The book shows $\frac{1}{4\pi}\int_S \frac{q}{\|x\|^3}\vec{x}\cdot\hat{n}\,dS = q$



If the charge was at x_0 rather than 0 we would simply have:

$\vec{x} = \vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\frac{1}{4\pi}\int_S \frac{q\,(\vec{x}-\vec{x}_0)\cdot\hat{n}\,dS}{\|x-x_0\|^3} = q$$

(b) Now consider a finite collection of charge particles: Part (a) applies to each of them

$$\sum_{j=1}^{N}\frac{1}{4\pi}\int_S \frac{q_j}{\|x-x_j\|^3}(\vec{x}-\vec{x}_j)\cdot\hat{n}\,dS = \sum^{N} q_j = Q$$

$$= \int_S \underbrace{\sum_{j=1}^{N}\frac{q_j}{4\pi}\frac{1}{\|x-x_j\|^3}(\vec{x}-\vec{x}_j)}_{E \text{ by superposition principle}}\cdot\hat{n}\,dS$$

(c) Now for a continuum (a charged solid body $\Omega\subset\mathbb{R}^3$ with charge density ρ)

analogously $$\frac{1}{4\pi}\int_\Omega\int_S \frac{\rho(y)}{\|x-y\|^3}(\vec{x}-\vec{y})\cdot\hat{n}_x dS\, d^3y = \int_\Omega \rho(y)\,d^3y = Q$$

$$= \int_S \underbrace{\left[\frac{1}{4\pi}\int_\Omega \frac{\rho(y)}{\|x-y\|^3}(\vec{x}-\vec{y})\,d^3y\right]}_{\vec{E}(x)}\cdot\hat{n}_x dS$$

$\square$

Marsden & Tromba VC

Ch 7.4

Based on book's ans p. 569

5/21/2003

(13) Let $\Omega \subset \mathbb{R}^3$ be a charged body and ρ the charge density fcn

Then $\rho \in C(\mathbb{R}^3 \to \mathbb{R})$ with $\operatorname{supp}(\rho) \subseteq \Omega$

The potential for this charge distribution is

$$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$x \longmapsto \int_\Omega \frac{1}{4\pi \|x-y\|} \rho(y)\, dV_y$$

$\vec{E} = \nabla\varphi$

(a) Show $\displaystyle\int_{\partial W} \vec{\nabla}\varphi \cdot \hat{n}\, dS = -\int_W \rho\, dV$ where W is any 3-mfd in $\mathbb{R}^3$ to which we can apply Gauss' Div Thm. (pretty general set).

This is just showing Gauss' Law for $Q = \int_\Omega \rho(y)\, dV$ Continuum body of charge.

$$\nabla\varphi(x) = \int_\Omega \frac{1}{4\pi} \nabla_x \left(\frac{1}{\|x-y\|}\right) \rho(y)\, dV_y$$

we can take ∇_x inside since everything is smooth

$$= -\int_\Omega \frac{1}{4\pi\|x-y\|^3} (\overrightarrow{x-y}) \rho(y)\, dV_y$$

vector of integrals

recall $\nabla(1/r) = -r^{-3}\vec{r}$ ch 3.5 #8a w/ $n=-1$

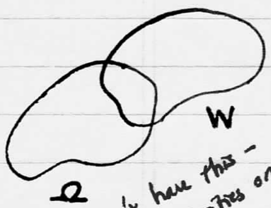
Thus

$$\int_{\partial W} \vec{\nabla}\varphi(x) \cdot \hat{n}\, dS_x = \int_{\partial W} -\int_\Omega \frac{1}{4\pi\|x-y\|^3} (\overrightarrow{x-y}) \cdot \hat{n}\, \rho(y)\, dV_y\, dS_x$$

$$= \int_\Omega -\frac{1}{4\pi} \underbrace{\int_{\partial W} \frac{1}{\|x-y\|^3} (\overrightarrow{x-y}) \cdot \hat{n}\, dS_x}\, \rho(y)\, dV_y$$

interchanging by Fubini

This is exactly the form p. 446–447 $\displaystyle\int_{S=\partial W} \frac{1}{r^3} \vec{r} \cdot \hat{n}\, dS = \begin{cases} 0 & y \notin W \\ 4\pi & y \in W \end{cases}$ no singularity / sing



we can't have this – no charge singularities on ∂W

$$= -\int_{\Omega \cap W} \frac{1}{\cancel{4\pi}} \cdot \cancel{4\pi}\, \rho(y)\, dV_y$$

$$= -\int_W \rho(y)\, dV_y$$

since $\rho \equiv 0$ outside of Ω we can integrate over all of W with no harm done. $\square$

(b) Show that φ satisfies Poisson's eq: $\nabla^2\varphi = -\rho$

Part (a) $\displaystyle\int_{\partial W} \vec{\nabla}\varphi \cdot \hat{n}\, dS = -\int_W \rho\, dV$

Div Thm $\displaystyle \| \quad \int_W \underbrace{(\nabla \cdot \nabla\varphi)}_{\nabla^2\varphi} dV$

$$\Rightarrow \int_W (\nabla^2\varphi - \rho)\, dV = 0 \Rightarrow \nabla^2\varphi = \rho$$

since the integrand is cont and the region W is arb $\square$

Marsden & Tromba VC
ch 7.4

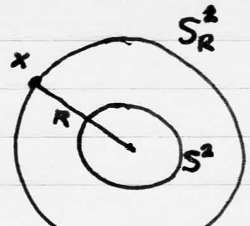
cf Ch 6.5 # 18
Calculating potential fcn w/out Gauss' Law for S^2

5/22/2003

(15) Consider a uniformly charged sphere S^2 carrying a charge Q. Show that the $\vec{E}$ field <u>outside</u> the sphere is just what it would be for a point charge of value Q at the center. Show that <u>inside</u> the sphere $\vec{E} = 0$.

This is just like p. 396–397 where we derived Coulomb's law for a pt. charge from Gauss' Law. The key idea is that the spherical symmetry of S^2 means that E must have spherical symmetry and thus $\vec{E} = E(r)\,\hat{e}_r$.

Consider a pt x outside S^2. It will lie on a concentric larger sphere S^2_R



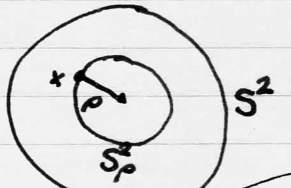
Apply Gauss' Law:

$$Q \stackrel{!}{=} \int_S \vec{E}\cdot\hat{n}\,dS = \int_{S^2_R} E(R)\,\hat{e}_R\cdot\hat{e}_R\,R^2\sin\varphi\,d\theta\,d\varphi$$

$$= E(R)\,R^2 \int_{[0,2\pi]\times[0,\pi]} \sin\varphi\,d\theta\,d\varphi = E(R)\,R^2\,4\pi$$

$\Rightarrow \vec{E}(R) = \dfrac{Q}{4\pi R^2}\,\hat{e}_R$ which is Coulomb's law for a unit test charge at radius R and a pt charge Q at R=0.

▷ Now let pt x be inside S^2
Again $\vec{E}(r) = E(r)\,\hat{e}_r$ by <u>Spherical symm.</u>
Let x be at radius ρ



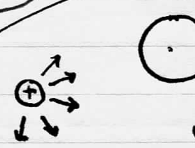
No charge

$$0 \stackrel{!}{=} \int_{S^2_\rho} E(\rho)\,\hat{e}_\rho\cdot\hat{e}_\rho\,dA$$

$$\Rightarrow 0 = E(\rho)\,\cancel{4\pi\rho^2}$$

$$\Rightarrow 0 = E(\rho)$$ and this applies for any interior radius ρ

□



This sphere also has no charge inside, but we can't say E=0 since no spherical sym

(16) For Gravity the exact same arg holds □

Marsden & Tromba
ch 7.4

Transport Eq.

(18) $\dot{x} = F(x)$ that is to say $\frac{d}{dt}(\varphi(t,x)) = F(\varphi(t,x))$

Variational eq: $D_x \frac{d}{dt}\varphi(t,x) = \frac{d}{dt} D(\varphi^t)_x = DF_{\varphi(t,x)} D(\varphi^t)_x$

(a) Show: $\frac{d}{dt}\det(D(\varphi^t)_x) = \operatorname{div}(F(\varphi(t,x)))\det D(\varphi^t)_x$

Step 1 $\frac{d}{dt}\det\begin{bmatrix} -a- \\ -b- \\ -c- \end{bmatrix} = \det\begin{bmatrix} -\dot{a}- \\ -b- \\ -c- \end{bmatrix} + \det\begin{bmatrix} -a- \\ -\dot{b}- \\ -c- \end{bmatrix} + \det\begin{bmatrix} -a- \\ -b- \\ -\dot{c}- \end{bmatrix}$

7/9/2022
I am not sure this arg is solid. See my write up of chain rule examples in ch2.4 or my write up of Avez DC ch 1
[The result here is correct, not the justification.]

To see this, let $g(a,b,c) = \det\begin{bmatrix} -a- \\ -b- \\ -c- \end{bmatrix}$

$\frac{d}{dt} g = D_1 g_{(a,b,c)}(\dot{a}) + D_2 g_{(a,b,c)}(\dot{b}) + D_3 g_{(a,b,c)}(\dot{c})$ chain rule

But if $G(a) := g(a,\underbrace{b,c}_{\text{fixed}}) \Rightarrow \underbrace{DG_a(\dot{a})}_{G(\dot{a})} = D_1 g_{(a,b,c)}(\dot{a})$

$G(\dot{a})$ since det is linear in each row.

Step 2 Thus we have

$$\frac{d}{dt}\det D(\varphi^t)_x = \begin{vmatrix} \dot{\varphi}^1_x & \dot{\varphi}^1_y & \dot{\varphi}^1_z \\ \varphi^2_x & \varphi^2_y & \varphi^2_z \\ \varphi^3_x & \varphi^3_y & \varphi^3_z \end{vmatrix} + \begin{vmatrix} \varphi^1_x & \varphi^1_y & \varphi^1_z \\ \dot{\varphi}^2_x & \dot{\varphi}^2_y & \dot{\varphi}^2_z \\ \varphi^3_x & \varphi^3_y & \varphi^3_z \end{vmatrix} + \begin{vmatrix} \varphi^1_x & \varphi^1_y & \varphi^1_z \\ \varphi^2_x & \varphi^2_y & \varphi^2_z \\ \dot{\varphi}^3_x & \dot{\varphi}^3_y & \dot{\varphi}^3_z \end{vmatrix}$$

But if we write out the variational eq (see top ↑):

$$\begin{bmatrix} \dot{\varphi}^1_x & \dot{\varphi}^1_y & \dot{\varphi}^1_z \\ \dot{\varphi}^2_x & \dot{\varphi}^2_y & \dot{\varphi}^2_z \\ \dot{\varphi}^3_x & \dot{\varphi}^3_y & \dot{\varphi}^3_z \end{bmatrix} = \begin{bmatrix} D_1F^1 & D_2F^1 & D_3F^1 \\ D_1F^2 & D_2F^2 & D_3F^2 \\ D_1F^3 & D_2F^3 & D_3F^3 \end{bmatrix} \begin{array}{c} \begin{bmatrix} \varphi^1_x & \varphi^1_y & \varphi^1_z \\ \varphi^2_x & \varphi^2_y & \varphi^2_z \\ \varphi^3_x & \varphi^3_y & \varphi^3_z \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{call this row } D\varphi^1 \\ \leftarrow D\varphi^2 \\ \end{array}$$

Then this becomes

$$\det\begin{bmatrix} D_1F^1(\vec{D\varphi^1}) + D_2F^1(\vec{D\varphi^2}) + D_3F^1(\vec{D\varphi^3}) \\ — \quad \vec{D\varphi^2} \quad — \\ — \quad \vec{D\varphi^3} \quad — \end{bmatrix} = D_1F^1 \det\begin{bmatrix} -D\varphi^1- \\ -D\varphi^2- \\ -D\varphi^3- \end{bmatrix}$$

And doing the same to the other terms, we obtain

$$\underbrace{(D_1F^1 + D_2F^2 + D_3F^3)}_{\nabla\cdot F}\det D(\varphi^t)_x \qquad \square$$

Cont'd →

18 cont'd

(b) Let $f: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, $(x,y,z,t) \mapsto$ and $\Omega_t := \varphi_t(\Omega_0)$ $\Omega_0 \subseteq \mathbb{R}^3$

$$\frac{d}{dt}\int_{\Omega_t} f(y,t)\,dy \overset{\text{COV Thm}}{=} \frac{d}{dt}\int_{\Omega_0} f(\varphi(t,x),t)\,\underbrace{|\det D(\varphi^t)_x|}_{J(x,t)}\,dx$$

$$= \int_{\Omega_0}\Big[D_1 f_{(\varphi(t,x),t)}\underbrace{(\dot\varphi(t,x))}_{F(\varphi(t,x))} + \underbrace{D_2 f_{(\varphi(t,x),t)}}_{\frac{\partial f}{\partial t}}\Big]J(x,t) + f(\varphi(t,x),t)\,\dot J(t,x)\,dx$$

Step 1 $\updownarrow$ $\operatorname{div}(F(\varphi(t,x)))J(t)$

$$= \int_{\Omega_0}\Big[D_1 f_{(\varphi(t,x),t)}(F(\varphi(t,x))) + \frac{\partial f}{\partial t}(\varphi,t) + f(\varphi,t)\operatorname{div}(F(\varphi))\Big]J(t,x)\,dt$$

$$\overset{\text{Cov Thm backwards}}{=} \int_{\Omega_t}\Big[\underbrace{D_1 f_{(y,t)}(F(y)) + \frac{\partial f}{\partial t}(y,t)}_{\text{“}\frac{Df}{Dt}\text{”}} + f(y,t)\operatorname{div}(F(y))\Big]\,dy$$

$\square$

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(c) Take $f \equiv 1$ and this formula becomes

$$\frac{d}{dt}\underbrace{\int_{\Omega_t} 1\,d^3x}_{Vol(\Omega_t)} = \int_{\Omega_t}\operatorname{div}(F)\,d^3x$$

Show TFAE: $\operatorname{div}(F) = 0 \overset{(i)}{\Rightarrow} Vol(\Omega_t) = Vol(\Omega_0) \overset{(ii)}{\Rightarrow} \det D\varphi^t_x = 1$ (iii)

(i) $\frac{d}{dt}Vol(\Omega_t) = \int_{\Omega_t}\operatorname{div}(F)\,d^3x = \int_{\Omega_t} 0 = 0 \Rightarrow Vol(\Omega_t) = \text{const} = Vol(\Omega_0)$

(ii) $Vol(\Omega_t) \overset{\text{def}}{=} \int_{\varphi_t(\Omega)} 1\,d^3y \overset{\text{COV}}{=} \int_\Omega |\det D\varphi^t_x|\,d^3x$ and $Vol(\Omega) = \int_\Omega 1\,d^3x$

but since we have $Vol(\Omega_t) = Vol(\Omega)$ we must have $\int_\Omega 1\,d^3x = \int_\Omega |\det D\varphi^t_x|\,d^3x \Rightarrow 1 = \det D\varphi^t_x$ since Ω arb.

Note we can drop the abs val since $\det D\varphi^t_x > 0$. This follows because $\varphi^t: \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeo. Thus $\det D\varphi^t_x \neq 0$ and it is pos since $\det D\varphi^0_x = I$

(iii) We established in (a): $\frac{d}{dt}\det(D\varphi^t_x) = \operatorname{div}(F)\det D\varphi^t_x$

Now if $\det D\varphi^t_x \equiv 1 \Rightarrow \frac{d}{dt}(\det D\varphi^t_x) = 0 \Rightarrow \operatorname{div}(F) = 0$

$\boxed{\text{QED}}$

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M&T ch 7.4

(19) Vector form of Transport Eq:

$$\frac{d}{dt}\int_{\Omega_t}(f\vec{F})\,d^3x = \int_{\Omega_t}\left[\frac{\partial}{\partial t}(f\vec{F}) + (\vec{F}\cdot\vec{\nabla})(f\vec{F}) + (f\vec{F})\,\mathrm{div}(F)\right]d^3x$$

I am going to rewrite this with ρ in place of f, as that is the particular form that we need:

$$\boxed{\frac{d}{dt}\int_{\Omega_t}(\rho\vec{F})\,d^3x = \int_{\Omega_t}\left[\frac{\partial}{\partial t}(\rho\vec{F}) + (\vec{F}\cdot\vec{\nabla})(\rho\vec{F}) + (\rho\vec{F})(\nabla\cdot F)\right]d^3x}$$

Pf. From prob 18, we have scalar version

$$\frac{d}{dt}\left(\int_{\Omega_t} f(t,x)\,d^3x\right) = \int_{\Omega_t}\left(f_t + \underbrace{D_xf(\vec{F})}_{\nabla_x f\cdot F \,=\, F\cdot\nabla f \,=\, (F\cdot\nabla)(f)} + f\,\mathrm{div}(F)\right)d^3x$$

This way of writing it is used below.

Now if we take $f = \rho F^{i}$ for $i=1,2,3$, we get 3 eqs:

$$\frac{d}{dt}\int_{\Omega_t}\begin{matrix}\rho F^{1}\\ \rho F^{2}\\ \rho F^{3}\end{matrix}\,d^3x = \int_{\Omega_t}\underbrace{\begin{matrix}(\rho F^{1})_t\\ (\rho F^{2})_t\\ (\rho F^{3})_t\end{matrix}}_{\frac{\partial}{\partial t}(\rho\vec{F})} + \boxed{\begin{matrix}D_x(\rho F^1)(\vec{F})\\ D_x(\rho F^2)(F)\\ D_x(\rho F^3)(F)\end{matrix}} + \underbrace{\begin{matrix}(\rho F^1)(\nabla\cdot F)\\ (\rho F^2)(\nabla\cdot F)\\ (\rho F^3)(\nabla\cdot F)\end{matrix}}$$

we need to work on this piece.

$$\Rightarrow \boxed{\frac{d}{dt}\int_{\Omega_t}\rho\vec{F}\,d^3x = \int_{\Omega_t}\left[\frac{\partial}{\partial t}(\rho\vec{F}) + (\vec{F}\cdot\vec{\nabla})(\rho\vec{F}) + (\rho\vec{F})\,\mathrm{div}(F)\right]d^3x}$$

we just need to establish the middle term:

First, we know that for any 2 fcns $\rho, h: \mathbb{R}^3 \to \mathbb{R}$ we can do prod rule:

$$D(\rho h)_x(\cdot) = h\,D\rho_x(\cdot) + \rho\,Dh_x(\cdot) \Rightarrow D(\rho h)_x(F) = h[\rho_x\ \rho_y\ \rho_z]\left[\vec{F}\right] + \rho[h_x\ h_y\ h_z]\left[\vec{F}\right]$$

$$= h\,\nabla\rho\cdot\vec{F} + \rho\,\nabla h\cdot\vec{F}$$

Now if we successively take $h = F^{1}, F^{2}, F^{3}$:

$$D_x(\rho F^1)(\vec{F}) = F^{1}\nabla\rho\cdot F + \rho\,\nabla F^{1}\cdot F = F^{1}\vec{F}\cdot\nabla\rho + \rho\,\vec{F}\cdot\nabla F^{1}$$

$$D_x(\rho F^2)(F) = F^{2}\nabla\rho\cdot F + \rho\,\nabla F^{2}\cdot F = F^{2}\vec{F}\cdot\nabla\rho + \rho\,\vec{F}\cdot\nabla F^{2}$$

$$D_x(\rho F^3)(F) = F^{3}\nabla\rho\cdot F + \rho\,\nabla F^{3}\cdot F = \underbrace{F^{3}\vec{F}\cdot\nabla\rho}_{\vec{F}(F\cdot\nabla)(\rho)} + \underbrace{\rho\,F\cdot\nabla F^{3}}_{\rho(F\cdot\nabla)(\vec{F})}$$

By p. 191 def of what this symbol means:

$$\vec{V} = (F\cdot\nabla)(\vec{G}) := \begin{bmatrix}F^1G^1_x + F^2G^1_y + F^3G^1_z\\ F^1G^2_x + F^2G^2_y + F^3G^2_z\\ F^1G^3_x + F^2G^3_y + F^3G^3_z\end{bmatrix}$$

a vector

$$= (F\cdot\nabla)(\rho)\vec{F} + \rho(F\cdot\nabla)(\vec{F})$$

$$= (F\cdot\nabla)(\rho\vec{F}) \text{ by product rule, backwards}$$

QED