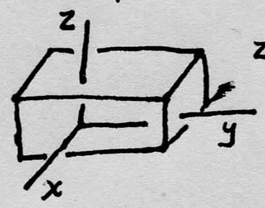
M&T ch 7.4

Gauss' Divergence Thm Summary and extensions ThanksGiving

We define a 'brick' to be a set in $\mathbb{R}^3$ where the ∂ can be expressed as the graph of a fcn over the xy, yz, and xz planes.

For example, a rectangular box qualifies

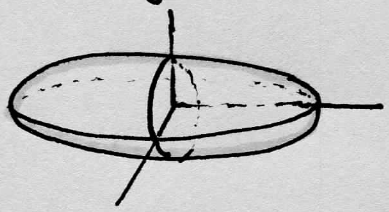


$z = f_{top}(x,y) = a \quad f_{bot}(x,y) = -a$

$x = f_{front}(y,z) = b \quad f_{back}(y,z) = -b$

$y = f_{right}(x,z) = c \quad f_{left}(x,z) = -c$

a whole ellipsoid also qualifies
(but even more useful to us later would be slicing it into smaller bricks along its planes of sym)



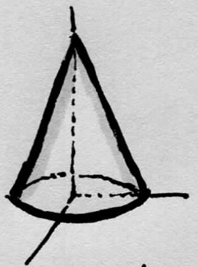
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

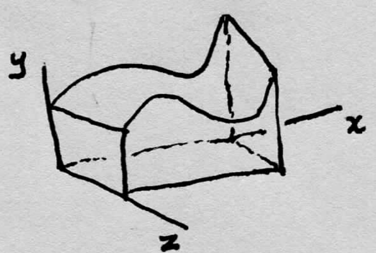
$$x = \pm\sqrt{a^2\left(1 - y^2/b^2 - z^2/c^2\right)}$$

$$y = \pm\sqrt{}$$

$$z = \pm\sqrt{}$$

and a cone





Note that a box like this does not qualify since the top would not be a graph over yz plane.

<u>Plan</u>: we will show (1) The Div Thm is true for a brick
(2) It is true for any region V that can be built from bricks
(this is more powerful than pfs regarding V to be a smooth cpt 3-mfd because we can easily handle shapes with sharp corners, etc...)

<u>Thm</u> <u>Gauss Div Thm</u> for a Brick

Let V be a brick in $\mathbb{R}^3$
∂V $\circlearrowleft$ with outward pointing normal
vf $F: V \to \mathbb{R}^3$ has <u>no singularities</u> in V

$$\Rightarrow \int_{\partial V} \vec{F}\cdot\hat{n}\,dS = \int_V \overset{\nabla\cdot F}{\mathrm{div}(F)}\,dV$$

Pf. $\boxed{\text{Step 1}}$ Let's write $\vec{F} = \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ for convenience

$$\underbrace{\int_{\partial V} F\cdot n\,dS}_{\int_{\partial V} P\hat{i}\cdot\hat{n}\,dS + \int_{\partial V} Q\hat{j}\cdot\hat{n}\,dS + \int_{\partial V} R\hat{k}\cdot n\,dS} = \underbrace{\int_V (\nabla\cdot F)\,dV}_{\int_V P_x\,dV + \int_V Q_y\,dV + \int_V R_z\,dV}$$

cont'd →

It is enough to show $\int_V R_z \, dV = \int_{\partial V} R \hat{k} \cdot \hat{n} \, dS$ because the other terms are the same idea

$\boxed{\text{Step 2}}$ $$\int_V \frac{\partial}{\partial z} R \, dV \overset{\text{Fubini}}{=} \int_{D_{xy}} \left[\int_{z=g(x,y)}^{z=f(x,y)} \frac{\partial}{\partial z} R(x,y,z) \, dz \right] dx\,dy = \int_{D_{xy}} \left[R(x,y,f(x,y)) - R(x,y,g(x,y)) \right] dx\,dy$$

This is the crucial step where we need $R_z(x,y,z)$ to exist $\forall$ pts $\langle x,y,z \rangle$ in the z line from $\langle x,y,f(x,y) \rangle$ to $\langle x,y,g(x,y) \rangle$ or else FTOC doesn't apply (Rudin POMA p.324)

$\boxed{\text{Step 3}}$ Expand $\int_{\partial V} R(\hat{k} \cdot \hat{n}) \, dS$ $\qquad \partial V = S_{top} \cup S_{bot} \cup \left[S_{north} \cup S_{South} \cup S_E \cup S_W \right]$

For all these faces, $\hat{k} \cdot \hat{n} = 0$ so the terms vanish

$$= \int_{S_{top}} R \, k \cdot n \, dS + \int_{S_{bot}} R \, k \cdot n \, dS$$

For S_{top} this graph is $\varphi : (x,y) \longmapsto \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ and we know $\hat{n}_f = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$

thus $k \cdot n_f = 1$

$$\int_{S_{top}} R(k \cdot n) \, dS = \int_{D_{xy}} R(\varphi(x,y)) \, 1 \, dx\,dy = \int_{D_{xy}} R(x,y,f(x,y)) \, dx\,dy$$

For S_{bot} we would have something very similiar, but $\hat{n}_{bot}$ must be outward pointing, so $\hat{k} \cdot \hat{n}_{bot} = -1$

$$\Rightarrow \int_{S_{top}} R \, k \cdot n_{top} \, dS + \int_{S_{bot}} R \, k \cdot n_{bot} \, dS = \int_{D_{xy}} \left[R(x,y,f(x,y)) - R(x,y,g(x,y)) \right] dx\,dy$$

Same

$\boxed{\text{Step 4}}$ Thus we see the results of <u>Step 2</u> and <u>Step 3</u> are identical

Thus $\int_{\partial V} R \hat{k} \cdot \hat{n} \, dS = \int_V R_z \, dV$ and we can apply the same arg to show

$$\int_{\partial V} P \hat{x} \cdot n \, dS = \int_V P_x \, dV$$

$$\int_{\partial V} Q \hat{j} \cdot n \, dS = \int_V Q_y \, dV$$

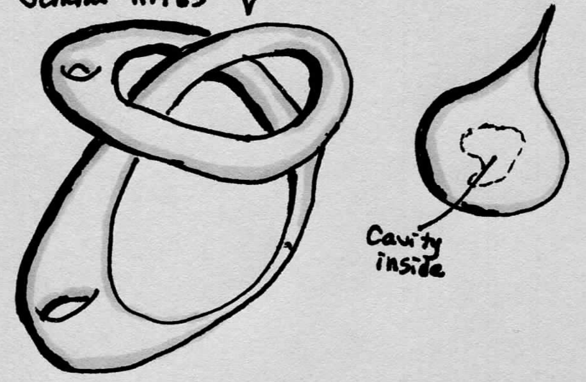
and the thm is proven for a brick $\square$

General Gauss Div Thm

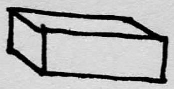
$\left.\begin{array}{l} V \text{ is a union of bricks, joined on their } \partial \\ \text{vf } \mathbf{F} \text{ has \underline{no singularities} \underline{in} } V \end{array}\right\} \Rightarrow \int_{\partial V} F\cdot n\,dS = \int_V (\nabla\cdot F)\,dV$

See counter example

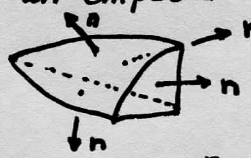
General mfds V



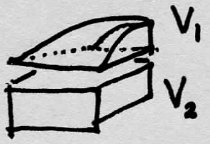
To build these types of things, the bricks we care about are rectangular boxes for interior



and surface pieces with 3 flat sides and 1 curved side (e.g. cutting an ellipsoid into 8^{th}s)



We can join 2 bricks together on their flat ∂s (they both are Θ^R with O.P.N.)
Thus we could join any finite number.



any flux going out the bot of V_1 is going in the top of V_2

Let the faces of V_1 be named T_i and V_2 named S_i

$$\int_{V_1\cup V_2} (\nabla\cdot F)\,dV = \int_{\partial(V_1\cup V_2)} \vec{F}\cdot\hat{n}\,dS = \int_{\partial V_1} F\cdot n\,dS + \int_{\partial V_2} F\cdot n\,dS$$

$$= \int_{T_{top}} + \int_{T_{sides}} + \int_{T_{bot}} + \int_{S_{top}} + \int_{S_{sides}} + \int_{S_{bot}}$$

These cancel because $\hat{n}_{bot} = -\hat{n}_{top}$

$$= \int_{\text{Only outer surfs of } (V_1\cup V_2)} F\cdot n\,dS \qquad \square$$

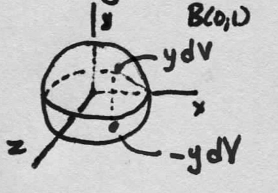
p.444

ex 3 Calculate $\int_S F\cdot n\,dS$ where $F = \begin{bmatrix} 2x \\ y^2 \\ z^2 \end{bmatrix}$ and $S = S^2_{(0,1)}$

we can apply Div Thm: $\int_{S^2} F\cdot n\,dS = \int_{B(0,1)} (\nabla\cdot F)\,dV \qquad \nabla\cdot F = [D_x\ D_y\ D_z]\begin{bmatrix} 2x \\ y^2 \\ z^2 \end{bmatrix} = 2+2y+2z$

$$\int_{B(0,1)} (2+2y+2z)\,dV = \underbrace{2\int_B dV}_{2\,\frac{4}{3}\pi(1)^3} + \underbrace{2\int_B y\,dV}_{0 \text{ by sym}} + \underbrace{2\int_B z\,dV}_{0} = \boxed{\frac{8}{3}\pi}$$

▷ why is $\int_{B(0,1)} y\,dV = 0$ by sym?



From p.305 #11

Sym $g: \mathbb{R}^3 \to \mathbb{R}^3$

$x \mapsto \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$g(B) = B$ and $f(g(x)) = -f(x)$

$$\int_{g(B)} f \overset{\text{cov}}{=} \int_B f(g(x))\,|\det Dg_x|\,d^3x = \int_B -f(x)\,d^3x$$

$\int_{g(B)} f = \int_B f$ since $g(B) = B$

$$\Rightarrow \int_B f = -\int_B f \Rightarrow \int_B f = 0 \qquad \square$$

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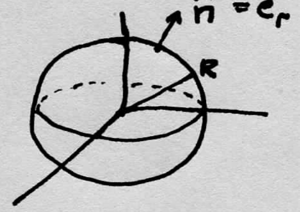
3a

Here is a counter example showing why we cannot have any singularities in the region where the Divergence Thm is applied



Inv squared vf $\vec{F} = \frac{1}{\|x\|^3}\vec{x} = \frac{1}{r^2}\hat{e}_r$

$$\int_{B(0,R)} (\nabla \cdot F)\, dV \overset{?}{=} \int_{\partial B} F \cdot n\, dA$$

$\nabla \cdot F = \nabla \cdot \left(\frac{1}{r^3}\vec{r}\right) = 0$

for all pts except $r = 0$

$\{0\}$ has m 0, so no effect on vol integral

$$= \int_{B - \{0\}} 0\, dV$$

$$= 0$$

$$\int_{\partial B} F \cdot n\, dA = \int_{\partial B} \frac{1}{r^2}\hat{e}_r \cdot \hat{e}_r\, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \frac{1}{\cancel{R^2}} \cancel{R^2} \sin\varphi\, d\varphi\, d\theta$$

$$= 4\pi$$

$\boxed{\text{ch 6.5 Sheet 11}}$

$0 \neq 4\pi \Rightarrow\!\Leftarrow$

Thus Div Thm does not hold if there is a singularity in the volume region $\square$

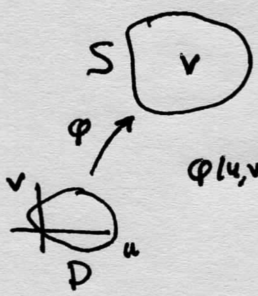
p. 877

19-17 Given a region V in $\mathbb{R}^3$ with $S = \partial V$ such that Div Thm applies

Show $\displaystyle vol(V) = \int_S x\, dy \wedge dz = \int_S y\, dz \wedge dx = \int_S z\, dx \wedge dy$

Gauss: $\displaystyle \int_V (\nabla \cdot F)\, dV = \int_S \vec{F} \cdot \hat{n}\, dS \overset{def}{:=} \int_D \vec{F}(\varphi) \cdot (\vec{\varphi}_u \times \vec{\varphi}_v)\, du\, dv$

$$\int_S F^{①} \underbrace{dy \wedge dz}_{\text{by } \boxed{def}} + F^{②} dz \wedge dx + F^{③} dx \wedge dy$$



$\varphi(u,v) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

This means the Jacobian det in co-ords

$$\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} = \begin{vmatrix} \varphi_u^{③} & \varphi_v^{②} \\ \varphi_u^{③} & \varphi_v^{③} \end{vmatrix}$$

The book writes this the old fashioned way (which is misleading)

$$\int_S F^{①} dy dz + F^{③} dx dz + F^{③} dx dy$$

$$\Rightarrow \int_V (\nabla \cdot F)\, dV = \int_S F^{①} dy \wedge dz + F^{②} dz \wedge dx + F^{③} dx \wedge dy$$

Let's make the clever choice $\vec{F} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$ Then $\nabla \cdot F = 1$

$$\int_V 1\, dV = \int_S x\, dy \wedge dz + 0 + 0$$

$$vol(V) = \int_S x\, dy \wedge dz$$

we can also do $F = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$ $\displaystyle vol(V) = \int_S y\, dz \wedge dx$

$F = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ $\displaystyle vol(V) = \int_S z\, dx \wedge dy$

$\square$

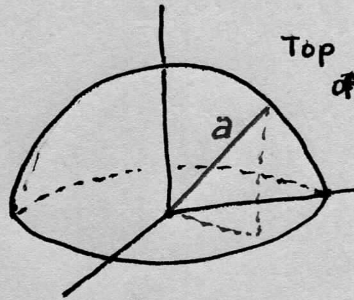
REA

11/1/2025

5

19-18 we want to compute $\text{Vol}(B(0,a))$ ans. $\frac{4\pi}{3}a^3$

From 19-17 we know $\text{vol} = \int_S z\, dx \wedge dy$



Top half of S^2

$$x^2 + y^2 + z^2 = a^2$$

$$z = \pm\left[a^2 - x^2 - y^2\right]^{1/2}$$

Use this parameterization: $\varphi(u,v) = \begin{bmatrix} u \\ v \\ +\left[a^2-u^2-v^2\right]^{1/2} \end{bmatrix} = \begin{bmatrix} u \\ v \\ p^{1/2} \end{bmatrix}$

$$\varphi_u = \begin{bmatrix} 1 \\ 0 \\ -u/p^{1/2} \end{bmatrix} \qquad \varphi_v = \begin{bmatrix} 0 \\ 1 \\ -v/p^{1/2} \end{bmatrix}$$

$$\varphi_u \times \varphi_v = \begin{bmatrix} u/p^{1/2} \\ v/p^{1/2} \\ 1 \end{bmatrix}$$

This points up, so we can see it is outward pointing normal on upper half sphere. φ is valid param

$$\Rightarrow \int_{S_{top}} z\, \overbrace{dx \wedge dy}$$

This symbol means Jacobian det

$$\begin{vmatrix} \varphi_u^1 & \varphi_v^1 \\ \varphi_u^2 & \varphi_v^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\circledast \quad \int_D \sqrt{a^2 - \underset{u}{x^2} - \underset{v}{y^2}} \cdot 1\, d\underset{u}{x}\, d\underset{v}{y} = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \left[a^2 - r^2\right]^{1/2} r\, dr\, d\theta$$

let $u = a^2 - r^2$

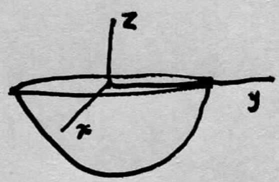
$du = -2r\, dr$

$$= \frac{2\pi}{-2} \int_{r=0}^{a} u^{1/2}\, du = -\pi\left[\frac{2}{3}\left[a^2-r^2\right]^{3/2}\right]_0^a$$

$$= \frac{-2\pi}{3}\left[0 - a^3\right] = \frac{2}{3}\pi a^3$$

What about the bot half?

If we make the obvious choice and take the neg root.



$$\Psi = \begin{bmatrix} u \\ v \\ -p^{1/2} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \Psi_u = \begin{bmatrix} 1 \\ 0 \\ u/p^{1/2} \end{bmatrix} \qquad \Psi_v = \begin{bmatrix} 0 \\ 1 \\ v/p^{1/2} \end{bmatrix}$$

$$\int_{S_{bot}} z\, \underbrace{dx \wedge dy} = \int_D -\sqrt{a^2 - u^2 - v^2} \cdot 1 \cdot du\, dv$$

$$\begin{vmatrix} \Psi_u^1 & \Psi_v^1 \\ \Psi_u^2 & \Psi_v^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

This is the neg of $\circledast$ so

$$\int_{S_{top}} + \int_{S_{bot}} = 0$$

This cannot be right, and the problem is that the $\mathcal{O}r$ given by Ψ is opposite that given by φ

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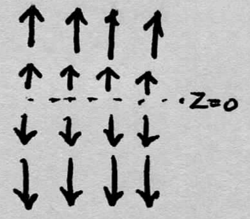
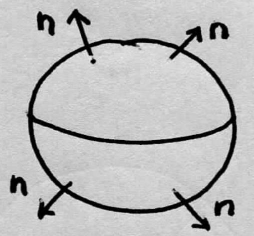
19-18 cont'd

To derive $\int_S z\,dx\wedge dz$ we used $\vec F = \begin{bmatrix}0\\0\\z\end{bmatrix}$





with proper outward pointing normal

$\vec F\cdot\hat n$ is always pos

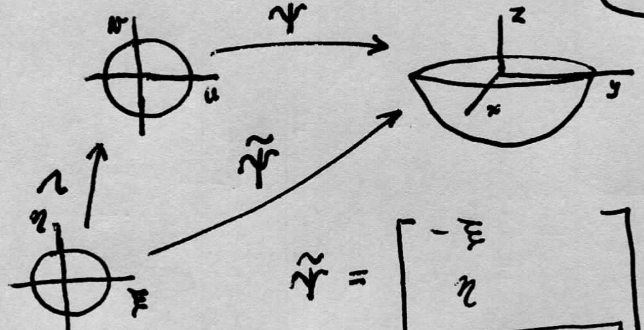
Here we can see the param Ψ does not give O.P.N.

$$\Psi_u \times \Psi_v = \begin{vmatrix} i & j & k \\ 1 & 0 & u/p^{1/2} \\ 0 & 1 & v/p^{1/2} \end{vmatrix} = \begin{bmatrix} -u/p^{1/2} \\ -v/p^{1/2} \\ 1 \end{bmatrix}$$

we can see this points up not down, at $= \langle 0,0,-a\rangle$

▷ To reverse the O^R we follow $\boxed{\text{Munkres AOM p. 286 §34}}$

$r = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$



$$\tilde\Psi(\xi,\eta) = \Psi(r(\xi,\eta)) = \Psi(-\xi,\eta)$$

$$D\tilde\Psi_\xi = D\Psi_{r(\xi,\eta)}\,Dr$$

$$\tilde\Psi = \begin{bmatrix} -\xi \\ \eta \\ -\sqrt{a^2-\xi^2-\eta^2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ u/p^{1/2} & v/p^{1/2} \end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -u/p^{1/2} & v/p^{1/2} \end{bmatrix}$$

$$\int_{S_{bot}} z\,\underbrace{dx\wedge dy} \qquad \begin{vmatrix}-1 & 0\\ 0 & 1\end{vmatrix} = -1$$

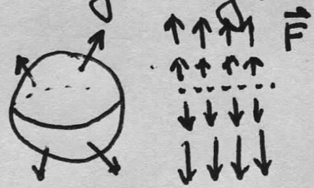
$$= \int_D -\sqrt{a^2-u^2-v^2}\,(-1)\,du\,dv = \int_D \sqrt{a^2-u^2-v^2}\,du\,dv = \frac{2}{3}\pi a^3$$

because this is identical to the one we did before

$$\Rightarrow \mathrm{Vol}(B(0,a)) = \int_{S_{top}} + \int_{S_{bot}} = \frac{2\pi a^3}{3} + \frac{2\pi a^3}{3} = \frac{4}{3}\pi a^3 \checkmark$$

▷ Can we do this with a sym arg?

From the picture



we could see that the flux thru the bot half would be the same as flux thru the top

$F\cdot n$ is pos in both hemispheres

But from just $\int_S z\,dx\wedge dy$ it isn't apparent how to use sym arg

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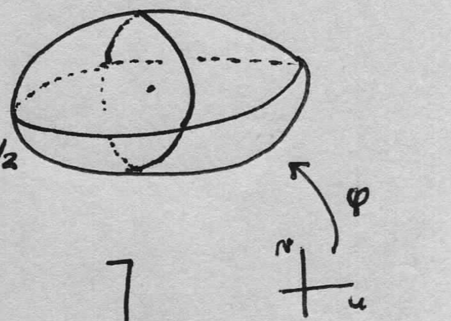
19-19 Find vol of ellipsoid $\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 + \frac{1}{c^2}z^2 = 1$

This is similiar to the prev problem and here we will appeal to sym and just calculate the top half.



$$\text{vol} = \int_S z \, dx \wedge dy \qquad z = \pm c\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{1/2}$$

$$\text{chart} \quad \varphi(u,v) = \begin{bmatrix} x \\ y \\ +c\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{1/2} \end{bmatrix}$$

I should be using u, v here but I am just going to call them x, y

Top half

$$\text{Vol} = \int_{x=-a}^{a} \int_{y=-b(1-\frac{x^2}{a^2})^{1/2}}^{+b(1-\frac{x^2}{a^2})^{1/2}} c\left(\underbrace{\left(1 - \frac{x^2}{a^2}\right)}_{P^2} - \frac{y^2}{b^2}\right)^{1/2} dy\, dx$$

The inner integral is

$$c \int_{y=-bP}^{bP} \left[P^2 - \frac{y^2}{b^2}\right]^{1/2} dy$$

let $u = \frac{1}{b}y$ then $du = \frac{1}{b}dy$

$y = -bP \Rightarrow u = -P$

$y = bP \Rightarrow u = P$

M&T Table of Integrals p. 523 #38

$$\int [P^2 - u^2]^{1/2} du = \frac{u}{2}\sqrt{P^2 - u^2} + \frac{P^2}{2}\sin^{-1}\left(\frac{u}{P}\right)$$

$$bc \int_{u=-P}^{P} [P^2 - u^2]^{1/2} du = bc\left[\frac{u}{2}\sqrt{P^2-u^2} + \frac{P^2}{2}\sin^{-1}\left(\frac{u}{P}\right)\right]_{-P}^{P}$$

$$= bc\left[\frac{P}{2}\underbrace{\sqrt{\cancel{P^2 - P^2}}}_{0} + \frac{P^2}{2}\underbrace{\sin^{-1}(1)}_{\pi/2} - \frac{-P}{2}\underbrace{\sqrt{\cancel{P^2 - P^2}}}_{0} - \frac{P^2}{2}\underbrace{\sin^{-1}(-1)}_{-\pi/2}\right]$$

$$= bc\frac{P^2}{2}\pi$$

Then

$$\underset{(\text{half})}{\text{vol}} = bc\frac{\pi}{2}\int_{-a}^{a}\left[1 - x^2/a^2\right]dx = bc\frac{\pi}{2}\left[x - \frac{1}{3a^2}x^3\right]_{-a}^{a}$$

$$= bc\frac{\pi}{2}\left[a - \frac{a^{\cancel{3}}}{3\cancel{a^2}} - \left(-a - \frac{-a^{\cancel{3}}}{3\cancel{a^2}}\right)\right]$$

$$a - \tfrac{1}{3}a + a - \tfrac{1}{3}a = \tfrac{4}{3}a$$

$$= \frac{2}{3}\pi abc$$

Mult by 2 for total vol

$$\text{Vol} = \frac{4}{3}\pi abc$$

□

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19-22 $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ show the vector integral $\boxed{\int_V \vec{\nabla} f \, dV = \int_S f \hat{n} \, dS}$

Important

We know Div Thm: $\int_V (\nabla \cdot F) dV = \int_S F \cdot n \, dS$

Let $\vec{F} := f\vec{a}$ for some const $\vec{a}$. Then $\nabla \cdot (f\vec{a}) = \nabla f \cdot a + f(\cancelto{0}{\nabla \cdot \vec{a}})$

$$\int_V (a \cdot \vec{\nabla} f) dV = \int_S f\vec{a} \cdot \hat{n} \, dS$$

$$a \cdot \int_V \vec{\nabla} f \, dV = a \cdot \int_S f \hat{n} \, dS \implies \int_V \vec{\nabla} f \, dV = \int_S f \hat{n} \, dS$$ since a was arb. $\square$

19-23 $\vec{F}$ C^1 smooth. show $\boxed{\int_V (\overrightarrow{\nabla \times F}) dV = -\int_S \vec{F} \times \hat{n} \, dS}$

Important

again let $\vec{a}$ be arb const vector. consider $\vec{G} = \vec{F} \times \vec{a}$

Div Thm $\int_V (\nabla \cdot G) dV = \int_S G \cdot n \, dS$

$$\int_V \nabla \cdot (F \times a) dV = \int_S (F \times a) \cdot n \, dS$$

$$\nabla \cdot (F \times a) = a \cdot (\nabla \times F) - F \cdot (\cancelto{0}{\nabla \times a}) = a \cdot \nabla \times F$$ M&T ch 3.5 p.190

whereas triple product $(F \times a) \cdot n = n \cdot (F \times a)$

$$= \det \begin{vmatrix} - n - \\ - F - \\ - a - \end{vmatrix}$$ transpose

$$\overset{(-1)^2}{=} \det \begin{vmatrix} - a - \\ - n - \\ - F - \end{vmatrix} = \vec{a} \cdot (\hat{n} \times \vec{F})$$

$$\implies \int_V a \cdot \nabla \times F \, dV = \int_S a \cdot (n \times F) \, dS$$

$$\vec{a} \cdot \int_V \overrightarrow{\nabla \times F} \, dV = \vec{a} \cdot \int_S (n \times F) \, dS$$

since a arb: $$\int_V (\nabla \times F) dV = \int_S (n \times F) \, dS = -\int_S (F \times n) \, dS$$ $\square$

19-24 $\boxed{\vec{F} \perp S \text{ for all pts } x \in S \implies \int_V (\nabla \times F) dV = \vec{0}}$

$\vec{F}(x) = \lambda(x) \hat{n}(x) \; \forall x \in S$

By prev prob $$\int_V (\nabla \times F) dV = -\int_S (\lambda \hat{n}) \times \hat{n} \, dS = \int_S \vec{0} = \vec{0}$$

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19-25 show $\displaystyle\int_S (\nabla \times F)\cdot \hat{n}\, dS = 0 \quad \forall F$

$$\int_S (\nabla\times F)\cdot n\, dS \overset{\text{Gauss}}{=} \int_V \nabla\cdot(\nabla\times F)\, dV = \int_V 0\, dV = 0 \quad \square$$

div (curl) $\equiv 0$

19-26 Generalized Integration by parts

$$\int_V \overset{u}{\varphi}\overset{dv}{(\nabla\cdot\vec{E})}\, dV = \int_{\partial V = S} \overset{uv}{(\varphi\vec{E})\cdot\hat{n}\, dS} - \int_V \overset{v\, du}{\vec{E}\cdot\vec{\nabla}\varphi}\, dV$$

(*) Important

we know $\nabla\cdot(\varphi\vec{E}) = \nabla\varphi\cdot E + \varphi(\nabla\cdot E)$

re-arrange: $\varphi(\nabla\cdot E) = -\nabla\varphi\cdot E + \nabla\cdot(\varphi E)$

$= \nabla\cdot(\varphi E) - E\cdot\nabla\varphi$

$$\int_V \varphi(\nabla\cdot E)\, dV = \int_V \nabla\cdot(\varphi E)\, dV - \int_V E\cdot\nabla\varphi\, dV$$

|| Gauss Div

$$\int_S \varphi E\cdot n\, dS \quad \square$$

19-27 Show $\displaystyle\int_V (a\nabla^2 f + b\nabla^2 g)\, dV = a\int_S \nabla f\cdot n\, dS + b\int_S \nabla g\cdot n\, dS$ $\quad a,b\in\mathbb{R}$ const

let $\vec{F} := a\vec{\nabla} f + b\vec{\nabla} g$

then $\nabla\cdot F = a\nabla^2 f + b\nabla^2 g$

$$\int_V (\nabla\cdot F)\, dV = \int_S F\cdot n\, dS$$

$$\int_V (a\nabla^2 f + b\nabla^2 g)\, dV = \int_S (a\nabla f + b\nabla g)\cdot n\, dS \quad \square$$

19-28 prequel Green's Identity I is, in fact, a case of <u>Int by Parts</u> (19-26)

$$\int_V \varphi(\nabla\cdot E)\, dV = \int_S \varphi E\cdot n\, dS - \int_V E\cdot\nabla\varphi\, dV$$

take $\varphi = f$ and $E = \nabla g$

$f(\nabla\cdot\nabla g)$ $\qquad f\nabla g$ $\qquad \nabla g\cdot\nabla f\, dV$

$$\int_V f\nabla^2 g\, dV = \int_S f\nabla g\cdot n\, dS - \int_V \nabla g\cdot\nabla f\, dV \Rightarrow \boxed{\int_V (f\nabla^2 g + \nabla g\cdot\nabla f)\, dV = \int_S f\nabla g\cdot n\, dS}$$

cont'd →

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(19-28) Green's 1st Id

M&T Ch 7.4 p.449 # 11

This is just a case of 'Integration by Parts' (19-26)

$$\boxed{\int_V \left[f\nabla^2 g + \nabla f\cdot\nabla g\right]dV = \int_{\partial V = S} f\,\underbrace{\vec{\nabla} g\cdot\hat{n}}_{\text{"}\frac{\partial g}{\partial n}\text{"}}\,dS}$$

Let $\vec{F} = f\vec{\nabla} g$ then $\nabla\cdot F = \nabla f\cdot\nabla g + f\nabla^2 g$

Just apply Div Thm:

$$\int_V (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV = \int_S (f\vec{\nabla} g)\cdot\hat{n}\,dS$$

(19-29) Green II (sym id)

$$\boxed{\begin{aligned}\int_V (f\nabla^2 g - g\nabla^2 f)\,dV &= \int_S (f\nabla g - g\nabla f)\cdot n\,dS \\ &= \int_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS\end{aligned}}$$

Here $\nabla f\cdot n = Df_x(n) = \text{"}\frac{\partial f}{\partial n}\text{"}$

Just take Green I and write it 2 ways, interchanging f, g

$$\int_V f\nabla^2 g + \nabla f\cdot\nabla g = \int_S f\nabla g\cdot n\,dS$$

$$-\int g\nabla^2 f + \nabla g\cdot\nabla f = \int_S g\nabla f$$

$$\int_V f\nabla^2 g - g\nabla^2 f = \int_S (f\nabla g - g\nabla f)\cdot n\,dS$$

(19-30)

Using $\nabla f\cdot n = Df_x(n) = \text{"}\frac{\partial f}{\partial n}\text{"}$ and the same for $\nabla g\cdot n = \text{"}\frac{\partial g}{\partial n}\text{"}$ we can express Green I and II:

$$\boxed{\int_V f\nabla^2 g + \nabla f\cdot\nabla g = \int_S f\frac{\partial g}{\partial n}\,dS}$$

$$\boxed{\int_V f\nabla^2 g - g\nabla^2 f = \int_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS}$$

(19-31)

φ harmonic in V $[\nabla^2\varphi = 0]$ $\Rightarrow$

(1) $\int_S \frac{\partial\varphi}{\partial n}\,dS = 0$

(2) $\int_S \varphi\frac{\partial\varphi}{\partial n}\,dS = \int_V \|\nabla\varphi\|^2$

(1) Take $g = \varphi$ and $f = 1$ in Green I:

$$\int_V \underbrace{\nabla^2\varphi}_{0} + 0\cdot\nabla\varphi = \int_S \nabla\varphi\cdot n\,dS \Rightarrow 0 = \int_S \frac{\partial\varphi}{\partial n}\,dS$$

(2) Take $f = \varphi$ and $g = \varphi$

$$\int_V \underbrace{\varphi\nabla^2\varphi}_{0} + \nabla\varphi\cdot\nabla\varphi = \int_S \varphi\frac{\partial\varphi}{\partial n}\,dS \Rightarrow \int_V \|\nabla\varphi\|^2\,dV = \int_S \varphi\frac{\partial\varphi}{\partial n}\,dS$$

$\square$

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(11)

Aside: Harmonic fcns

Extrema on ∂

1. Max Princ : Let cpt $K \subset \mathcal{U}$ open in $\mathbb{R}^n$

   f harmonic $\Rightarrow$ f has its max & min on ∂K

2. Mean Value Property $\quad f(x) = \dfrac{1}{n\,\mathrm{Vol}(B_{(x,r)})\, r^{n-1}} \displaystyle\int_{\partial B_r} f\, dS \quad$ Avg over surf of ball

   $f(x) = \dfrac{1}{\mathrm{Vol}(B_r)\, r^n} \displaystyle\int_{B_r} f\, dV \quad$ Avg over whole solid ball

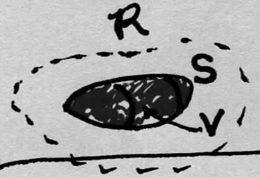
3. Harnack's Ineq $\quad \sup_V (f) \leq C \cdot \inf_V (f)$

4. Liouville Thm

   $\left.\begin{array}{l}\text{f harmonic are defined on all of } \mathbb{R}^n \\ \text{f bdd above OR bdd below}\end{array}\right\} \Rightarrow f = \text{const.}$

   [In $\mathbb{C}$ analysis, a bdd analytic entire fcn is const.]

19-32



S is surf of region V contained in open set $R \subseteq \mathbb{R}^3$

$\boxed{\left.\begin{array}{l}\nabla^2 \varphi = 0 \text{ on } R \\ \varphi = 0 \text{ on } S = \partial V\end{array}\right\} \Rightarrow \varphi \equiv 0 \text{ in } V \text{ as well}}$

Behaviour on ∂ determines values in interior

From prob 19-31 $\quad \displaystyle\int_S \underbrace{\varphi \frac{\partial \varphi}{\partial n}}_{\text{This is } 0} dS = \int_V \|\nabla\varphi\|^2 dV \Rightarrow 0 = \int_V \|\nabla \varphi\|^2$

if $\displaystyle\int_V f^2 = 0$ then $f \equiv 0$ (Since it is cont)

$\Rightarrow \|\nabla\varphi\| = 0$

$\Rightarrow \nabla\varphi = 0 \Rightarrow \varphi = a$ (some const)

and since $\varphi = 0$ on S $\Rightarrow a = 0 \Rightarrow \varphi \equiv 0$ in V

19-33

(1) $\left.\begin{array}{l}\nabla^2\varphi = 0 \\ \nabla^2\psi = 0 \\ \varphi = \psi \text{ on } S\end{array}\right\} \text{in } R \Rightarrow \varphi \equiv \psi \text{ in all of } V$

(2) $\left.\begin{array}{l}\nabla^2\varphi = 0 \text{ in } R \\ \dfrac{\partial\varphi}{\partial n} = 0 \text{ on } S\end{array}\right\} \Rightarrow \varphi = \text{const in all o } V$

(1) Consider fcn $\varphi - \psi$. Since φ, ψ are harmonic, so is $\varphi - \psi$ and $\varphi - \psi \equiv 0$ on S

From prev prob 19-32, $\varphi - \psi \equiv 0$ on all of V $\Rightarrow \varphi \equiv \psi$ on V

(2) Using (19-31) $\displaystyle\int_S \underbrace{\varphi \frac{\partial\varphi}{\partial n}}_{0} dS = \int_V \|\nabla\varphi\|^2 \Rightarrow \nabla\varphi = 0 \Rightarrow \varphi = \text{const on } V$

$\square$

19-34 φ, ψ harmonic

(1) $\dfrac{\partial\varphi}{\partial n} = \dfrac{\partial\psi}{\partial n} \Rightarrow \varphi = \psi + c$ on V

(2) $\left.\begin{aligned}\dfrac{\partial\varphi}{\partial n} &= -\varphi + f \\ \dfrac{\partial\psi}{\partial n} &= -\psi + f\end{aligned}\right\} \Rightarrow \varphi \equiv \psi$

∂

(1) $\dfrac{\partial}{\partial n}(\varphi - \psi) \equiv 0$ on S From (19.33) $\varphi - \psi = \text{const}$ on V $\Rightarrow \varphi = \psi + \text{const}$ on V.

(2) Hypoth $\Rightarrow \dfrac{\partial\varphi}{\partial n} - \dfrac{\partial\psi}{\partial n} = (-\varphi + f) - (-\psi + f) = \psi - \varphi = -(\varphi - \psi)$

$\Rightarrow \dfrac{\partial}{\partial n}(\varphi - \psi) = -(\varphi - \psi)$

Let $g := \varphi - \psi$ then $\dfrac{\partial}{\partial n} g = -g$

From (19-31) we know $\displaystyle\int_S g \frac{\partial g}{\partial n} dS = \int_V \|\nabla g\|^2$

$$\int_S g \frac{\partial g}{\partial n} dS = \underbrace{-\int_S (\varphi - \psi)(\varphi - \psi)\, dS}_{-\int_S (\varphi-\psi)^2 dS} = \int_V \|\nabla(\varphi - \psi)\|^2 \geq 0$$

LHS neg or 0 but RHS pos or 0 $\Rightarrow \displaystyle\int_S (\varphi - \psi)^2 dS \overset{!}{=} 0 \Rightarrow (\varphi - \psi)^2 \equiv 0$ since cont $\Rightarrow \varphi \equiv \psi$ on V

19-35 Prev hypothesis all in effect

$S = \partial V$ $V \subset R$ open in $\mathbb{R}^3$

(1) φ, ψ harmonic in $R \Rightarrow \displaystyle\int_S \left(\varphi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\varphi}{\partial n}\right) dS = 0$

φ, ψ not harm then

(2) $\left.\begin{aligned}\nabla^2\varphi &= f\varphi \\ \nabla^2\psi &= f\psi\end{aligned}\right\} \Rightarrow \displaystyle\int_S \left(\varphi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\varphi}{\partial n}\right) dS = 0$

EVs of ∇^2

(1) Green II (19-29) $\Rightarrow \displaystyle\int_S \left(\varphi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\varphi}{\partial n}\right) dS = \int_V \Big(\varphi \underbrace{\nabla^2\psi}_{0} - \psi \underbrace{\nabla^2\varphi}_{0}\Big) dV = 0$ ✓

(2) Green II $\displaystyle\int_S \left(\varphi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\varphi}{\partial n}\right) dS = \int_V \Big(\varphi \underbrace{\nabla^2\psi}_{f\psi} - \psi \underbrace{\nabla^2\varphi}_{f\varphi}\Big) dV = \int_V \underbrace{(\varphi f \psi - \psi f \varphi)}_{0} dV = 0$

19-36 φ, ψ satisfy <u>same</u> Poisson eq:

$\left.\begin{aligned}\nabla^2\varphi &= -4\pi f \\ \nabla^2\psi &= -4\pi f \\ \varphi &\equiv \psi \text{ on } S\end{aligned}\right\} \Rightarrow \varphi \equiv \psi$ on all of V

$\nabla^2\varphi - \nabla^2\psi = -4\pi f - (-4\pi f) = 0 \Rightarrow \nabla^2(\varphi - \psi) = 0$ so $\varphi - \psi$ is harmonic

$\varphi \equiv \psi$ on $S \Rightarrow \varphi - \psi \equiv 0$ on S From (19-32) $\varphi - \psi \equiv 0$ on V

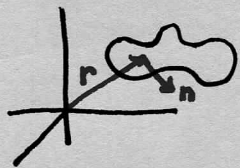
so $\varphi \equiv \psi$ on V

$\square$

19-37 Let S be a clsd cpt surf $\Rightarrow \int_S \hat{n}\, dS = 0$

From **19-22** $\int_V \vec{\nabla} f\, dV = \int_S f \hat{n}\, dS$ Take $f \equiv 1$ $\underbrace{\int_V 0\, dV}_{0} = \int_S \hat{n}\, dS$ □

19-38 $\int_S \vec{r} \times \hat{n}\, dS = 0$



"Total torque on S due to $\hat{n}$ is 0"

From **19-23** $\int_V (\nabla \times F)\, dV = \int_S (n \times F)\, dS$ Take $\vec{F} = \vec{r}$

$$\int_S (n \times r)\, dS = \int_V \underbrace{(\nabla \times r)}_{\vec{0}}\, dV$$

$$\nabla \times r = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ x & y & z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-\int_S (r \times n)\, dS = 0$ or $\int_S (r \times n)\, dS = 0$ □

19-39 $\int_V (\nabla \cdot n)\, dV = \text{area}(S)$

$$\int_V (\nabla \cdot n)\, dV \overset{\text{Div}}{=} \int_S \hat{n} \cdot \hat{n}\, dS = \int_S 1\, dS = \text{area}(S)$$ □

19-40 $\int_V \frac{1}{r^2}\, dV = \int_S \frac{1}{r^2} \vec{r} \cdot n\, dS$

Let $F := \frac{1}{r^2} \vec{r} = \frac{1}{\|x\|^2} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

M&T ch 3.5 p. 196

$$\nabla \cdot (r^n \vec{r}) = (n+3) r^n$$

Here $n = -2$ $\nabla \cdot F = \nabla \cdot (r^{-2} \vec{r})$ $= (-2+3) r^{-2} = \frac{1}{r^2}$

Now apply Div Thm □

19-41 $\int_V r^3\, dV = p \Rightarrow \int_S r^3 \vec{r} \cdot n\, dS = 6p$

Let $F := r^3 \vec{r}$ Then $\nabla \cdot F = \nabla \cdot (r^3 \vec{r}) = (n+3) r^n = (3+3) r^3 = 6 r^3$

Div Thm: $\int_S F \cdot n\, dS = \int_V 6 r^3\, dV = 6p$ □

19-42 $\int_S r^m \hat{n}\, dS = \int_V m r^{m-2} \vec{r}\, dV$

M&T p. 196

$$\nabla (r^n) = n r^{n-2} \vec{r}$$

From **19-22** $\int_V \vec{\nabla} f\, dV = \int_S f \hat{n}\, dS$ Take $f = r^m$

$\Rightarrow \int_V m r^{m-2} \vec{r}\, dV = \int_S r^m \hat{n}\, dS$ □

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(19-43) $\boxed{\text{Show } \operatorname{div}(F)(x) = \lim_{\operatorname{diam}\{U_\varepsilon\}\to 0} \frac{1}{\operatorname{vol}(U_\varepsilon)} \int_{\partial U_\varepsilon} \vec{F}\cdot\hat{n}\,dS}$ co-ord free def

M&T p. 445 Let us work with a ball $B(x,\varepsilon)$ and we will assume this arg works for any region for which Div Thm would apply. non-pathological

$$\int_{\partial B_\varepsilon} F\cdot n\,dS \overset{\text{Div}}{=} \int_{B_\varepsilon} \operatorname{div}(F)\,dV \overset{\text{MVT}}{=} \operatorname{div}(F)(y_\varepsilon)\int_{B_\varepsilon} dV \quad \text{For some } y_\varepsilon \in B(x,\varepsilon)$$

$$\Rightarrow \frac{1}{\operatorname{vol}(B_\varepsilon)}\int_{\partial B_\varepsilon} F\cdot n\,dS = \operatorname{div}(F)(y_\varepsilon) \qquad y_\varepsilon \to x \text{ as } \varepsilon \to 0$$

$$\Rightarrow \operatorname{div}(F)(x) = \lim_{\varepsilon\to 0}\frac{1}{\operatorname{vol}(B_\varepsilon)}\int_{\partial B_\varepsilon} F\cdot n\,dS$$

rate of net outward flux at x per unit volume.

[For example, consider $F = \frac{1}{r^3}\vec{r}$ and $x=0$. When we discuss solid angle shortly we will show flux thru $S^2_{(0,\varepsilon)}$ is $4\pi\varepsilon^2$ (surf area of sphere)

$\lim_{\varepsilon\to 0} \frac{4\pi\varepsilon^2}{\frac{4}{3}\pi\varepsilon^3} \to \infty$ Divergence <u>at</u> a point src is ∞ ratio of surf area to vol. Surf area dominates as radius $\to 0$.]

(19-44) $\boxed{\text{show } \operatorname{grad}(\varphi)_{(x)} = \lim_{\operatorname{diam}(V)\to 0} \frac{1}{\operatorname{vol}(V)}\int_{S=\partial V} \varphi\,\hat{n}\,dS}$

By (19-22) $\int_V \vec{\nabla} f\,dV = \int_S f\,\hat{n}\,dS$ Here $\int_{V_\varepsilon} \vec{\nabla}\varphi\,dV = \int_{S_\varepsilon} \varphi\,\hat{n}\,dS$

Dot prod both sides with $\hat{\imath}$: $\hat{\imath}\cdot\int_{V_\varepsilon} \nabla\varphi\,dV = \hat{\imath}\cdot\int_{S_\varepsilon}\varphi\,\hat{n}\,dS$

$$\underbrace{\int_{V_\varepsilon} (\nabla\varphi)^{①}\,dV}_{\text{MVT } \nabla\varphi^{①}(y_\varepsilon)\operatorname{vol}(V_\varepsilon)} = \int_{S_\varepsilon} \varphi\, n^{①}\,dS$$

$$\Rightarrow \nabla\varphi^{①}(y_\varepsilon) = \frac{1}{\operatorname{vol}(V_\varepsilon)}\int_{S_\varepsilon}\varphi\,n^{①}\,dS$$

$y_\varepsilon \to x$

$$\nabla\varphi^{①}(x) = \lim_{\varepsilon\to 0}\frac{1}{\operatorname{vol}(V_\varepsilon)}\int_{S_\varepsilon}\varphi\,n^{①}\,dS$$

Likewise for $\nabla\varphi^{②}$ and $\nabla\varphi^{③}$. The MVT pts y_ε would depend on the component $\varphi^{(i)}$ but for all they converge to same x $\quad y_\varepsilon \to x$

$$\operatorname{grad}(\varphi)_{(x)} = \begin{bmatrix}\nabla\varphi^1\\ \nabla\varphi^2\\ \nabla\varphi^3\end{bmatrix}_{(x)} = \lim_{|V|\to 0}\int_{\partial V_\varepsilon}\varphi\,\hat{n}\,dS \qquad \square$$