Sheet 1 Poincaré Lemma

Thm 7 is a converse to $\nabla \times \nabla f \equiv 0$ (vector identity) i.e. $F = \nabla f \Rightarrow \nabla \times F = 0$

Another vector relation we have is $\operatorname{div}(\operatorname{Curl} G) \equiv 0$ for vf $\vec{G}$ C^2 smooth

Converse: $\operatorname{div}(F) = 0 \overset{?}{\Longrightarrow} F = \operatorname{Curl}(G)$ for some $\vec{G}$?

Thm 8 $\mathcal{U} \subseteq \mathbb{R}^3$ is rectangular box $(a,b) \times (c,d) \times (e,f)$ possibly infinite intervals

$F: \mathcal{U} \to \mathbb{R}^3$ C^1 smooth, NO singularities in $\mathcal{U}$

$\nabla \cdot F = 0$ in $\mathcal{U}$

$\Rightarrow$ $\exists$ vf $G: \mathcal{U} \to \mathbb{R}^3$ such that $F = \nabla \times G$

Pf. Define $\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ and $\vec{G} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ $\qquad \nabla \cdot F = 0$ means $P_x + Q_y + R_z = 0$

Then $\nabla \times G \overset{!}{=} F \Rightarrow \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ A & B & C \end{vmatrix} = \begin{bmatrix} C_y - B_z \\ A_z - C_x \\ B_x - A_y \end{bmatrix} \overset{!}{=} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ $\forall (x,y,z)$

Now we can simplify by taking $C \equiv 0 \Rightarrow$ $B_z \overset{!}{=} -P$

$A_z = Q$

$B_x - A_y = R$

$$\Rightarrow B(x,y,z) = \int_c^z -P(x,y,t)\,dt + h(x,y)$$

From base pt $\langle a,b,c \rangle$ no relation to previous intervals $(a,b)\times(c,d)\times(e,f)$

$$A(x,y,z) = \int_c^z Q(x,y,t)\,dt + k(x,y)$$

we must have $B_x - A_y = R$:

If we tried to take $h \equiv 0$, $k \equiv 0$ we'd get $\frac{\partial}{\partial x}\int_c^z -P(x,y,t)\,dt - \frac{\partial}{\partial y}\int_c^z Q(x,y,t)\,dt$

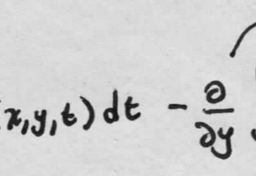
$$= \int_c^z \underbrace{\left(-P_x(x,y,t) - Q_y(x,y,t)\right)}_{R_z(x,y,t)}\,dt \quad \text{because } P_x + Q_y + R_z = 0$$

$$= \int_c^z \frac{\partial}{\partial z} R(x,y,t)\,dt \overset{\text{FTOC}}{=} R(x,y,z) - R(x,y,c)$$ we need to eliminate this term.

Try again, taking $k(x,y) := -\int_b^y R(x,t,c)\,dt$

Then $A_y = \int_c^z Q_y(x,y,t)\,dt - R(x,y,c)$

Then $B_x - A_y = \underbrace{\int_c^z R_z(x,y,t)\,dt}_{R(x,y,z) - \cancel{R(x,y,c)}} + R\cancel{(x,y,c)} = R(x,y,z)$ ✓



why did we need the restrictions on the domain?

For example, when we integrate $\int_c^z P(x,y,t)\,dt$, since x,y,z is an arb pt, we are really integrating up all these vertical lines.

So we fill up the box with lines.

3/2/2016

This is really Munkres AOM p. 260-261 #5

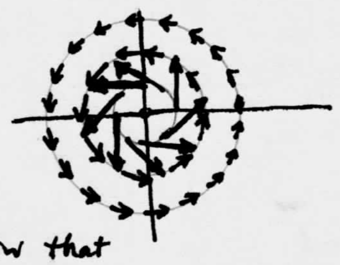
(12) Given a vf F defined on a domain $\mathcal{U}$, let's show that the topology of $\mathcal{U}$ determines whether or not a potential fcn $f: \mathcal{U} \to \mathbb{R}$ can exist. That is, we can have $\nabla \times F = 0$, but still no $f \ni \nabla f = F$.

In general, we find f by integrating along a path σ from a base pt a to the pt we are interested in x: $\quad f(x) := \int_\sigma F \cdot ds$

Here we will show that $\mathcal{U}$ not being simply conn allows us to get contradictory results by going along different paths from a to x.

For this example, we take $\vec{F}(x,y) = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$ defined on $\mathcal{U} = \mathbb{R}^2 - \{0\}$

Then $\omega := F \cdot ds = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$



[This is the famous "angle form"; for any curve σ, $\int_\sigma \omega$ measures the 'winding' angle of curve σ wrt origin]

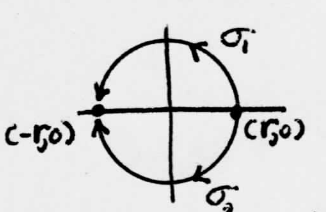
Therefore it is customary to call ω '$d\theta$' even though we shall show that <u>no</u> angle <u>fcn</u> θ can exist $\theta: \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}$

(i) Here we are working in $\mathbb{R}^2$, so the condition $\nabla \times F = 0$ becomes $D_2 F^{(1)} = D_1 F^{(2)}$ $(P_y = Q_x)$
[really the exterior derivative $d\omega = 0$]

$$D_2 F^{(1)} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2 - y^2 + 2y^2}{\cdot} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$D_1 F^{(2)} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \longleftarrow \text{SAME}$$

(ii) Now show that integrating along a path to 2 end points depends on the path in this vf:



$\sigma_1(t) = \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix}$ $0 \le t \le \pi$

$\sigma_1'(t) = \begin{bmatrix} -r\sin t \\ r\cos t \end{bmatrix}$

$$\int_{\sigma_1} F \cdot ds = \int_0^\pi \frac{-r\sin t}{r^2}(-r\sin t) + \frac{r\cos t}{r^2}(r\cos t)\, dt$$

$$= \int_0^\pi \left(\frac{r^2}{r^2}\sin^2 + \frac{r^2}{r^2}\cos^2 t\right) dt = \pi$$

Lower path is mirror image

$\sigma_2(t) = \begin{bmatrix} r\cos t \\ -r\sin t \end{bmatrix}$ $0 \le t \le \pi$

$\sigma_2'(t) = \begin{bmatrix} -r\sin t \\ -r\cos t \end{bmatrix}$

$$\int_{\sigma_2} F \cdot ds = \int_0^\pi \frac{+r\sin t}{r^2}(-r\sin t) + \frac{+r\cos t}{r^2}(-r\cos t)$$

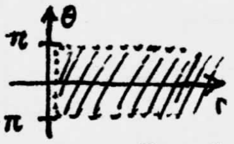
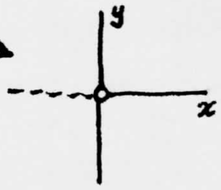
$$= \int_0^\pi -\frac{r^2}{r^2}\left(\sin^2 t + \cos^2 t\right) dt = -\pi$$

Most books want to show $\oint F \cdot ds = 2\pi \neq 0$, but I think this makes the issue clearer.

cont'd →

12 cont'd 3/2/2016

(iii) Now we show that there **is** a smooth angle fcn $\theta : \mathbb{R}^2 - N \longrightarrow \mathbb{R}$ that satisfies $d\theta = \omega = F \cdot ds$ but note this domain is **not** $\mathbb{R}^2 - \{0\}$ [and also $\mathbb{R}^2 - N$ is **not** $(a,b)\times(c,d)$] (N is neg x axis and 0)

Polar-co-ords  $\xrightarrow{\varphi}$ 

Restricted to the r,θ strip, φ is One-to-One and Onto $\mathbb{R}^2 - N$

$$D\varphi_{r\theta} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det(D\varphi_{r\theta}) = rc^2 + rs^2 = r$ so $\operatorname{Det}(D\varphi_{r\theta}) \neq 0$ on strip.

Inv Fcn Thm: For any pt $(x,y) \in (\mathbb{R}^2 -$ [redacted] $N)$, $\exists$ nbhd η_{xy} and a smooth map $g : \eta_{xy} \longrightarrow \eta_{r\theta}$ such that $g = \varphi^{-1}$ in η_{xy}.

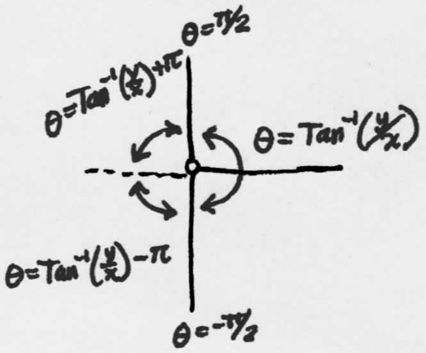
But since φ is One-to-One globally, all these g's make up a global inverse φ^{-1}

$\varphi^{-1} : \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} r \\ \theta \end{bmatrix}$ ← In particular, the fcn $\theta(x,y)$ is C^∞ smooth.

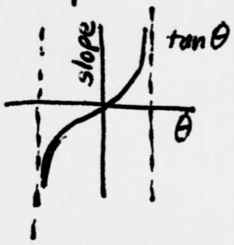
▷ Now we can give an explicit formula for $\theta : \mathbb{R}^2 - N \longrightarrow \mathbb{R}$ [we know it is the 'same' θ since inverses are unique. What about branches of the Arg fcn discussed in $\mathbb{C}$-analysis? Not to worry — we are on a fixed branch with fixed domain $r\theta$ strip where pos x axis has angle $\theta = 0$.]

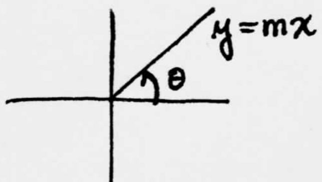
$$\text{Define } \theta(x,y) := \begin{cases} \operatorname{Tan}^{-1}(y/x) & \text{Quad I, IV} \\ \frac{\pi}{2} & \text{pos y axis} \\ \operatorname{Tan}^{-1}(\frac{y}{x}) + \pi & \text{Quad II} \\ \frac{-\pi}{2} & \text{neg y axis} \\ \operatorname{Tan}^{-1}(y/x) - \pi & \text{Q III} \end{cases}$$

[obviously Continuous, and in fact C^∞ by prior argument.]



Note: principal branch Tan





$m = \frac{y}{x}$ $\quad \tan\theta = \text{slope} = m$

we map $(x,y) \longmapsto \frac{y}{x} = m$ and then map $m \longmapsto$ angle θ with the Principal branch of the angle fcn specified by $\theta = 0$ on pos x axis.

▷ Now show $d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $\mathbb{R}^2 - N$

$\theta(x,y) = \operatorname{Tan}^{-1}(\frac{y}{x})$ we drop the consts (π or $-\pi$) since they disappear anyway under d, and θ is smooth crossing y axis.

$$\operatorname{Tan}^{-1}(\tfrac{y}{x}) = \int_0^{y/x} \frac{1}{1+t^2}\,dt$$

$$d\theta = D_x\theta\,dx + D_y\theta\,dy$$

By Leibniz Rule $\quad D_x\theta = \frac{\partial}{\partial x}\int_0^{y/x} \frac{1}{1+t^2}dt = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{\frac{x^2+y^2}{\cancel{x^2}}} \cdot \frac{1}{\cancel{x^2}} = \frac{-y}{x^2+y^2}$

$$D_y\theta = \frac{1}{1+(\frac{y}{x})^2}\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

and we recover our original 1-form.

cont'd →

3/2/2016

(iv) Now show that θ can not be extended to $\mathbb{R}^2 - \{0\}$:

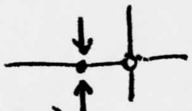
Suppose there was a fcn $f : \mathbb{R}^2 - \{0\}$ such that $df = \omega$. Then on $\mathbb{R}^2 - N$ we would have $df = d\theta \Rightarrow d(f - \theta) = 0$

$\Rightarrow f = \theta + c$ on $\mathbb{R}^2 - N$

> **Lemma:** If some arb fcn $h: \mathbb{R}^2 - N \to \mathbb{R}$ has $Dh_x = 0 \ \forall x \Rightarrow h = c$
>
> pf. MVT
>
> $h(x) - h(y) = Dh_\xi (y - x)$ for some pt ξ on line segment $[x, y]$.
>
> We may have to use 3 segments:
>
> 

But we said f was defined on N. What is it there?

Let's examine the pt $(-1, 0)$



We must have $\lim\limits_{y \nearrow 0} f(-1, y) \overset{!}{=} \lim\limits_{y \searrow 0} f(-1, y)$

$= \lim\limits_{y \nearrow 0} \theta(-1, y) + c \overset{!}{=} \lim\limits_{y \searrow 0} \theta(-1, y) + c$

$= -\pi + \cancel{c} \overset{?}{=} \pi + \cancel{c}$

$\Rightarrow\Leftarrow$ Thus no continuous (let alone smooth) f can exist on $\mathbb{R}^2 - \{0\}$ that satisfies $df = \omega$ $\square$

ch 7.3 Easy special case of Poincaré Lemma

(8) Given vf $\vec{F} = \begin{bmatrix} F^{①} \\ F^{②} \\ F^{③} \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ where (1) Each component fcn of $\vec{F}$ is homogeneous of degree n $(n \neq -1)$:

$$F^{(i)}(tx, ty, tz) = t^n F^{(i)}(x,y,z) \quad t>0$$

i.e. $\boxed{\vec{F}(tx) = t^n \vec{F}(x)}$

(2) $\boxed{\nabla \times \vec{F} = 0}$ [th

Defined on all of $\mathbb{R}^3$ by the homog condition — See remarks below! NOT NEC DEFINED AT ORIGIN!

__Show__ $\vec{F} = \nabla f$ where $f(x) = \frac{1}{(n+1)}\left[x F^{①}(x,y,z) + y F^{②}(x,y,z) + z F^{③}(x,y,z)\right]$

This says that we can immediately write down the potential fcn f without the need for any integration operation (quadrature).

__pf__ Lets show that the given f works. Rewrite it as:

$$(n+1) f = xP + yQ + zR$$

observe $\nabla \times F = 0 \Rightarrow \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ P & Q & R \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} R_y - Q_z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} R_y = Q_z \\ R_x = P_z \\ Q_x = P_y \end{matrix}$

Do partial derivs on f:

$$(n+1) f_x = P + x P_x + y \underset{P_y}{Q_x} + z \underset{P_z}{R_x}$$

$$= P + [x \ y \ z] \cdot \nabla P$$

$$= P + nP \quad \text{by Euler's homog thm: } x \cdot \nabla h = n h(x)$$

p.148 #21 ch 2 writeup sheet 6

$$= (n+1) P$$

$$(n+1) f_y = Q + x P_y + y Q_y + z R_y$$

$$= Q + x Q_x + y Q_y + z Q_z = (n+1) Q$$

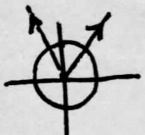
$$(n+1) f_z = (n+1) R$$

See my write up of ODEs

$$\Rightarrow \cancel{(n+1)} \nabla f = \cancel{(n+1)} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \Rightarrow \boxed{\nabla f = F}$$

__Remarks:__ (a) The prototype of a homogeneous fcn is a poly like $(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3$

(b) Given a homog fcn $h(tx, ty, tz) \overset{t>0}{=} t^n h(x,y,z)$, this shows that h is determined by its values on the unit sphere $S(0,1)$ [or any $S(0,r)$ for that matter]



Along every ray out from the origin, its value scales by t^n

At the origin, $h(0,0,0) = t^n h(0,0,0)$ Thus $h(0,0,0) = 0$ unless it has a singularity there like $h(x,y) = \frac{1}{x^2} + \frac{1}{y^2}$

For a homog vf $F(0) = 0$ origin is a FP

(c) If we follow the idea in Ince ODE p. 18-19 he has $P\,dx + Q\,dy = 0$ as a diff eq.

For us here, that would correspond to $P\,dx + Q\,dy + R\,dz = 0$

Then $df = f_x\,dx + f_y\,dy + f_z\,dz$

$$= P\,dx + Q\,dy + R\,dz = 0$$

$\Rightarrow f(x,y,z)$

Re-written on next page →

If we follow Ince ODE p.18–19 he has the diff eq $P\,dx + Q\,dy = 0$

Generalizing to the 3 variables we have here:

$$P\,dx + Q\,dy + R\,dz = 0$$

This is saying $df = F\cdot ds = 0 \Rightarrow f(x,y,z) = c$

$$\Rightarrow \frac{1}{n+1}\left[xP + yQ + zR\right] = c$$ a conserved quantity or const of motion

$F \perp \sigma'$

$$F\cdot ds = F(\sigma(t))\cdot\sigma'(t) = 0$$

if we parameterized $x = x(t)$, $y = y(t)$, $z = z(t)$ $= \sigma(t)$

which is not the usual flow of a vector field

$$\dot{\varphi}(x,t) = \vec{F}(\varphi(x,t))$$

It makes more sense in 2-dim:

$$P\,dx + Q\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-P}{Q}$$ provided $Q(x,y) \neq 0$

$\vec{F}\cdot ds = 0 \qquad F(\sigma(t))\cdot\sigma'(t) = 0$

$$xP + yQ = c$$ 1-parameter family of solutions

$\boxed{\text{BUT}}$ from sheet ③ of ch 7.1 Green's Thm

The vf $F = \begin{array}{c} P \\ Q \end{array}\begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$ is homog for degree $= 1$ $F(t\vec{x}) = tF(\vec{x})$ But Green's Thm fails for punctured disc $\Delta^*(0,1)$

~~I think the idea is if we extend to 3-dim (with $R = 0$) then it works fine.~~ NO

$$\oint_{\partial S} F\cdot ds = \int_S \nabla\times F\,ds$$ if $F = \nabla f$ then RHS would be 0

$$\nabla\times\nabla f = 0$$

$\boxed{\text{WORK THIS OUT}}$

$$F = \begin{bmatrix} \frac{-y}{r^2} \\ \frac{x}{r^2} \\ 0 \end{bmatrix} \qquad \nabla\times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ P & Q & 0 \end{vmatrix} = \begin{bmatrix} Q_z \\ P_z \\ Q_x - P_y \end{bmatrix} \neq \vec{0}$$

But if we don't extend to $\mathbb{R}^3$ then no z dependence

$$= \begin{bmatrix} 0 \\ 0 \\ Q_x - P_y \end{bmatrix}$$

So the resolution seems to be that this homogeneous Poincare Lemma does not apply to this F. Yes, F is homog of degree 1, but $\nabla\times F \neq 0$

$$Q_x = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{\ell h' - h\ell'}{\ell^2} = \frac{r^2\cdot 1 - x(2r}{}$$