cf. Hamiltonian $H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$X_H = J\nabla H$ where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

ch 7.3

Conservative Fields

∇V is direction of greatest rate of increase of V. Thus $-\nabla V$ direction of greatest decrease

1/17/2016

7.3.1

We call a vf **Conservative** if it comes from a potential fcn ('vf generating fcn') $\vec{F} = \nabla f$

p.243 **Classic Example**

Note: Potential fcns are very useful. McCauley CMTFAICD p.37 uses $F = -\nabla V$ to derive the most general invariance of Newton $m\ddot{\sigma} = F(\sigma)$

(i) Given $\vec{F}$ and a curve $\sigma: [a,b] \to \mathbb{R}^3$ that is the trajectory of a particle of mass m moving according to Newton's law

$$m\vec{\sigma}''(t) = \vec{F}(\sigma(t))$$

(ii) F comes from a potential fcn V

$$F = -\nabla V$$

physics likes this neg sign — particle moves toward decreasing potential

$$\Rightarrow \underbrace{\tfrac{1}{2}m\|\dot{\sigma}\|^2}_{KE} + \underbrace{V(\sigma(t))}_{PE} = \text{Const} \quad \forall t \in [a,b]$$

↑ total Energy

Conservation of Energy

Pf Note $\|\sigma'\|^2 = \sigma' \cdot \sigma'$

$$\frac{d}{dt}\left[\tfrac{1}{2}m\sigma' \cdot \sigma' + V(\sigma(t))\right] = \tfrac{1}{2}m\sigma'' \cdot \sigma' + \tfrac{1}{2}m\sigma' \cdot \sigma'' + DV_{\sigma(t)}(\sigma'(t))$$

$$= \underbrace{m\sigma''}_{F(\sigma(t))} \cdot \sigma' + \underbrace{\nabla V_{\sigma(t)}}_{-F(\sigma(t))} \cdot \sigma'(t)$$

$$= F(\sigma(t)) \cdot \sigma' - F(\sigma(t)) \cdot \sigma'(t) = 0$$

Thus $\tfrac{1}{2}m\sigma' \cdot \sigma' + V(\sigma(t)) = C$ $\square$

See my Fowles AM ch 7 write up sheets where I showed the total mechanical energy of a system of N particles can **only** be changed by an **external** force doing work on the system.

Really M&T are proving it for a rectangular box $[a,b] \times [c,d] \times [e,f]$

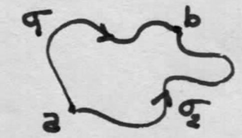
Can be an open convex subset $\mathcal{U}$ Rudin POMA

Really $\mathcal{U}$ being simply conn (Every loop is contractive in $\mathcal{U}$ to a pt)

actually I think only no limit pts

BIG THM 7

$F: \mathbb{R}^3 \to \mathbb{R}^3$ ($\mathcal{U}$) C^1 smooth vf (F may have finitely many singularities)



TFAE (i) For any clsd, C^1 curve σ $\int_\sigma \vec{F} \cdot d\vec{s} = 0$

⇓

(ii) **Indep of Path**: For any 2 curves σ_1, σ_2 connecting endpts a, b

$$\int_{\sigma_1} F \cdot ds = \int_{\sigma_2} F \cdot ds$$

⇓

(iii) $\exists$ potential fcn (vf generating fcn)

$$F = \nabla f$$

⇓

(iv) **irrotational** $\nabla \times F = 0$

contains Poincaré Lemma (iv) ⇒ (iii)

G&P ch 4

If a clsd curve γ can be homotopically contracted to a pt a, then

$\oint_\gamma \omega = \int_{\{a\}} \omega = 0$ Thus no holes can be in domain

• $\oint_\gamma \omega = 0 \Rightarrow$ path indep

• path indep ⇒ we can define a primitive $F(x) = \int_a^x f$ $\quad df = \omega$

Pf The idea is to show

(i) ↗ (ii) ↘ (iii) ↙ (iv) ↖ (i)

Thm is Partial converse to $\nabla \times \nabla f = 0$: $\nabla \times F = 0 \Rightarrow \exists f \ni F = \nabla f$ (except for topological obstructions i.e. a bad domain)

(i) ⇒ (ii) Given two curves σ_1, σ_2 connecting a & b

Let $\sigma := \sigma_1 * (-\sigma_2)$ (Join the curves by tracing σ_1 and then going backwards on σ_2)

$$\text{Then } \int_\sigma \vec{F} \cdot d\vec{s} = 0 \Rightarrow \int_{\sigma_1} + \int_{-\sigma_2} = 0 \Rightarrow \int_{\sigma_1} F \cdot ds = \int_{\sigma_2} F \cdot ds$$

(ii) $\Rightarrow$ (iii) $\int_\sigma F\cdot ds$ is indep of path σ $\Longrightarrow$ $\exists$ potential fcn $f:\mathbb{R}^3\to\mathbb{R}$ $\ni$ $\vec{F}=\nabla f$

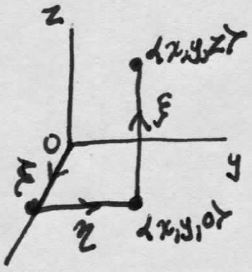
Easy case – no singularities in $\mathbb{R}^3$

Fix base pt (say origin $\vec{0}$). choose a pt X. $X=\langle x,y,z\rangle$. Define $f_0(x) := \int_\sigma \vec{F}\cdot d\vec{s}$

f_0 does not depend on σ because of indep of path.

We need to compute $D_x f$, $D_y f$, $D_z f$

Let us choose ~~seven~~ 3 different paths σ that will make it easy to compute each partial deriv. Lets start with $\frac{\partial}{\partial z} f$



This path σ consists of 3 straight line segments, parallel to co-ord axes. The key is that all the z variation is in the last segment

$$\int_\sigma F\cdot ds = \int_a^b F(\sigma(t))\cdot\sigma'(t)\,dt = \int_a^b \left(F^{①}(\sigma)\dot{\sigma}_1 + F^{②}(\sigma)\dot{\sigma}_2 + F^3\dot{\sigma}_3\right)dt$$

$\zeta(t) = \begin{bmatrix} t\\0\\0\end{bmatrix}$ for $t\in[0,x]$ $\qquad \eta(t) = \begin{bmatrix} x\\t\\0\end{bmatrix}$ $0\le t\le y$ $\qquad \xi(t) = \begin{bmatrix} x\\y\\t\end{bmatrix}$ $0\le t\le z$

$$f_0(x) = \int_0^x F^{①}(t,0,0)\,1\,dt + \int_0^y F^{②}(x,t,0)\,1\,dt + \int_0^z F^{③}(x,y,t)\,1\,dt$$

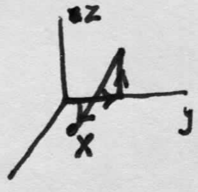
$$\frac{\partial}{\partial z} f_0(x) = 0 + 0 + \underbrace{\frac{\partial}{\partial z}\int_0^z F^3(x,y,t)\,dt}_{F^{③}(x,y,z)}$$

If we tried to do $\frac{\partial}{\partial x} f_0$ on this σ, it's valid, but we'd get a messy expression.

FTOC, requires no singularities on this line segment

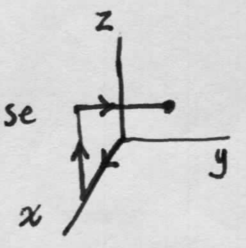
Rudin POMA p. 324

▷ For $\frac{\partial}{\partial x} f_0$, do the same thing with path



$$\frac{\partial}{\partial x} f_0(x) = F^{①}(x)$$

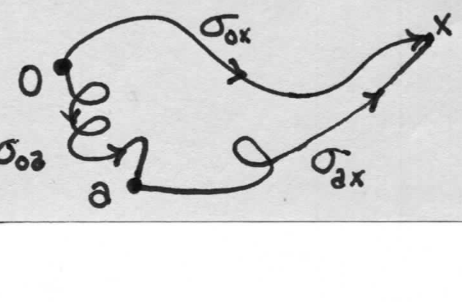
▷ For $\frac{\partial}{\partial y} f_0$, use



$\Rightarrow \frac{\partial}{\partial y} f_0(x) = F^{②}(x)$ $\Longrightarrow$ $\boxed{\vec{\nabla} f_0(x) = \vec{F}(x) \text{ for any } X \quad f_0 \text{ is the potential fcn we seek (one of many).}}$

NOTE The path could actually be arb in the plane of the other 2 variables, we just need a straight line path for the variable we differentiate with respect to. and for this final path to have the final values of the other 2 vars.

▷ What is we chose a different base pt rather than 0? Lets take a



$f_0(x) = \int_{\sigma_{0x}} F\cdot ds$

$f_a(x) = \int_{\sigma_{ax}} F\cdot ds$

what is the relationship?

$\int_{\sigma_{0x}} + \int_{-\sigma_{ax}} + \int_{-\sigma_{0a}} = 0$ because $\sigma_{0x} * (-\sigma_{ax}) * (-\sigma_{0a})$ closed loop

$\Rightarrow \int_{\sigma_{0x}} = \int_{\sigma_{ax}} + \int_{\sigma_{0a}}$ $\Rightarrow$ $f_0(x) = f_a(x) + C$

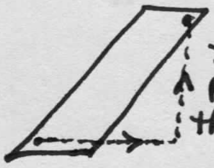
$\boxed{\nabla(f_0) = \nabla(f_a)}$

Since $\nabla(f_b) = \nabla(f_a)$ the base pt is irrelevant for pf

1/20/2016

② (7.3.3)

Now what about Singularities in the domain? Or $\mathcal{U}$ being an open, convex set?

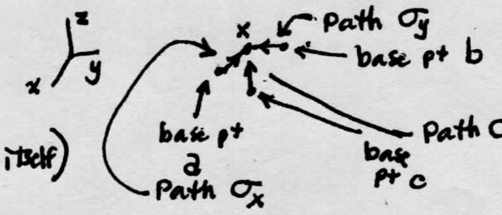


can't do the same path arg in this set!

Here is an idea that does <u>not</u> work:

From the previous arg we see we really only need to approach pt $x = \langle x,y,z \rangle$ along extremely short paths in the direction of the co-ord we want to differentiate.

Thus we can avoid all singularities of F (unless x is one itself)



Thus we can get $f_a(x) = \int_{\sigma_x} F \cdot ds$

$f_b(x) := \int_{\sigma_y} F \cdot ds$

$f_c(x) := \int_{\sigma_z} F \cdot ds$

Here we have a problem that we get a different fcn (3 different fcns) for each <u>x</u>

Then it is true that

$$\nabla(f_a) = \nabla(f_b) = \nabla(f_c) = \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix}$$

but we <u>don't</u> have <u>a</u> potential fcn, we have 3

$$f_a(x) = f_b(x) + C_0 = f_c(x) + C_1$$ and a, b, c depend on x.

~~but we could just take any one of them!~~

The real answer is that the domain $\mathcal{U}$ must be simply conn (Every closed loop in $\mathcal{U}$ is contractible over $\mathcal{U}$ to a pt) ~~DONE~~

~~Then~~ Any 2 paths σ_1, σ_2 with the same fixed end pts are homotopic to each other

Boothby AITDMARG p. 265

mfd M <u>Simply Conn</u> — Every loop γ can be contracted <u>over M</u> to a one pt 'loop'.

M <u>Contractible</u> — The identity map $Id: M \longrightarrow M$ is homotopic to a const map $c: M \longrightarrow \{m_0\}$

The whole mfd M can be contracted <u>over itself</u> to a pt.

Contractible $\Longrightarrow$ Simply Conn

Contractible $\not\Longleftarrow$ Simply Conn

Counter example S^2 in $\mathbb{R}^3$

- we could contract it to the center, but
- we leave S^2 and go into the ambient space
- If we try to contract it over itself, we must ~~tear it~~ puncture it.

p. 268 <u>Thm</u>

$$\left.\begin{array}{l} \omega \text{ 1-form on } M \\ d\omega = 0 \text{ (clsd)} \\ \gamma_1, \gamma_2 \text{ any 2 paths from } p \text{ to } q \\ \gamma_1 \sim \gamma_2 \text{ homotopic} \end{array}\right\} \Longrightarrow \int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

(iii) $\Rightarrow$ (iv) $f \in C^2$

▷ show $F = \nabla f \Rightarrow \nabla \times F = 0$

This is just computing $\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ D_x f & D_y f & D_z f \end{vmatrix} = \begin{bmatrix} f_{yz} - f_{zy} \\ -(f_{xz} - f_{zx}) \\ f_{xy} - f_{yx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ we can interchange the order of partial derivs cf Ch 2.6 Sheets

▷ (iv) $\Rightarrow$ (i) choose any closed curve σ in $\mathcal{U}$

hand waving

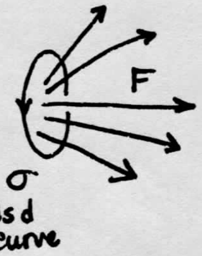
Let S be a surface in $\mathcal{U}$ that has σ as its ∂
(If S passes through a singularity of F, then dent S slightly to avoid that pt — this should always work for isolated sings)

apply Stokes: $\int_\sigma F \cdot ds = \int_S (\nabla \times F) \cdot n \, dS = \int_S 0 = 0$ $\square$

COR This Thm holds in $\mathbb{R}^2$ but there $\mathcal{U}$ can have no holes or else not simply conn.

See example further on. #12

▷ we know work $W = \int_\sigma F \cdot ds$ but it also has other interpretations.



Let F be the velocity field of a moving fluid

$\oint_\sigma F \cdot ds \neq 0$ means F induces circulation around loop σ
(yet σ is not nec a flowline of F — no fluid may actually traverse σ)

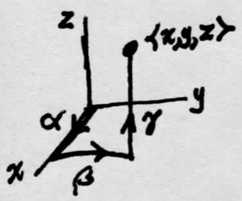
For electric field E, $EMF = \oint_\sigma E \cdot ds$ If this integral is non-zero, E causes circulation of charge in a wire loop tracing σ.

Irrotational flow: $\text{Curl}(F) = 0$

example Given vf F defined on all of $\mathbb{R}^3$ with $\nabla \times F = 0$, here are 2 methods to find a potential fcn f ('vf generating fcn'):

$$F(x,y,z) = \begin{bmatrix} y \\ z\cos(yz) + x \\ y\cos(yz) \end{bmatrix} \quad \text{verify} \quad \nabla \times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ y & z\cos(yz) + x & y\cos(yz) \end{vmatrix} = \begin{bmatrix} \cos(yz) - yz\sin(yz) - [\cos(yz) - yz\sin(yz)] \\ 0 - 0 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Method 1 Integrate along straight lines parallel to co-ord axes:



$$f(x,y,z) = \int_0^x F^{①}(t,0,0)\,dt + \int_0^y F^{②}(x,t,0)\,dt + \int_0^z F^{③}(x,y,t)\,dt$$

$$= \int_0^x 0 \cdot 1t + \int_0^y x\,dt + \int_0^z y\cos(yt)\,dt \qquad u = yt,\ du = y\,dt$$

$$= c + xy + \sin(yt)\Big]_0^z$$

$$= c + xy + \sin(yz)$$

choosing base pt 0 typically makes terms vanish

extraneous

In general, this method might not be this easy.

See also my discussion in Boyce & DiPrima ODE Summary sheets $\mathbb{R}^2$ case

Prob 24 p. 296 from solutions 2/27/2016

Here is a special case of Poincaré Lemma from Rudin POMA

Let $\mathcal{U}$ open, convex set in $\mathbb{R}^3$

$\vec{F} = \begin{bmatrix} F^{1} \\ F^{2} \\ F^{3} \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ Satisfies $\nabla \times F = 0 \rightsquigarrow \begin{vmatrix} i & j & k \\ D_1 & D_2 & D_3 \\ P & Q & R \end{vmatrix} = \begin{bmatrix} R_y - Q_z \\ R_x - P_z \\ Q_x - P_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $D_2R = D_3Q$, $D_1R = D_3P$, $D_1Q = D_2P$

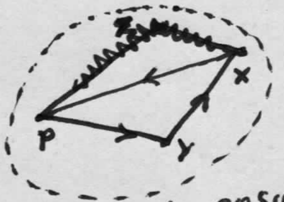
Need this

Show $\exists\ f : \mathcal{U} \to \mathbb{R} \ni F = \nabla f$ in $\mathcal{U}$

This is showing we don't need to do straight line integration parallel to co-ord axes.

Choose base pt $p \in \mathcal{U}$. Define $f_p(x) := \int_{[p\,x]} F \cdot ds$

where $[p\,x]$ denotes straight line path



Choose any pt $y \in \mathcal{U}$ not on line seg $[p, x]$, such that $[p, y, x]$ is pos OR

Let K be the planar triangular surf $[p, y, x]$

Convexity ensures that this triangle in $\mathcal{U}$ for any p, x, y.

Apply Stokes $\underbrace{\int_K (\nabla \times F)}_{0} \cdot \hat{n}\, dS = \oint F \cdot ds$

$$0 = \underbrace{\int_p^y F \cdot ds}_{f_p(y)} + \int_y^x F \cdot ds + \underbrace{\int_x^p F \cdot ds}_{-f_p(x)} \Rightarrow f_p(x) - f_p(y) = \int_y^x F \cdot ds$$

The path $[y\ x] = \sigma(t) = (1-t)y + tx$ for $0 \le t \le 1$

then $\sigma'(t) = x - y$

$$f_p(x) - f_p(y) \int_\sigma F \cdot ds = \int_0^1 P((1-t)\vec{y} + t\vec{x})(x_1 - y_1) + Q(\sigma(t))(x_2 - y_2) + R(\sigma(t))(x_3 - y_3)\, dt$$

▷ Now show $\frac{\partial}{\partial x_1} f_p(x) = F^{1}(x)$

$$\frac{\partial}{\partial x_1}\Big(f_p(x) - \underbrace{f_p(y)}_{0}\Big) = \frac{\partial}{\partial x_1} \int_0^1 \Big[P((1-t)y_1 + tx_1, (1-t)y_2 + tx_2, (1-t)y_3 + tx_3)(x_1 - y_1) + Q(\sigma)(x_2 - y_2) + R(\sigma)(x_3 - y_3)\Big]$$

$$\frac{\partial}{\partial x_1}\Big[P((1-t)y_1 + tx_1, \sigma^{2}, \sigma^{3})(x_1 - y_1)\Big] = D_1P(\sigma)\,t(x_1 - y_1) + P(\sigma)\cdot 1$$

$$\frac{\partial}{\partial x_1}\Big[Q((1-t)y_1 + tx_1, \sigma^{2}, \sigma^{3})(x_2 - y_2)\Big] = \underbrace{D_1Q}_{= D_2P}(\sigma)\,t(x_2 - y_2)$$ from $\nabla \times F = 0$

$$\frac{\partial}{\partial x_1}\Big[R(\sigma)(x_3 - y_3)\Big] = \underbrace{D_1R}_{= D_3P}(\sigma)\,t(x_3 - y_3)$$

$$\Rightarrow \frac{\partial}{\partial x_1} f_p(x) = \int_0^1 D_1P(\sigma)\,t(x_1 - y_1) + P(\sigma) + D_2P(\sigma)\,t(x_2 - y_2) + D_3P(\sigma)\,t(x_3 - y_3)\, dt$$

But observe $\frac{d}{dt} P(\sigma^1, \sigma^2, \sigma^3) = D_1P\dot{\sigma}^{1} + D_2P\dot{\sigma}^{2} + D_3P\dot{\sigma}^{3} = D_1P[x_1 - y_1] + D_2P[x_2 - y_2] + D_3P[x_3 - y_3]$

Int by parts $u = t$, $du = dt$, $dv = \frac{d}{dt}P$, $v = P(\sigma)$

Thus we have $\frac{\partial}{\partial x_1} f_p(x) = \int_0^1 P(\sigma) + t\left(\frac{d}{dt}P(\sigma)\right) dt = \cancel{\int_0^1 P(\sigma)\, dt} + \underbrace{\int_0^1 t\left(\frac{d}{dt}P(\sigma)\right) dt}_{tP(\sigma)\big]_0^1 - \cancel{\int_0^1 P(\sigma(t))\,dt}}$

$$= 1\,P(\sigma(1)) - 0 = P(\sigma(1)) = P(x)$$

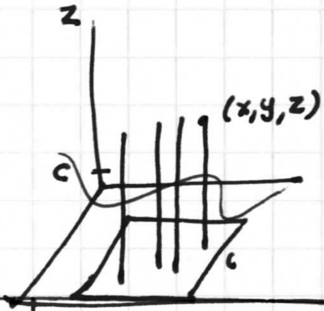
$$\frac{\partial}{\partial x_1} f_p(x) = F^{1}(x)$$

and likewise for $\frac{\partial f}{\partial x_2} = F^{2}$ $\quad \frac{\partial}{\partial x_3} f = F^{3}(x)$

$\square$

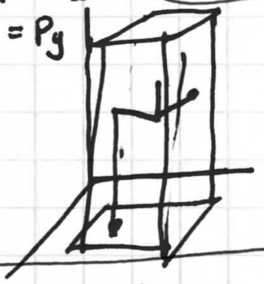
Why did we need restriction on the domain?

~~when we integrate~~ $\cancel{B(x,y,z) = \int_c^z P(x,y,t)\,dt + h(x,y)}$



$$\begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ P & Q & R \end{vmatrix} = \left.\begin{matrix} R_y - Q_z = 0 \\ R_x - P_z = 0 \\ Q_x - P_y = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} R_y = Q_z \\ R_x = P_z \\ Q_x = P_y \end{matrix}$$

Romantic dream



Let's revisit Method 2

If $\nabla \times F = 0$ we seek $f \ni F = \nabla f$

we like to take the base pt $= (0,0,0)$ to eliminate terms

Directly solve PDEs $\quad \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$

Solve $\frac{\partial f}{\partial x} = P(x,y,z) \Rightarrow f(x,y,z) = \int_{x_0}^{x} P(t,y,z)\,dt + h(y,z)$

$\frac{\partial f}{\partial y} = Q(x,y,z) \rightarrow$

Then $\frac{\partial}{\partial y}\int_{x_0}^{x} \underbrace{P_y(t,y,z)}_{Q_x(t,y,z)}\,dt + h_y(y,z) \stackrel{!}{=} Q(x,y,z)$

FTOC $\Rightarrow Q(\cancel{x,y,z}) - Q(x_0,y,z) + h_y \stackrel{!}{=} Q\cancel{(x,y,z)}$

$\Rightarrow h_y(y,z) = Q(x_0,y,z) \Rightarrow h(y,z) = \int_{y_0}^{y} Q(x_0,t,z)\,dt + k(z)$

$\Rightarrow f(x,y,z) = \int_{x_0}^{x} P(t,y,z)\,dt + \int_{y_0}^{y} Q(x_0,t,z)\,dt + k(z)$

and once again we compare to the next term. $\frac{\partial}{\partial z} f \stackrel{!}{=} R$

$\Rightarrow \int_{x_0}^{x} \underbrace{P_z(t,y,z)}_{R_x}\,dt + \int_{y_0}^{y} \underbrace{Q_z(x_0,t,z)}_{R_y}\,dt + k'(z) \stackrel{!}{=} R(x,y,z)$

$\cancel{R(x,y,z)} - \cancel{R(x_0,y,z)} + \cancel{R(x_0,y,z)} - R(x_0,y_0,z) + k'(z) = \cancel{R(x,y,z)}$

$k'(z) = R(x_0,y_0,z)$

Thus $f(x,y,z) = \int_{x_0}^{x} P(t,y,z)\,dt + \int_{y_0}^{y} Q(x_0,t,z)\,dt + \int_{z_0}^{z} R(x_0,y_0,t)\,dt \Rightarrow k(z) = \int_{z_0}^{z} R(x_0,y_0,t)\,dt + c$

<u>NOTE</u> we could have started with $\frac{\partial}{\partial y} f = Q$ or $\frac{\partial}{\partial z} f = R$ and gotten a <u>similar</u> formula. See my ODE write up.

I think this is the same method as before, just disguised. 7.3.5

Method 2 Directly solve PDEs : $\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} F^{①} \\ F^{②} \\ F^{③} \end{bmatrix}$

some fcn that depends only on y, z

Rewrite with ideas from TG Pollard p.75 DONE ON NEXT PAGE

integrating along the segment where y,z are const

Solve $\frac{\partial f}{\partial x} = y \implies yx + g(y,z)$ call this f^a

$\frac{\partial f}{\partial y} = z\cos(yz) + x \implies \sin(yz) + xy + h(x,z) =: f^b$

$\frac{\partial f}{\partial z} = y\cos(yz) \implies \sin(yz) + k(x,yz =: f^c$

Take f^a : $\frac{\partial}{\partial y} f^a \overset{!}{=} F^{②}$

$\cancel{x} + \frac{\partial g}{\partial y} \overset{!}{=} z\cos(yz) + \cancel{x} \implies \frac{\partial g}{\partial y} = z\cos(yz) \implies g(y,z) = \sin(yz) + l(z)$

So now $f^a = yx + \sin(yz) + l(z)$

Compare to $F^{③}$: $\frac{\partial f^a}{\partial z} \overset{!}{=} y\cos(yz) \implies y\cos(yz) + l'(z) \overset{!}{=} y\cos(yz)$

$\implies l'(z) = 0$

$\implies l(z) = C$ we don't care about constants for potential fcns.

$F = \nabla f$

$\implies f(x,y,z) = f^a = yx + \sin(yz) + C$

Same thing we found by previous method.

we could have started with the $\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$ eq also

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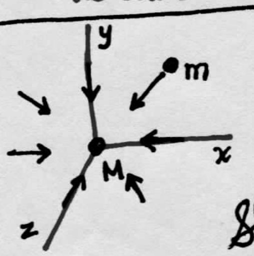


Gravitational field Force of mass M on mass m

$$\vec{F}(x) = \frac{-GMm}{\|x\|^3}\vec{x} = \frac{-GMm}{r^3}\vec{r} = \frac{-GMm}{r^2}\hat{e}_r$$

$\vec{r} := \vec{x}$ here

Note that this only has the domain $\mathbb{R}^3 - \{0\}$ but that is still simply conn (every loop can be contracted by going to the side of the origin, unlike $\mathbb{R}^2 - \{0\}$)

Show F is irrotational ($\nabla \times F = 0$) and find potential fcn

ch 3.5

$$\nabla \times F = \nabla \times \left(\frac{GMm}{r^3}\vec{r}\right) = GMm\left(\nabla\left(\tfrac{1}{r^3}\right) \times r + \tfrac{1}{r^3}(\nabla \times r)\right)$$

we know $\nabla(r^n) = n r^{n-2}\vec{r}$ so here $\nabla(r^{-3}) = -3r^{-5}\vec{r}$

$$= GMm\left(\underbrace{\cancel{\tfrac{-3}{r^5}\vec{r} \times \vec{r}}}_{0} + \tfrac{1}{r^3}\nabla \times r\right)$$

$$\nabla \times r = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ x & y & z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\nabla \times F = 0$

[Fowles AM does this calculation in spherical co-ords] ch 6 p.141

$$\nabla \times F = \frac{1}{r^2 \sin\theta}\begin{vmatrix} e_r & re_\theta & r\sin\theta e_\varphi \\ D_r & D_\theta & D_\varphi \\ F^r & rF^\theta & r\sin\theta F^\varphi \end{vmatrix}$$

Fowles switches θ & φ from M&T

In fact, Fowles shows

$F = f(r)\hat{e}_r$ has $\nabla \times F = 0$

$V(r) := -\int_{r_0}^{r} F \cdot ds = -\int_{r_0}^{r} f(r)\, dr$

take $r_0 = \infty$

① Thus we can observe if $n = -1$

$$\nabla(1/r) = -\frac{1}{r^3}\vec{r} \quad \text{so} \quad \nabla\left(\frac{GMm}{r}\right) = \frac{-GMm}{r^3}\vec{r}$$

Thus $V(x) = \frac{GMm}{\|x\|}$ is the potential fcn we seek.

② Or we could do it by $W = \int_\sigma F \cdot ds$

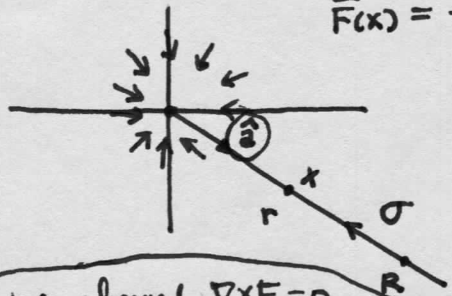
Fowles AM

$$V(r) = -\int_{-r_{ref}}^{r} F \cdot ds = -\int_{r_{ref}}^{r} f(r)\, dr$$

Lets integrate to find the potential fcn $f \ni F = \nabla f$

In more detail, since $\vec{F}$ is radial (spherical symm), lets take our path to be a radial line. we fix a unit vector $\hat{a}$ for direction, then specify 2 pts on the line by $x = r\hat{a}$ and reference (base) pt is $R\hat{a}$.

$$\vec{F}(x) = -\frac{GMm}{\|x\|^3}\vec{x} = -k\frac{x}{\|x\|^3}$$



The path from $R\hat{a}$ to $r\hat{a}$ is

$$\sigma(t) = (1-t)\underbrace{R\hat{a}}_{p_0} + t\underbrace{r\hat{a}}_{x} \quad t\in[0,1]$$

we showed $\nabla \times F = 0$ and $\mathbb{R}^3 - \{0\}$ is Simply Conn. Thus $\int_\sigma F\cdot ds$ is indep of path σ and depends only on the endpts p_0, q. Fix base pt p_0 and take the easist path (a straight line) to q.

Define $f_p(x) = \int_\sigma F\cdot ds$

~~The natural choice would if we duplicated prev~~ be ~~arg~~, we'd take $p_0 = 0$. But this has problems as will be apparent.

Note that $\|\sigma\| = (1-t)R + tR$ always pos since $r, R > 0$ we are not going thru the origin

$$\sigma' = (r-R)\hat{a}$$

$$f(x) = \int_\sigma F(\sigma)\cdot\sigma' dt = \int_0^1 \frac{-k}{\|\sigma\|^3}\sigma\cdot\sigma' dt$$

$$= \int_0^1 \frac{-k}{[tr+(1-t)R]^{\cancel{3}\,2}}\cancel{(tr+(1-t)R)}(r-R)\,dt$$

COV
$u = tr + (1-t)R$
$du = (r-R)dt$

$$= -k\int_{u=R}^{r}\frac{1}{u^2}du = -k\left[-\frac{1}{u}\right]_R^r$$

$$= k\left[\frac{1}{r} - \frac{1}{R}\right]$$

Thus if we let $R \to 0$ to take origin as base pt, we'd get $-\infty$. So ~~we~~ the convention is we let $R \to \infty$ and we take ∞ as the base pt.

$$f(x) = \frac{GMm}{r} \text{ or } \frac{GMm}{\|x\|}$$ is gravitational (well inv squared) potential fcn.

Then $\nabla f = \nabla\left(\frac{GMm}{r}\right) = GMm\nabla\left(\frac{1}{r}\right) = \frac{-GMm}{r^3}\vec{r}$ and we recovered $\vec{F}$ $\quad F = \nabla f = -\nabla(-f)$

But the convention in physics is $F = -\nabla V$ so we define $V = -f$

The idea is that $\vec{F}$ should be the direction of the steepest decrease of V

$\vec{a}\cdot\vec{b} = \|a\|\,\|b\|\cos\theta$

Thus $\vec{\nabla V}\cdot\hat{n} = \|\nabla V\|\, 1 \cos\theta$

This is a max when $\cos = 1$ $(\hat{n} = \widehat{\nabla V})$ and a min when $\cos = -1$ $(\hat{n} = -\nabla V)$

$\square$