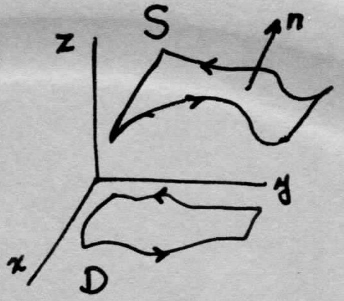
Summary of ch 7.2

Stokes' Thm For Graphs (single co-ord patch)



$$S = \{ \langle x, y, f(x,y) \rangle \mid (x,y) \in D \} \quad \left.\begin{array}{l} f\ C^2 \\ S\ \Theta^R \\ F\ C^1\ vf \end{array}\right\} \Rightarrow \int_{+\partial S} \vec{F} \cdot d\vec{s} = \int_S (\nabla \times F) \cdot d\vec{S}$$

This is like lifting the plane Green region up.

<u>pf</u> The whole trick is to expand the LHS (line integral) into a form to which we can apply Green's Thm to D.

$$\int_{+\partial S} \vec{F} \cdot d\vec{s} = \int_\eta F^1 dx + F^2 dy + F^3 dz \qquad \eta : t \longmapsto \begin{bmatrix} x(t) \\ y(t) \\ f(x(t), y(t)) \end{bmatrix}$$

$$= \int_a^b \left(F^1 \dot{\eta}^1 + F^2 \dot{\eta}^2 + F^3 \dot{\eta}^3 \right) dt$$

$$= \int_a^b F^1 \dot{x} + F^2 \dot{y} + F^3 \left(\frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \right) dt$$

$$= \int_a^b \left(F^1 + F^3 \frac{\partial f}{\partial x} \right) \dot{x} + \left(F^2 + F^3 \frac{\partial f}{\partial y} \right) \dot{y}\, dt$$

Now by some formalism $\dot{x}\, dt = \frac{dx}{dt} dt = dx$

$\langle x(t), y(t) \rangle$ trace out $+\partial D$ because it is a graph (just project along z axis).

$$= \oint_{\partial D} \underbrace{\left(F^1 + F^3 \frac{\partial f}{\partial x} \right)}_{P} dx + \underbrace{\left(F^2 + F^3 \frac{\partial f}{\partial y} \right)}_{Q} dy$$

apply Green

$$= \int_D \frac{\partial}{\partial x} \left(F^2 + F^3 \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(F^1 + F^3 \frac{\partial f}{\partial x} \right) dA$$

just dxdy in plane, right?

$D_1 F^3 \cdot f_y + D_3 F^3 \cdot f_x \cdot f_y + F^3 D(f_y)$

should be here

rename "f" as "z" to make the classical notation work.

$$= \frac{\partial F^2}{\partial x} + \frac{\partial F^2}{\partial z}\frac{\partial z}{\partial x} + \cancel{\frac{\partial F^3}{\partial x}\frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial y}\right)} + \cancel{F^3 \frac{\partial^2 z}{\partial x \partial y}} + \frac{\partial F^3}{\partial x}\frac{\partial z}{\partial y}$$

$$- \left[\frac{\partial F^1}{\partial y} + \frac{\partial F^1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F^3}{\partial y} \cdot \frac{\partial z}{\partial x} + \cancel{\frac{\partial F^3}{\partial z}\frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial x}} + \cancel{F^3 \frac{\partial^2 z}{\partial y \partial x}} \right]$$

mixed partials are equal by Smoothness C^2.

If the param is
$\Phi : D \longrightarrow S$
$(x,y) \longmapsto [x, y, f(x,y)]^T$
$\vec{n} = \vec{\Phi}_x \times \vec{\Phi}_y = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \\ 1 \end{bmatrix}$

$$= \int_D \left[\left(\frac{\partial F^2}{\partial z} - \frac{\partial F^3}{\partial y} \right) \frac{\partial z}{\partial x} + \left(\frac{\partial F^3}{\partial x} - \frac{\partial F^1}{\partial z} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y} \right) \right] dA$$

cont'd →

$$\nabla \times F = \begin{vmatrix} i & j & k \\ D_1 & D_2 & D_3 \\ F^1 & F^2 & F^3 \end{vmatrix} = \begin{bmatrix} D_2 F^3 - D_3 F^2 \\ -(D_1 F^3 - D_3 F^1) \\ D_1 F^2 - D_2 F^1 \end{bmatrix}$$

and I have (pulling minus signs)

$$\int_D \left[\left(\frac{\partial F^3}{\partial y} - \frac{\partial F^2}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) + \left(\frac{\partial F^1}{\partial z} - \frac{\partial F^3}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right)(1)\right] dA$$

$$= \int_D (\nabla \times F) \cdot \vec{n}\, dA$$

$$\overset{\text{def}}{=} \int_S (\nabla \cdot F) \cdot \vec{n}\, dA$$

I'm still a little worried about that whole $\vec{n}\, dA$ business!

Here "dA" just means $dx\,dy$

Here it is just a formal symbol

$\square$

Thm 6 Stokes' for Parameterized Surfs $\displaystyle\int_{\partial S} \vec{F} \cdot d\vec{s} = \int (\nabla \times F) \cdot d\vec{S}$

No pf.

---

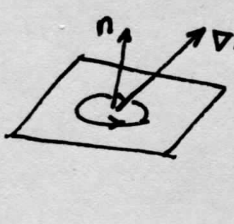
p.423 Use Stokes to justify $\nabla \times F$ as rotation of fluid.

Let F be the velocity field of a fluid

<u>i.e.</u> $\dot{x} = F(x)$

Fix a pt X, and let a plane pass thru it with normal $\hat{n}$.

Let there be a <u>small</u> disk in the plane Δ_ρ centered at X, with nice oriented $\partial \Delta_\rho$



$$\int_{\partial \Delta_\rho} F \cdot ds \overset{\text{stokes}}{=} \int_{\Delta_\rho} (\nabla \times F) \cdot \hat{n}\, dS$$

$$\overset{\text{MVT p.289}}{=} (\nabla \times F)(\xi) \cdot \hat{n} \int_{\Delta_\rho} dS = (\nabla \times F)_{(\xi)} \cdot \hat{n}\, \underset{\pi\rho^2}{A(\Delta_\rho)}$$

For some $\xi \in \Delta_\rho$

Take $\lim_{\rho \to 0}$

$\Rightarrow$

Then $\xi \to x$

$$\boxed{\lim_{\rho \to 0} \frac{1}{A(\Delta_\rho)} \underbrace{\int_{\partial \Delta_\rho} F \cdot ds}_{\text{"circulation"}} = (\nabla \times F)_{(x)} \cdot \hat{n}}$$

Some kind of an average, but not a terribly nice one.

If we choose n to coincide with $(\nabla \times F)(x)$,

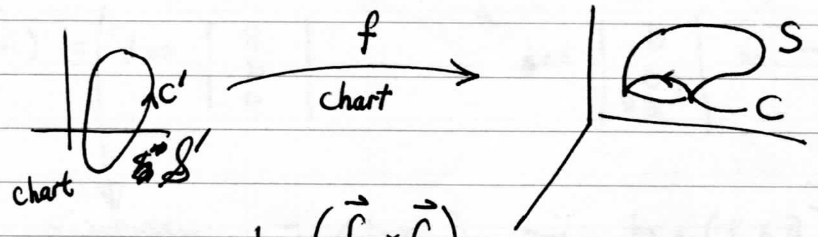
RHS $= \|(\nabla \times F)(x)\|$

The one thing left in ch 7.2 is Faraday's Law on a Möbius Str

REA
p.992-994

12/31/2015

Stoke's for Param Surfs (Still image of one chart)



Pull it down to chart

$$\vec{n} = \frac{1}{\|f_u \times f_v\|}(\vec{f_u} \times \vec{f_v})$$

Stokes $$\int_S (\text{Curl } F) \cdot n\, dS = \int_{\mathcal{D}'} (\nabla \times F) \cdot (\vec{f_u} \times \vec{f_v})\, du\, dv$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot (\vec{f_u}\, du + \vec{f_v}\, dv)$$

Use Green Thm to Show: $$\int_{\mathcal{D}'} (\nabla \times F) \cdot (f_u \times f_v)\, du\, dv = \int_{C'} \vec{F} \cdot (\vec{f_u}\, du + \vec{f_v}\, dv)$$

Now we must simplify $(\nabla \times F) \cdot (f_u \times f_v) =$

$$\nabla \times F = \hat{\imath} \times \vec{F_x} + \hat{\jmath} \times \vec{F_y} + \hat{k} \times \vec{F_z}$$

$\boxed{\text{ID}}$ $$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) = \begin{vmatrix} (a \cdot c) & (a \cdot d) \\ (b \cdot c) & (b \cdot d) \end{vmatrix}$$

p.95

after a lot of work:

See next page

$$(\nabla \times F) \cdot (\vec{f_u} \times \vec{f_v}) = \frac{\partial}{\partial u}(\vec{f_v} \cdot \vec{F}) - \frac{\partial}{\partial v}(\vec{f_u} \cdot F)$$

then $$\int_{\mathcal{D}'} (\nabla \times F) \cdot (\vec{f_u} \times \vec{f_v})\, du\, dv = \int_{\mathcal{D}'} \underbrace{\frac{\partial}{\partial u}(\vec{f_v} \cdot \vec{F})}_{Q_u} - \underbrace{\frac{\partial}{\partial v}(\vec{f_u} \cdot \vec{F})}_{P_v}\, du\, dv$$

Green $$= \oint_{C'} (\vec{f_u} \cdot \vec{F})\, du + (\vec{f_v} \cdot \vec{F})\, dv = \oint_{C'} \vec{F} \cdot \underbrace{(\vec{f_u}\, du + \vec{f_v}\, dv)}_{d\vec{s}}$$

$\boxed{\text{QED}}$

p.98 $(a\times b)\cdot(c\times d) = \begin{vmatrix} a\cdot c & a\cdot d \\ b\cdot c & b\cdot d \end{vmatrix}$ they call it Lagrange Id 12/31/2015

Use $a\cdot(b\times c) = b\cdot(c\times a)$ prob 3-26

triple product

$a\cdot(b\times c) = \det\begin{vmatrix} a \\ b \\ c \end{vmatrix}$ $\quad \det\begin{vmatrix} b \\ c \\ a \end{vmatrix} = -\det\begin{vmatrix} b \\ a \\ c \end{vmatrix} = (-1)^2 \det\begin{vmatrix} a \\ b \\ c \end{vmatrix}$

$$a\cdot(b\times c) = b\cdot(c\times a)$$

Subs

$$(a\times b)\cdot(c\times d) = c\cdot(d\times(a\times b))$$

bacab rule

Since $A\times(B\times C) = (A\cdot C)\vec{B} - (A\cdot B)\vec{C}$

$d \quad a \quad b \qquad (d\cdot b)\vec{a} \qquad (d\cdot a)\vec{b}$

$$= c\cdot\left((d\cdot b)\vec{a} - (d\cdot a)\vec{b}\right)$$

$$= (d\cdot b)(c\cdot a) - (d\cdot a)(c\cdot b)$$

$$\Rightarrow (a\times b)\cdot(c\times d) = \begin{vmatrix} (a\cdot c) & (a\cdot d) \\ (b\cdot c) & (b\cdot d) \end{vmatrix}$$ QED

Now we want to simplify $(\nabla\times F)\cdot(f_u\times f_v)$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big[$$

$$(\nabla\times F)\cdot(\vec{f}_u\times\vec{f}_v) = \left[(\vec{f}_u\cdot\hat{\imath})(\vec{f}_v\cdot\vec{F}_x) - (\vec{f}_u\cdot\vec{F}_x)(\vec{f}_v\cdot\hat{\imath})\right]$$
$$+ \left[(\vec{f}_u\cdot\hat{j})(\vec{f}_v\cdot\vec{F}_y) - (\vec{f}_u\cdot\vec{F}_y)(\vec{f}_v\cdot j)\right]$$
$$+ \left[(\vec{f}_u\cdot\hat{k})(\vec{f}_v\cdot\vec{F}_z) - (\vec{f}_u\cdot\vec{F}_z)(\vec{f}_v\cdot\hat{k})\right]$$

$\vec{f}_u\cdot\hat{\imath} = \frac{\partial}{\partial u} f^{(1)}$

$\vec{f}_u\cdot j = \frac{\partial}{\partial u} f^{(2)}$

$\vec{f}_u\cdot\hat{k} = \frac{\partial}{\partial u} f^{(3)}$

$$= \left[f_u^{(1)}(\vec{f}_v\cdot\vec{F}_x) - (\vec{f}_u\cdot\vec{F}_x) f_v^{(1)}\right] + \left[f_u^{(2)}(\vec{f}_v\cdot\vec{F}_y) - (\vec{f}_u\cdot\vec{F}_y) f_v^{(2)}\right] + \left[f_u^{(3)}(f_v\cdot F_z) - (f_u\cdot F_z) f_v^{(3)}\right] = \vec{f}_v\cdot\vec{F}_u - \vec{f}_u\cdot\vec{F}_v$$

$=$

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

$$\overset{a}{\vec{f}_u} \times \overset{b}{\vec{f}_r} \cdot (\overset{c}{\hat{\imath}} \times \overset{d}{\vec{F}_x}) = (f_u \cdot \hat{\imath})(f_r \cdot F_x) - (f_u \cdot F_x)(f_r \cdot \hat{\imath})$$ ok

12/31/2015

$$F(x)$$
$$= F(\vec{f}(u,v))$$

$$\frac{\partial}{\partial u}\left(\vec{f}_v \cdot \vec{F}\right) - \frac{\partial}{\partial v}\left(\vec{f}_u \cdot \vec{F}\right)$$

$$\frac{\partial}{\partial u}\left(f_v^1 F^1 + f_v^2 F^2 + f_v^3 F^3\right) - \frac{\partial}{\partial u}\left(f_u^1 F^1 + f_u^2 F^2 + f_u^3 F^3\right)$$

$$\vec{f}_v \cdot \vec{F}_u = f_v^1 F$$

$$\vec{F}_u = DF(\vec{f}_u)$$
$$= \begin{bmatrix} F_x^1 & F_y^1 & F_z^1 \\ F_x^2 & F_y^2 & F_z^2 \\ F_x^3 & F_y^3 & F_z^3 \end{bmatrix} \begin{bmatrix} f_u^1 \\ f_u^2 \\ f_u^3 \end{bmatrix}$$
$$= \vec{F}_x f_u^1 + \vec{F}_y f_u^2 + \vec{F}_z f_u^3$$

Then $\vec{f}_v \cdot \vec{F}_u =$

$$f_v^1 \vec{F}_x f_u^1 + f_v^2 \vec{F}_y f_u^2 + f_v^3 \vec{F}_z f_u^3$$

$$\vec{F}_v = DF(f_v)$$

$$\begin{bmatrix} \; \end{bmatrix}$$

$$= \vec{F}_x f_v^1 + \vec{F}_y f_v^2 + \vec{F}_z f_v^3$$