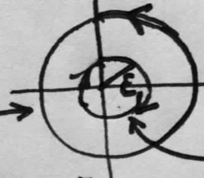
Counterexample cont'd

Let's show that this F works perfectly fine for Green's Thm if we cut out the singularity with another ∂



$D = \overline{B}(0,1) - \mathring{B}(0,\varepsilon)$

$\sigma_1(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

$\sigma_2(t) = \varepsilon \begin{bmatrix} \cos(2\pi - t) \\ \sin(2\pi - t) \end{bmatrix}$ then $F(\sigma_2(t)) = \frac{1}{\varepsilon} \begin{bmatrix} -\sin(2\pi - t) \\ \cos(2\pi - t) \end{bmatrix}$

$0 \le t \le 2\pi$

$\sigma_2'(t) = \varepsilon \begin{bmatrix} -\sin(2\pi - t)(-1) \\ \cos(2\pi - t)(-1) \end{bmatrix} = \varepsilon \begin{bmatrix} \sin(2\pi - t) \\ -\cos(2\pi - t) \end{bmatrix}$

$$\int_\sigma F \cdot ds = \underbrace{\int_{\sigma_1} F(\sigma_1)\cdot\sigma_1'\,dt}_{2\pi \text{ from prev page}} + \underbrace{\int_{\sigma_2} F(\sigma_2(t))\cdot\sigma_2'(t)\,dt}$$

$$\int_0^{2\pi} \frac{1}{\varepsilon}\begin{bmatrix} -\sin(2\pi - t) \\ \cos(2\pi - t) \end{bmatrix} \cdot \varepsilon \begin{bmatrix} \sin(2\pi - t) \\ -\cos(2\pi - t) \end{bmatrix} dt$$

$$= \int_0^{2\pi} -\left(\sin^2(2\pi - t) + \cos^2(2\pi - t)\right) dt = -2\pi$$

Thus $\int_\sigma F \cdot ds = 0$ and we already showed $\int_D (\nabla \times F)\cdot\hat{k} = 0$ so Green holds for this domain.

p. 409 **Thm** Let D be a region to which Green's Thm applies and let ∂D be oriented by RHR $\Rightarrow$

$\text{Area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$

$\text{Area}(D) = \oint_{\partial D} x\,dy$

$\text{Area}(D) = \int_{\partial D} -y\,dx$

Prob #20 p. 415

Pf. We know $\oint_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx\,dy$ ⊛

(1) Take $F = \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ and mult ⊛ by $\frac{1}{2}$: $\frac{1}{2}\oint -y\,dx + x\,dy = \frac{1}{2}\int_D (1 - -1)\,dx\,dy = \int_D 1\,dx\,dy$ ✓

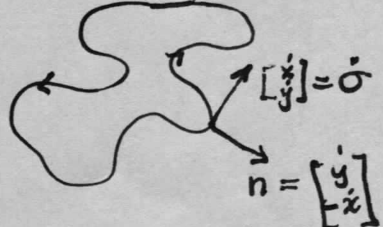
(2) Take $\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$ in ⊛: $\oint x\,dy = \int_D (1-0)\,dx\,dy$

(3) Take $\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$ in ⊛: $\oint -y\,dx = \int_D (0 - -1)\,dx\,dy$ □

▷ Given a domain D to which Green applies and a vf $F: D \to \mathbb{R}^2$ we can give a 2-Dim version of the Divergence Thm:

want: $\int_{\partial D} \vec{F}\cdot\hat{n}\,ds = \int_D (\nabla\cdot F)\,dA$

$\int_{\partial D} \vec{F}\cdot\hat{n}\,ds = \int (F^{(1)}n^{(1)} + F^{(2)}n^{(2)})\,ds$, $\quad (\nabla\cdot F) = D_1F^1 + D_2F^2$



can't say if this is Outward pointing

But Abraham, Marsden, Ratiu does p. 504

Define the Green components P, Q: $\begin{bmatrix} Q \\ -P \end{bmatrix} := \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$ from predetermined $\vec{F}$

$$\int_{\partial D} \vec{F}\cdot\hat{n}\,ds = \int_{\partial D} \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \underbrace{\frac{1}{\|\dot{\sigma}\|}\begin{bmatrix} \dot{y} \\ -\dot{x} \end{bmatrix}}_{\hat{n}} \underbrace{\|\dot{\sigma}\|\,dt}_{ds} = \int_{\partial D} (Q\dot{y} + P\dot{x})\,dt = \int_{\partial D} P\,dx + Q\,dy \overset{\text{Green}}{=}$$

$$\overset{\text{Green}}{=} \int_D (Q_x - P_y)\,dx\,dy = \int_D (D_1F^1 + D_2F^2)\,dx\,dy = \int_D (\nabla\cdot F)\,dA \quad \square$$

$Q_x - P_y = D_xF^1 - D_y(-F^2)$

$D \subseteq \mathbb{R}^2$ open region

$u : \overline{D} \to \mathbb{R}$ cont

u is C^2 on $\overset{\circ}{D} = D$

$B(p,R) \subseteq D$

$$\sigma(t) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix}$$

$$I(r) := \frac{1}{r} \int_{\partial B(p,r)} u \, ds = \frac{1}{\cancel{r}} \int_0^{2\pi} u(\sigma(t)) \underbrace{\cancel{\|\sigma'(t)\|}}_{r} \, dt$$

(21) show $\lim_{r \to 0} I(r) = 2\pi\, u(p)$

u is a cont fcn on the cpt set $\overline{B}(p,R)$, thus it is unif cont.

Given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $\|p - q\| < \delta \Rightarrow |u(p) - u(q)| < \varepsilon$

Now I claim: $r < \delta \Rightarrow \left| \int_0^{2\pi} u(\sigma(t))\,dt - 2\pi u(p) \right| < 2\pi\varepsilon$ and we are done.

($2\pi u(p) = u(p)\int_0^{2\pi} dt$)

To see this:

$$\left| \int_0^{2\pi} u(\sigma(t)) - u(p)\, dt \right| \le \int_0^{2\pi} \left| u(\sigma(t)) - u(p) \right| dt < \int_0^{2\pi} \varepsilon \, dt$$

$\square$

(22) It follows directly from p. 412-413 (Div Thm in plane) that:

$$\int_{\partial B_r} \nabla u \cdot n \, ds = \int_{B_r} \nabla^2 u \, dA$$

$\square$

(23) show $I'(r) = \frac{1}{r} \int_{B_r} \nabla^2 u \, dA$

$$\frac{d}{dr} I(r) = \frac{d}{dr} \int_0^{2\pi} u(\sigma(t))\,dt$$

Interchange is justified since fcns are smooth & bdd
If you like, you can see Apostol MA p.283

$$= \int_0^{2\pi} D_1 u \frac{\partial \sigma_1}{\partial r} + D_2 u \frac{\partial \sigma_2}{\partial r} \, dt \qquad \sigma'(r;t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \hat{n} \text{ for disc}$$

$$= \frac{1}{r} \int_0^{2\pi} \nabla u \cdot \hat{n} \, \overbrace{r\,dt}^{ds}$$

Prob 22 →

$$= \frac{1}{r} \int_{B_r} \nabla^2 u \, dA$$

$\square$

(24) Assume now $\nabla^2 u = 0$ in D [u is harmonic]

Show $u(p) = \frac{1}{2\pi r} \int_{\partial B_r} u \, ds$ Average over a circle.

From the Mean Value Thm $I(r) = I(0) + I'(\xi) r$ for some $\xi \in (0,r)$.

From prob 21, we define $I(0) := 2\pi u(p)$

From prob 23, $I'(r) = \frac{1}{r} \int_{B_r} \nabla^2 u \, dA$

Thus if $\nabla^2 u = 0$ we have $I'(\xi) = 0$ irregardless of ξ

Thus $I(r) = 2\pi u(p)$

$$u(p) = \frac{1}{2\pi r} \int_0^{2\pi} u(\sigma(t)) \, dt \qquad \square$$

(25) $\nabla^2 u = 0$ Show $u(p) = \frac{1}{\pi R^2} \int_{B(p,R)} u \, dA$ Average over a disc

In (24) we showed $u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(\sigma(\theta)) \, d\theta$ renaming "t" as "θ".

← this depends on r, but the dependence vanishes when we integrate.

integrate both sides:

$$\int_0^R r \, u(p) \, dr = \frac{1}{2\pi} \int_0^R r \int_0^{2\pi} u(\sigma(\theta)) \, d\theta \, dr$$

$$\frac{1}{2} R^2 u(p) = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(\sigma(\theta)) \underbrace{r \, d\theta \, dr}_{dA}$$

$$\Rightarrow u(p) = \frac{1}{\pi R^2} \int_{B_R} u \, dA$$

$\square$

(26) $\nabla^2 u = 0$ on D

u has a local extrema (max or min) at $p^* \in D$ [This implies $\nabla(u(p^*)) = 0$]

ⓐ Let's assume it is a max (very similiar arg for local min)

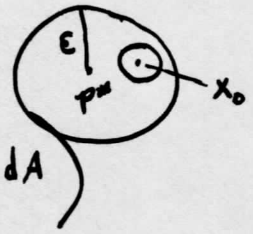
Show u must be const on some disc $B(p^*, \lambda) \subseteq D$.

p^* local max $\Rightarrow \exists\ B(p^*, \varepsilon) \ni u(x) \leq u(p^*)\ \forall x \in B(p^*, \varepsilon)$.

Suppose $\exists\ x_0 \in B(p^*, \varepsilon)$ where $u(x_0) < u(p^*)$

By continuity of u, $\exists$ disc $B(x_0, \eta)$ where $u < u(p^*)$

This disc $\Delta := B(x_0, \eta)$ has pos measure.



$$u(p^*) \overset{\text{Prob 25}}{=} \frac{1}{\pi\varepsilon^2} \int_{B_\varepsilon} u\, dA = \frac{1}{\pi\varepsilon^2}\left(\underbrace{\int_\Delta u\, dA}_{< u(p^*) \int_\Delta dA} + \underbrace{\int_{B_\varepsilon - \Delta} u\, dA}_{\leq u(p^*) \int_{B_\varepsilon - \Delta} dA} \right)$$

$$< \frac{1}{\pi\varepsilon^2} u(p^*)\, \pi\varepsilon^2$$

$\Rightarrow\Leftarrow$

Therefore no ~~local mins~~ pts where $u(x) \neq u(p^*)$ in $B(p^*, \varepsilon)$.

The crux of the arg is that since the value of u is the avg over a disc (for an interior pt) then the value can't be an extremum.

ⓑ If D path conn (actually I just need conn.) $\Rightarrow$ u const on D

and u has at least one local extrema.

Suppose u not const,

classic Disconn Arg.

Let $A := \{ \text{all } x \in D \mid u(x) \text{ local extrema} \}$ open in D because every $x \in A$ has $\mathring{B}(x, \varepsilon)$ where u is const. $\Rightarrow \mathring{B}(x, \varepsilon) \subseteq A$.

$B := \{ \text{all } x \in D \mid \nabla u \neq 0 \}$ open in D because ∇u is cont.

let $F(x) := \nabla u(x)$ $\quad B := \underbrace{D - \underbrace{F^{-1}(\{0\})}_{\text{clsd set}}}_{\text{Open in } D}$.

$D = A \cup B$

We have disconnected D

$\Rightarrow\!\!\!\!\times\!\!\Leftarrow$

□

(27) $\nabla^2 u \geq 0$ sub harmonic

(a) Show: If u has a max in $D \Rightarrow u = \text{const}$ in D

Just like before we can write $I(r) = I(0) + I'(\xi) r$

$$I(r) \to \frac{1}{r}\int_{\partial B_r} u\,ds, \qquad I(0) \to 2\pi u(p), \qquad I'(\xi) r \to \underbrace{\int_{B_r} \nabla^2 u\, dA}_{\geq 0} \leftarrow \text{call the value } \lambda$$

$$u(p) = \frac{1}{2\pi r}\int_{\partial B_r} u\,ds - \lambda$$

$$\Rightarrow u(p) \leq \frac{1}{2\pi r}\int_{\partial B_r} u\,ds$$

By prob #25 arg $\quad u(p) \leq \frac{1}{\pi r^2}\int_{B_r} u\,dA$

Then by prob #26a
if $u(p)$ is the max on $B(p,\varepsilon)$

$$u(p) \leq \frac{1}{\pi\varepsilon^2}\int_{B_\varepsilon} u\,dA = \frac{1}{\pi\varepsilon^2}\Bigg(\underbrace{\int_{\Delta} u\,dA}_{< u(p)\int_\Delta dA} + \underbrace{\int_{B_\varepsilon - \Delta} u\,dA}_{\leq \int_{B_\varepsilon-\Delta} dA}\Bigg) < u(p)$$

$\Rightarrow\Leftarrow$ □

(28) If $u : \overline{B}(0,1) \to \mathbb{R}$ cont ($\mathbb{R}^2$)
u sub harmonic on $\mathring{B}(0,1) =: \Delta$

Show $\exists$ harmonic fcn v on $\Delta \ni u \leq v$ and $u|_{\partial\Delta} = v|_{\partial\Delta}$

This is trivial from the assumption they give us.
This is trivial from the assumption and then we assume the existence of ~~#~~ v

Take $f = u|_{\partial\Delta}$ and then

$$u \leq \frac{1}{2\pi}\int_{\partial\Delta} u\,ds = \frac{1}{2\pi}\int_{\partial\Delta} u|_{\partial\Delta}\,ds = v$$

□