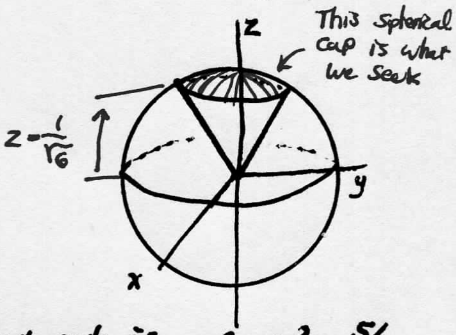
SOLID ANGLE

7/12/2025

8a

REA VA p.654

(14-12) The <u>solid angle</u> is the area that is obscured on the unit sphere by some object as seen from the center (origin)



Here we consider a solid cone $x^2+y^2 \le 5z^2$ $(z>0)$

$$\partial\text{ cone}\quad \left.\begin{matrix} x^2+y^2+z^2=1 \\ x^2+y^2 = 5z^2 \end{matrix}\right\} \Rightarrow \begin{aligned} 5z^2+z^2 &= 1 \\ 6z^2 &= 1 \\ z &= \frac{+1}{\sqrt6} \end{aligned}$$

and at this height, the circle cut out is $x^2+y^2 = 5/6$

$z = f(x,y) = \sqrt{1-(x^2+y^2)} =: q^{1/2}$ The base in x,y plane under this spherical cap is $x^2+y^2 \le 5/6$

$\varphi(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ area $A = \int_D \sqrt{f_x^2+f_y^2+1}\, dx\,dy$ $\quad f_x = \frac{-x}{q^{1/2}} \quad f_y = \frac{-y}{q^{1/2}}$

cf Sheet 8

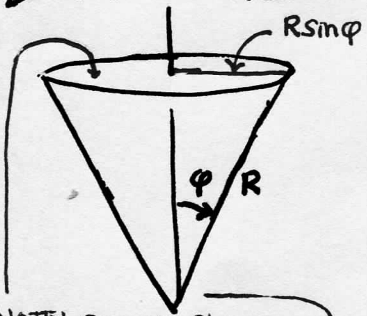
$$= \int_D \sqrt{\frac{x^2}{q}+\frac{y^2}{q}+\frac{q}{q}}\, dx\,dy = \int_D \frac{1}{\sqrt{1-(x^2+y^2)}}\, dx\,dy$$

Convert to polar

$$\int_{r=0}^{\sqrt{5/6}} \int_{\theta=0}^{2\pi} \frac{1}{\sqrt{1-r^2}}\, r\,d\theta\,dr = 2\pi \int_{r=0}^{\sqrt{5/6}} \frac{1}{\sqrt{1-r^2}}\, r\,dr \qquad \begin{aligned} u &= 1-r^2 \\ du &= -2r\,dr \end{aligned} \qquad u^{-1/2} \rightsquigarrow 2u^{1/2}$$

$$= \frac{-2\pi}{\cancel{2}} \left[\cancel{2}u^{1/2}\right]_{u=1}^{1/6} = -2\pi\left[\frac{1}{\sqrt6}-1\right] = 2\pi\left[1-\frac{1}{\sqrt6}\right] \quad ✓$$

▷ Now consider the cone specified by angle φ in a sphere of radius R



$z = f(x,y) = \sqrt{R^2-(x^2+y^2)} =: q^{1/2}$ $\qquad D = x^2+y^2 \le R^2\sin^2\varphi$

$$A = \int_D \sqrt{\frac{x^2}{q}+\frac{y^2}{q}+\frac{q}{q}}\, dx\,dy = \int_D \sqrt{\frac{x^2+y^2+R^2-x^2-y^2}{R^2-x^2-y^2}}\, dx\,dy$$

$$= \int_D \frac{R}{\sqrt{R^2-r^2}}\, r\,d\theta\,dr = 2\pi R \int_{r=0}^{R\sin\varphi} \frac{1}{\sqrt{R^2-r^2}}\, r\,dr \qquad \begin{aligned} u &= R^2-r^2 \\ du &= -2r\,dr \end{aligned}$$

NOTE: area of flat cap (disc) $= \pi r^2 = \pi R^2\sin^2\varphi$
If we normalize by $\frac{1}{R^2}$
disc area $= \pi\sin^2\varphi$
$= \pi(1-c^2)$
$= \pi(1-c)(1+c)$

$$= -2\pi R\left[u^{1/2}\right]_{R^2}^{R^2-R^2\sin^2\varphi} = +2\pi R\Big[\underbrace{\sqrt{R^2}-\sqrt{R^2-R^2\sin^2\varphi}}_{R\,-\,R\underbrace{\sqrt{1-\sin^2\varphi}}_{\cos\varphi}}\Big]$$

$$= 2\pi R^2\left[1-\cos\varphi\right]$$

Then to get solid angle, we divide by R^2

$$\boxed{\text{Solid angle } \Omega = 2\pi(1-\cos\varphi)}$$

Remark $\Omega = \frac{(\text{disc area})\cdot 2}{1+\cos\varphi}$

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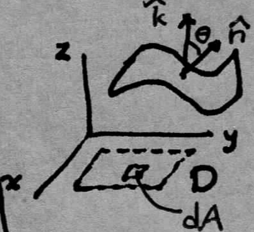
p.383

From sheet 10 we know $\displaystyle\int_S f\,dS := \int_D f(\varphi(u,v))\,\overbrace{\|\varphi_u\times\varphi_v\|}^{\|\vec n\|}\,du\,dv$

where
- $S=\varphi(D)$
- Take $f=1$ and get Surf area



Now let S be the graph of $z=g(x,y)$

CLAIM In this case we can write $dS=\frac{1}{\cos\theta}\,dA$

meaning $\displaystyle\int_S f\,dS = \int_D (f\circ\varphi)\frac{1}{\cos\theta}\,dx\,dy$

Pf. For a graph $\varphi(x,y)=\begin{bmatrix}x\\y\\g(x,y)\end{bmatrix}$ $\quad \vec n=\begin{bmatrix}1\\0\\g_x\end{bmatrix}\times\begin{bmatrix}0\\1\\g_y\end{bmatrix}=\begin{bmatrix}-g_x\\-g_y\\1\end{bmatrix}$ Key

For any unit $\hat u$ $\quad \vec n\cdot\hat u=\|n\|\cdot 1\cdot\cos\theta \Rightarrow \dfrac{\vec n\cdot u}{\cos\theta}=\|\vec n\|=\|\varphi_x\times\varphi_y\|$

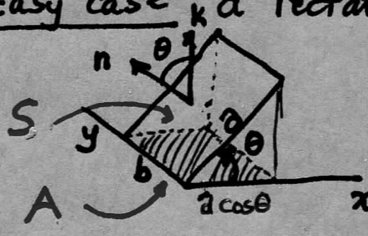
Here if we take $\hat u=\hat k=\hat e_3$ $\quad \dfrac{n\cdot k}{\cos\theta}=\dfrac{n^{(3)}}{\cos\theta}=\dfrac{1}{\cos\theta}\stackrel{*}{=}\|\varphi_x\times\varphi_y\|$

$$\Rightarrow \int_D f(\varphi(x,y))\|\varphi_x\times\varphi_y\|\,dx\,dy=\int_D f(\varphi(x,y))\frac{1}{\cos\theta}\,dx\,dy \qquad \square$$

[More generally this arg works if $\exists\,\hat u \ni \vec n\cdot\hat u=$ const c $\Rightarrow \|\vec n\|=\dfrac{c}{\cos\theta}$]

For intuition of $dS=\frac{1}{\cos\theta}dA$ or $\Delta S=\frac{1}{\cos\theta}\Delta A$ consider a rectangle:

<u>Easy case</u> a rectangle rotated up around x axis or y axis



$S=ab$

$A=(\cos\theta)ab$

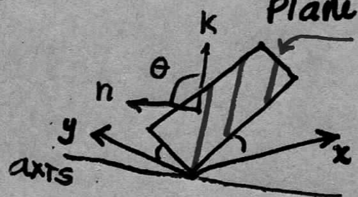
$\Rightarrow \frac{1}{\cos}A=ab=S$

NOTE: length b along axis of rotation is unchanged

length a is $\perp$ to axis and becomes $a(\cos\theta)$

$\hat n$ tilts away from $\hat k$ by angle θ

<u>More generally</u> rect could be tipped up on a corner. This is equivalent to lifting the plane of the rect by rotating it about another axis in x-y plane thru corner



The tilted rect is made of inf strips $\perp$ to axis

we know from EasyCase that a $\perp$ line of length ℓ projects to $\ell\cos\theta$

while its width dw is unchanged.

$S=\int \ell(w)\,dw$

$A=\int \ell(w)\cos\theta\,dw=\cos\theta\int\ell(w)\,dw=\cos\theta\,S$ or $\frac{1}{\cos\theta}A=S$

$\square$

This also shows it for any weird planar shape that can be sliced into $\perp$ lines.