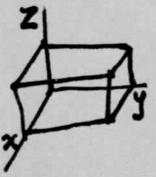
(ex 4) Find vol of largest box subj to constraint surf area of sides must be 10.



$V(x,y,z) = xyz \qquad g(x,y,z) = \cancel{2}(xy + xz + yz) = \cancel{10}\,5$

$\nabla V = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \qquad \nabla g = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix} \Rightarrow \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \overset{!}{=} \lambda \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix}$

we know $x \neq 0,\ y \neq 0,\ z \neq 0$ and are pos

Take first 2 eqs: $\dfrac{yz}{y+z} = \lambda = \dfrac{xz}{x+z} \Rightarrow \dfrac{y}{y+z} = \dfrac{x}{x+z}$

$y(x+z) = x(y+z)$

$\cancel{yx} + yz = \cancel{xy} + xz \Rightarrow y = x$

similiarly we get $x = z$

$g(x,x,x) = 5 \Rightarrow 3x^2 = 5 \Rightarrow x = +\sqrt{\frac{5}{3}}$

Thus $V\left(\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right) = \sqrt{\frac{5^3}{3^3}}$ is the max. □

Thm Lagrange Multipliers for multiple constraints

$f: \mathbb{R}^n \to \mathbb{R}$

$g_i: \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, k$ $\quad k < n$

then $g: \mathbb{R}^n \to \mathbb{R}^k$ and $g^{-1}(0) =: S$

again we require $\vec{\nabla} g_i(a) \neq 0$ when $f|_S$ has extrema at a

$\Rightarrow \vec{\nabla} f(a) = \sum_{i=1}^{k} \lambda_i \vec{\nabla} g_i(a)$

This means $Dg_a: \mathbb{R}^n \to \mathbb{R}^k$ is Onto $\forall a$

Pf Sheets for Avez DC p124
Ch 10 Thm 10.3 writeup. Relies on G&P.

Cheney Apl Math I
Ch 4.5 p.23-24
Valid in Banach sp.

□

Before going on, lets give a couple problem solutions:

(13) Let A be 3×3 symm matrix and consider quadratic form $f: \mathbb{R}^3 \to \mathbb{R}$, $x \longmapsto \frac{1}{2} x^T A x$

(a) what is $\vec{\nabla} f$? Earlier in this section (ch 4.2) we computed $Df_x = x^T A$
so $\nabla f = (Df_x)^T = A^T x = Ax$ since A symm

(b) Now restrict f to $S^1 = g^{-1}(1)$ where $g(x) = x^T x$. Derive eigenvalue eq for A.
S^1 is cpt so f attains max and min. Say x^* is the min.
Lagrange multipliers: $\nabla f(x^*) \overset{!}{=} \lambda \nabla g(x^*) \Rightarrow Ax^* = \lambda x^*$ because $g(x) = x^T I x$, $Dg_x = x^T I$

(14) Now suppose A not symm

~~again $\nabla f(x) = A^T x$~~ NO! $Df_x = (A + A^T)x \neq 2Ax$ because A not symm.
Then we would have $(A + A^T)x = \lambda x$

~~The EW eq would be $A^T x^* = \lambda x^*$ from Lagrange multipliers~~

~~we know, for any square matrix B $\det(B^T) = \det(B)$ so let $B = A - \lambda I$~~

~~and we see A and A^T have same EWs,~~

~~so $A^T x^* = \lambda x^*$ has a $\lambda \in \mathbb{R}$ as a soln $\rightarrow$ this λ is also EW of A~~

~~$\Rightarrow Ay = \lambda y$ $(A - \lambda I)y = 0$ has a soln so $\vec{y} \in \mathbb{R}^3$~~

~~f would have a min and a max on S^1 (possibly same)~~

M&T ch 4.4 6/28/2024

Imp Fcn Thm (and Inv Fcn Thm)

The complete discussion is in my sheets for Avez DC ch 3 with proper pfs of Inv Fcn Thm in Banach spaces and using this for Imp FT. More examples written up there too.

M&T only give a special case Imp FT and attempt to prove it with no tools, like contraction mapping thm, only elementary techniques like MVT and IVT. The pf is very difficult to make sense of, so I am not writing it up.

Thm 8 Imp F.T. special case

$f : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ C^1 partials
$\langle x, y, z \rangle$, with $x = \langle x, y\rangle$

$f(x_0, z_0) = 0$

$D_2 f = \frac{\partial f}{\partial z}(x_0, z_0) \neq 0$

$\Longrightarrow$

$\exists$ open balls $\mathcal{U}$, $x_0 \in \mathcal{U}$; $\mathcal{V}$, $z_0 \in \mathcal{V}$

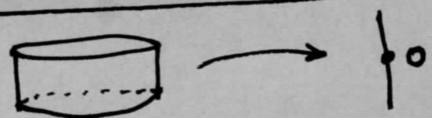
$\exists !\ g : \mathcal{U} \to \mathcal{V}$ which satisfies

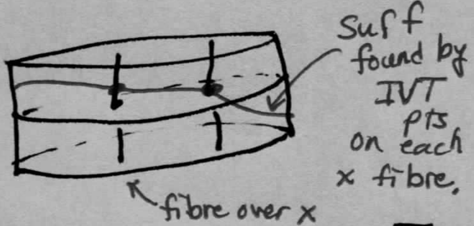
$f(x, g(x)) = 0 \quad \forall x \in \mathcal{U}$

and $Dg_x = \dfrac{-1}{\frac{\partial f}{\partial z}(x, g(x))} D_x f_{\langle x, z\rangle}$

that is $[g_x \ g_y] = \dfrac{-1}{\frac{\partial f}{\partial z}} [f_x \ f_y]$

pf. some ideas $\Delta_a \times I_a$

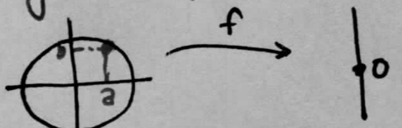




□

Geometrically, Imp FT is a local Pre-Image Thm (G&P) combined with representing a mfd locally as a graph.

(ex) $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $\langle x, y\rangle \longmapsto x^2 + y^2 - 1$ $\quad f^{-1}(0)$

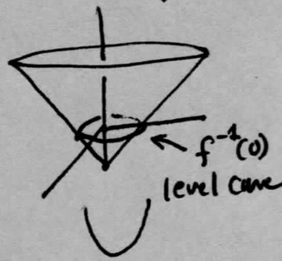


See Avez DC Ch 3 writeup sheet 11 for more details

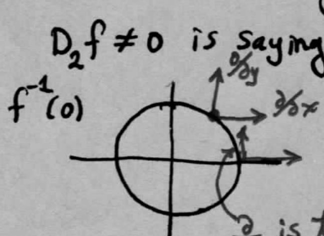
For a $\langle a, b\rangle$ on the circle as shown, we have $f(a,b) = 0$ and we want a fcn $y = y(x) \ni f(x, y(x)) = 0$

$D_2 f_{ab} = \frac{\partial}{\partial y}(x^2 + y^2 - 1)\Big|_{ab} = 2y\Big|_b = 2b \neq 0$

To illustrate the geometric meaning by $z = f(x,y) = x^2 + y^2 - 1$ The graph of f



$D_2 f \neq 0$ is saying $\frac{\partial z}{\partial y} \neq 0$



$\frac{\partial}{\partial y}$ is tangent to level curve at $\langle 1, 0\rangle \Rightarrow \frac{\partial}{\partial y} f\Big|_{(1,0)} = 0$

$\frac{dy}{dx} = \dfrac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \dfrac{-\partial z/\partial x}{0}$ ← at $\langle 1, 0\rangle$

so y' would be undefined there
y can't be smooth at that pt.

□

M&T ch 4.4 6/28/2024

8

p. 289 ex 3

$$xu + yvu^2 = 2$$
$$xu^3 + y^2v^4 = 2$$

can we solve for $u = u(x,y)$, $v = v(x,y)$ near $\langle 1,1,1,1\rangle$?

and can we find $\frac{\partial u}{\partial x}(1,1) = ?$

Domain split

Define $f: \mathbb{R}^2 \times \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$

$$\langle x,y,u,v\rangle \longmapsto \begin{bmatrix} f^{1} \\ f^{2} \end{bmatrix} = \begin{bmatrix} xu + yvu^2 - 2 \\ xu^3 + y^2v^4 - 2 \end{bmatrix}$$

$$D_2 f = \begin{bmatrix} f^{1}_u & f^{1}_v \\ f^{2}_u & f^{2}_v \end{bmatrix} = \begin{bmatrix} (x + 2yvu) & yu^2 \\ 3xu^2 & 4v^3y^2 \end{bmatrix}$$

$$D_2 f_{\langle 1,1,1,1\rangle} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}$$

$\det(D_2 f) = 9 \neq 0$ so we can apply Imp FT

$$\Rightarrow \exists\, g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\langle x,y\rangle \longmapsto \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

$$Dg_x = -(D_2 f)^{-1} D_1 f$$

$f(x, g(x)) = 0$

In Avez write up, I did prb 10 and computed these things. Here we are going to use implicit differentiation to solve for $u_x(1,1)$.

$f(x,g(x)) = \vec{0}$

$$\frac{\partial f^1}{\partial x} = u + xu_x + yv_x u^2 + 2yvuu_x = 0$$
$$\frac{\partial f^2}{\partial x} = u^3 + 3xu^2u_x + 4y^2v^3v_x = 0$$

plug in $\langle x,y,u,v\rangle = \langle 1,1,1,1\rangle$

$$1 + u_x + v_x + 2u_x = 0 \Rightarrow 3u_x + v_x = -1$$
$$1 + 3u_x + 4v_x = 0 \Rightarrow 3u_x + 4v_x = -1$$

Solve this system of linear eqs

$$12u_x + 4v_x = -4$$
$$-(3u_x + 4v_x = -1)$$
$$\overline{9u_x + 0 = -3}$$

$$\Rightarrow u_x(1,1) = \frac{-1}{3} \quad \square$$

Example from Rudin POMA p.224

Linear Imp FT

$A: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\langle x, y\rangle \longmapsto A\begin{bmatrix} x \\ y \end{bmatrix}$

$A(0,0) = 0$

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [0] \Rightarrow A_1x + A_2y = 0 \Rightarrow A_2y = -A_1x$$

if A_2^{-1} exists, we can solve

$$y = \underbrace{-A_2^{-1}A_1x}_{y(x)}$$

Then $A(x, y(x)) = 0 \quad \forall x \in \mathbb{R}^m$ (global Imp FT)

$A^{-1}(0) = \ker A$ is a subsp of $\mathbb{R}^{m+n}$ m dim i.e submfd.

$Dy_x(h) = -A_2^{-1}A_1h$ since y linear fcn of x

$\square$

Let's recall Thm 6 (Lagrange multipliers) from sheet 4 and give further discussion:

Thm 6
$$\left.\begin{array}{l} f: \mathcal{U}^{\mathbb{R}^n} \to \mathbb{R} \text{ smooth} \\ g: \mathcal{U} \to \mathbb{R} \quad " \\ S = g^{-1}(0),\ a \in S \\ \nabla g(a) \neq 0 \\ f|_S \text{ has extrema at } a \end{array}\right\} \Rightarrow \nabla f(a) = \lambda \nabla g(a)$$

0 is a Regular value: Dg_x maps onto $\mathbb{R}$ for $x \in S$ ONLY LOCALLY NEAR a $\longleftrightarrow \nabla g(a) \neq 0$

$x_0 = a$

Here we want to show $\boxed{T_xS = (\nabla g(x))^\perp \text{ for } x \in S,\ x \text{ near } a}$ Maybe only $T_{x_0}S = (\nabla g(x_0))^\perp$

M&T have only shown what tangent plane T_xS is for S being a graph until now

Let $S = g^{-1}(c)$ [we will take $n=3$ for convenience]

S is level surf of g thru pt $\langle x_0, y_0, z_0\rangle = x_0$

$\nabla g(x_0) \neq 0$
So at least one partial $\neq 0$
assume it is $\frac{\partial g}{\partial z}(x_0) \neq 0$

Define $F := g(x,y,z) - c$ $\quad F: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$

$\boxed{\text{Imp FT}} \Rightarrow \exists\ k(x,y) \ni z = k(x,y)$ in nbhd of x_0
i.e. $S = \text{graph}(k)$ in nbhd $\mathcal{U}$ of $\langle x_0, y_0\rangle$ $\quad F(x,y,k(x,y)) = 0$ in $\mathcal{U}$

For a graph, we know tan plane T_xS is given by

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - \underbrace{k(x_0,y_0)}_{z_0} \end{bmatrix} \cdot \begin{bmatrix} -k_x \\ -k_y \\ 1 \end{bmatrix} = 0 \Rightarrow z = z_0 + k_x(x - x_0) + k_y(y - y_0)$$

From $\boxed{\text{Imp FT}}$ $\quad k_x(x_0) = \dfrac{-g_x(x_0)}{g_z(x_0)} \qquad k_y(x_0) = \dfrac{-g_y(x_0)}{g_z(x_0)}$

$$\Rightarrow z = z_0 + \frac{-g_x}{g_z}(x - x_0) + \frac{-g_y}{g_z}(y - y_0)$$

$$\Rightarrow \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0 \quad \text{i.e. } \vec{\nabla} g \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

This defines $T_{x_0}S$
i.e. $T_{x_0}S = (\nabla g(x_0))^\perp$

▷ Now show every tan vector to S at x_0 is the tan vector of a curve c in S.
[This is supposed to complete the pf of Thm 6 – although maybe not nec for me]

From prev work, we need only show this in graph(k)

Say $\vec{v} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ is tangent to $S = Gr(k)$

then define $c: \mathbb{R} \to \overset{\mathbb{R}^3}{S}$

$$t \mapsto \begin{bmatrix} x_0 + t(x - x_0) \\ y_0 + t(y - y_0) \\ k(x_0 + t(x - x_0),\ y_0 + t(y - y_0)) \end{bmatrix}$$

$$c'(t) = \begin{bmatrix} x - x_0 \\ y - y_0 \\ k_x(x - x_0) + k_y(y - y_0) \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = \vec{v}$$

□

From my Avez DC writeup ch 3 sheet (6) see pf given there.
Based on Rudin POMA ch 9 p. 221-223

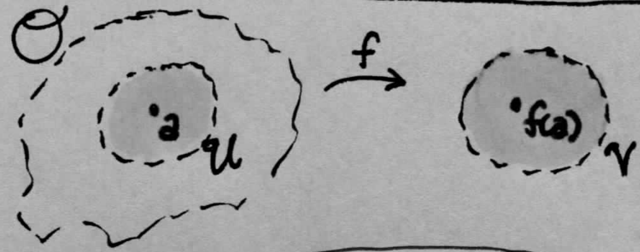
Inv Fcn Thm

Df_a dominates the behavior of f near a

$$\left.\begin{array}{l} f : \underset{}{\overset{\mathbb{R}^n}{\mathcal{O}}} \longrightarrow \mathbb{R}^n \quad C^1 \text{smooth} \\ [Df_a]^{-1} \text{ exists for one } a \in \mathcal{O} \end{array}\right\} \Rightarrow \begin{array}{l} \exists \text{ open sets } \mathcal{U}, \mathcal{V} \ni \\ f : \mathcal{U} \to \mathcal{V} \text{ is a diffeo} \end{array}$$

In more detail:

- f^{-1} exists in $\mathcal{V}$
- $D(f^{-1})_{f(x)}$ exists in $\mathcal{V}$
- $D(f^{-1})_{f(x)} = [Df_x]^{-1}$
- f C^N smooth $\Rightarrow f^{-1}$ C^N smooth



Problem (11) Can we use Inv FT to express the roots of a poly as fcns of the coeffs? Answer p. 555

Let's just focus on the $n=3$ case. $p(x) = x^3 + a_2x^2 + a_1x + a_0$

roots form $(x-r_1)(x-r_2)(x-r_3)$

Regard $a_i = a_i(r_1, r_2, r_3)$ Use Inv FT $r_i = r_i(a_0, a_1, a_2)$

$$x^3 + a_2x^2 + a_1x + a_0 \stackrel{!}{=} (x-\overset{r}{r_1})(x-\overset{s}{r_2})(x-\overset{t}{r_3}) = x^3 - (r+s+t)x^2 + [rs+rt+st]x - rst$$

$a_2(r,s,t) = -(r+s+t) \longleftarrow e_1(r,s,t)$

$a_1(r,s,t) = rs + rt + st \longleftarrow e_2(r,s,t)$

$a_0(r,s,t) = -rst \longleftarrow e_3(r,s,t)$

these are the "elementary symm polys" except perhaps for the signs.

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\underset{\text{roots}}{\langle r,s,t\rangle} \longmapsto \underset{\text{coeffs}}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}$$

$$Df_{rst} = \begin{bmatrix} \frac{\partial a^{(0)}}{\partial r} & \frac{\partial a_0}{\partial s} & \frac{\partial a^{(0)}}{\partial t} \\ a^{(1)}_r & a^{(1)}_s & a^{(1)}_t \\ a^{(2)}_r & a^{(2)}_s & a^{(2)}_t \end{bmatrix} = \begin{bmatrix} -st & -rt & -rs \\ (s+t) & (r+t) & (r+s) \\ -1 & -1 & -1 \end{bmatrix}$$

$$\begin{aligned} \det[Df_{rst}] &= -st[-(r+t)+(r+s)] + rt[-(s+t)+(s+r)] - rs[-(s+t)+(t+r)] \\ &= -st(s-t) + rt(r-t) - rs(r-s) \\ &= -s^2t + st^2 + r^2t - rt^2 - r^2s + rs^2 \quad -srt + srt \\ &= (t-s)[r^2 - rt - sr + st] \\ &= (t-s)(r-s)(r-t) \quad \text{this is nonzero if roots are distinct.} \end{aligned}$$

So for distinct roots, say $\langle r_0, s_0, t_0\rangle =: p_0$ $\det Df_{p_0} \neq 0$

apply Inv FT

$\exists$ nbhd $\mathcal{U}$ of p_0 and $\mathcal{V}$ of $f(p_0) \ni \exists f^{-1} : \mathcal{V} \longrightarrow \mathcal{U}$

$$\underset{\text{coeffs}}{\langle a_0, a_1, a_2\rangle} \longmapsto \underset{\text{roots}}{\langle r,s,t\rangle}$$

$\square$