5/30/2016

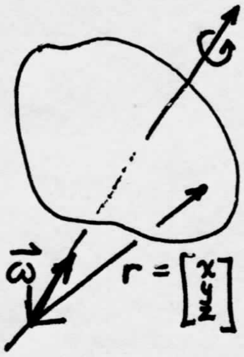
Ch 3.4 Div and Curl of a Vector Field

Let $F: \mathcal{U} \to \mathbb{R}^3$ be a C^1 smooth vf. (with $\mathcal{U} \subset \mathbb{R}^3$)

Define $\text{Curl}(F) := \nabla \times F := \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ D_x & D_y & D_z \\ F^{①} & F^{②} & F^{③} \end{bmatrix} = \begin{bmatrix} F^{③}_y - F^{②}_z \\ F^{①}_z - F^{③}_x \\ F^{②}_x - F^{①}_y \end{bmatrix}$

$\boxed{\text{p.179 } \underline{\text{Thm}} \text{ Let } f \in C^2 \implies \vec{\nabla} \times (\vec{\nabla} f) = \vec{0}}$ $\boxed{\text{Example of } d^2 = 0}$

__pf.__ $\nabla \times \nabla f = \det \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ f_x & f_y & f_z \end{vmatrix} = \begin{bmatrix} f_{yz} - f_{zy} \\ -(f_{xz} - f_{zx}) \\ f_{xy} - f_{yx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by equality of mixed partials (see ch 2.6) $\square$



Consider a rigid body $\mathcal{B}$ rotating about an axis defined by the angular velocity vector $\vec{\omega}$

Then the velocity vf for pts on $\mathcal{B}$ is given by $\vec{V} = \vec{\omega} \times \vec{r}$ (see my work in Beer & Johnston VMFE)

For simplicity, they assume $\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$ (along z axis)

What is $\text{Curl}(\vec{V})$?

$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \begin{bmatrix} -\omega y \\ -(0 - \omega x) \\ 0 \end{bmatrix}$

$\text{Curl}(V) = \nabla \times V = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ -\omega y & \omega x & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega + \omega \end{bmatrix} = 2\vec{\omega}$

Intuitively, $\text{Curl}(F)(x)$ measures the rotation of a microscopic paddlewheel in the flow of $\vec{F}$. $\text{Curl}(F) = 0$ means F is <u>irrotational</u>. This is justified by Stokes Thm in Ch 7

<u>Def</u> <u>Divergence</u> $\text{div}(F) := \nabla \cdot F := \sum D_i F^{ⓘ}$

$\boxed{\underline{\text{Thm}} \ \text{div}(\text{curl}(F)) \equiv 0 \quad \text{i.e. } \nabla \cdot \nabla \times F = 0}$ Another example of $d^2 = 0$

__pf__ $\nabla \cdot (\nabla \times F) = \nabla \cdot \begin{bmatrix} F^{③}_y - F^{②}_z \\ F^{①}_z - F^{③}_x \\ F^{②}_x - F^{①}_y \end{bmatrix} = F^3_{xy} - F^2_{xz} + F^1_{yz} - F^3_{yx} + F^2_{zx} - F^1_{zy} = 0$ $\square$

equality of mixed partials

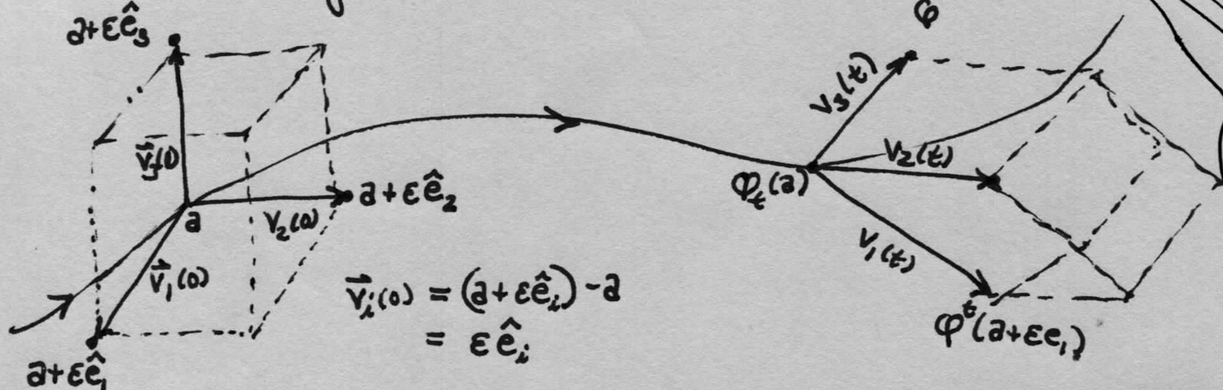
If $\text{div}(F) = 0$ we call the flow <u>incompressible</u>. This will be justified shortly.

<u>Laplace operator</u> $\nabla^2 f = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}$

6/9/2016 (1)

$\dot{x} = F(x)$ with flow φ. First we want to explore the relationship between the rate of change of volume of an element of fluid and div(F) of the ~~[illegible]~~ vf.

Consider a tiny cube with sides parallel to co-ord axes and let it be moved and deformed by the flow.



$$\vec{v}_i(0) = (a+\varepsilon\hat{e}_i) - a = \varepsilon\hat{e}_i$$

We can approximate the cube at some time t ~~after it has been moved by~~ the corner pts have been moved by the flow by the Taylor series approx:

$$\varphi_t(a+\varepsilon v) = \varphi_t(a) + D(\varphi_t)_a(\varepsilon v) + O(\varepsilon^2)$$

This approx makes this discussion not rigorous, but see the remarks after.

Then $$\boxed{v_i(t) = \varphi_t(a+\varepsilon v_i(0)) - \varphi^t(a) \approx D(\varphi^t_a)(\varepsilon v_i(0)) \quad \text{for } i=1,2,3} \quad (*)$$

As the cube moves with the flow, its end pts define a parallelipiped. An expression for the volume of this is $Vol(t) = V(t) = v_1(t)\cdot[v_2(t)\times v_3(t)]$ triple product

p.184

$$\boxed{\textbf{Thm} \quad \text{For such a cube,} \quad \frac{V'(0)}{V(0)} = \operatorname{div}(F)^a_{(x)}}$$

Pf The idea is that if we calculate $V'(0)$, the divergence $\nabla\cdot F$ falls out.

$$V'(t) = \dot{v}_1(t)\times[v_2\times v_3] + v_1\cdot[\dot{v}_2\times v_3] + \ldots$$

To see this, note triple prod $a\cdot(b\times c) = \det\begin{vmatrix} -a- \\ -b- \\ -c- \end{vmatrix}$

Thus $= b\cdot(c\times a) = c\cdot(a\times b)$

$$= \dot{v}_1\cdot[v_2\times v_3] + \dot{v}_2\cdot[v_3\times v_1] + \dot{v}_3\cdot[v_1\times v_2]$$

Evaluate at $t=0$

$$V'(0) = \dot{v}_1(0)\cdot \varepsilon^2\underbrace{[\hat{e}_2\times\hat{e}_3]}_{e_1} + \dot{v}_2(0)\cdot \varepsilon^2\underbrace{[e_3\times e_1]}_{e_2} + \dot{v}_3(0)\cdot \varepsilon^2\underbrace{[e_1\times e_2]}_{e_3}$$

By Lemma which appears below

$$= \varepsilon^3(DF_a(e_1))^T e_1 + \varepsilon^3[DF_a(e_2)]^T e_2 + \varepsilon^3[DF_a(e_3)]^T e_3$$

$$= \varepsilon^3 e_1^T DF_a e_1 + \varepsilon^2 e_2^T DF_a e_2 + \varepsilon^3 e_3^T DF_a e_3$$

$$= \varepsilon^3 [1\ 0\ 0]\begin{bmatrix} F^1_x & F^1_y & F^1_z \\ F^2_x & F^2_y & F^2_z \\ F^3_x & F^3_y & F^3_z \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} + \cdot + \cdot$$

$$= \varepsilon^3\left[F^1_x + F^2_y + F^3_z\right]$$

$$\Rightarrow \frac{V'(0)}{\varepsilon^3} = \frac{V'(0)}{V(0)} = (\nabla\cdot F)(a) \qquad \boxed{\text{QED}}$$

Now for the lemma →

Lemma with our previous approximation,

$$\dot{V}_i(0) = DF_a(V_i(0))$$

By ⊛ $\vec{V}_i(t) = D_x(\varphi_t)_a(V_0)$

then $\frac{d}{dt}\vec{V}_i = \frac{d}{dt} D_x(\varphi_t)_a(V_0) = D_x(\dot{\varphi}_t)_a(V_0)$

$\dot{\varphi}_t = F(\varphi_t)$

$= DF_{\varphi_t(a)} \cdot D(\varphi_t)_a(V_0)$

Evaluate at $t=0$

$\varphi_{t=0} = Id$

$= DF_a(V_i(0))$

$$\dot{V}_i(0) = DF_a(V_i(0))$$

Now we note $V_i(0) = \varepsilon \hat{e}_i$ so $\dot{V}_i(0) = \varepsilon DF_a(e_i)$ □

Now let's compare this with some other results: The definite result (transport thm) is at the end.

▷ On ch 7.4 p.445-446, M&T use the Divergence thm to show

$$(\nabla\cdot F)(x) = \lim_{\rho\to 0} \frac{\int_{\partial B} \vec{F}\cdot\hat{n}\,dA}{Vol(B_\rho)}$$ where we have ball $B = B(x,\rho)$

▷ Arnold's MMOCM p. gives a more general argument. He has a vf $F: \mathbb{R}^n \to \mathbb{R}^n$ and he considers the volume of an arb @able set D (not only a parallelepiped).

He defines $Vol_D(t) := V_D(t) = \int_{\varphi_t(D)} 1\, d^n y \overset{\text{CoV Thm, apostol MA p.421}}{=} \int_D |\det D\varphi^t_x|\, d^n x$

Thm $div(F) = 0 \Rightarrow V_D(t) = V_D(0)$ [volume preserving flow]

This uses Key Lemma $V_D'(0) = \int_D div(F)\, d^n x$

Step 1 Note that for any fixed x, φ_x has the following Taylor expansion about $t=0$:

$$\underbrace{\varphi_x(t)}_{\varphi(x,t)} = \underbrace{\varphi_x(0)}_{} + \underbrace{\dot{\varphi}_x(0)}_{}(t-0) + \underbrace{\frac{\ddot{\varphi}_x(\xi(x))(t-0)^2}{2!}}_{}$$

Now allow x to vary and this relationship

$$\boxed{\varphi_t(x) = x + F(x)\,t + h(x)\,t^2}$$

$$D(\varphi^t)_x = I + t\,DF_x + t^2 Dh_x$$

$$\det(D(\varphi_t)_x) = \det(I + tDF_x + t^2 Dh_x)$$
$$= 1 + t\,\mathrm{Tr}(DF_x) + \mathcal{O}(t^2)$$ ← I can't justify this step despite a work thru in A&P AITDS p. 46

$$\Rightarrow \quad V_D(t) = \int_D \left| 1 + t\,\mathrm{Trace}(DF_x) + \mathcal{O}(t^2) \right| d^n x$$

$$\frac{d}{dt} V_D(t) = \frac{d}{dt} \int_D \left| 1 + t\,\mathrm{Trace}(DF_x) + \mathcal{O}(t^2) \right| d^n x = \int_D \left| 0 + \mathrm{Tr}(DF_x) + \mathcal{O}(t) \right| d^n x$$

Evaluate at $t = 0$ $\quad V_D'(0) = \int_D \underbrace{\left| \mathrm{Trace}(DF_x) \right|}_{\substack{\sum D_i F^i \\ \nabla \cdot F}} d^n x$

Thus if $\mathrm{div}(F) = 0 \Longrightarrow V_D'(0) = \int_D 0 = 0 \Rightarrow$ $V_D(t) = \text{const}$ $\square$

p. 70 In the special case of Hamilton's eqs, n vector $\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{bmatrix}$ form $\dot{x} = h(x)$

$$\nabla \cdot h = \sum \frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) = \sum 0 = 0 \Rightarrow \mathrm{div}(h) = 0$$

▷ The <u>Transport Thm</u> of ch 7.4 prob 18 ~~difinitevly~~ difinitevely asues this question, and the pf is rigorous.

Given vf F with flow φ, consider a region Ω in the flow and a time dependent fcn $f: \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$. Let $\Omega_t := \varphi_t(\Omega)$

<u>Transport thm</u>: $$\frac{d}{dt} \int_{\Omega_t} f(x,t)\, d^3x = \int_{\Omega_t} \left[\frac{d}{dt}(f \circ \varphi) + f\,\mathrm{div}(F) \right] d^3x$$

<u>Key Lemma</u> (which involves $\mathrm{div}(F)$): $\frac{d}{dt} \det(D\varphi_x^t) = \mathrm{div}(F) \det(D\varphi_x^t)$

Take $f \equiv 1$ to get the relevant results:

$$\frac{d}{dt} \underbrace{\int_{\Omega_t} 1\, d^3x}_{Vol(\Omega_t)} = \int_{\Omega_t} \mathrm{div}(F)\, d^3x$$

Then we get $\quad \mathrm{div}(F) = 0 \Longleftrightarrow Vol(\Omega_t) = Vol(\Omega) \Longleftrightarrow \det D\varphi_x^t = 1$

<u>One issue remains</u>: (Maybe this should be addressed in ch 7?)

If I have something like $V_D(t) = \int_{\varphi_t(D)} 1\, d^3y \overset{COV}{=} \int_D \left| \det D\varphi_x^t \right| d^3x$

I wanted to say since $\varphi^t: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a <u>diffeo</u>, $\det D\varphi_x^t \neq 0$ for any x or t

Thus $\det D\varphi_x^t > 0$ always, because $\varphi^0 = Id$, $\det I = 1$

How do we know it is a diffeo?

I thought if $\dot{x} = F(x)$ no dependence on t, then flowlines can't cross, but consider weight loss 6/12/2016

Marsden & Chorin p. 19

Velocity vf of fluid: $\vec{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$$\vec{w} \quad \vec{\xi} = \nabla \times u = \begin{bmatrix} \partial_y w - \partial_z v \\ \partial_z u \quad \partial_x w \\ \partial_x v - \partial_y u \end{bmatrix}$$ vorticity field

claim: In nbhd of each pt of fluid, $\vec{u}$ = translation + deformation + rigid rotation with rot vector $\vec{\xi}$
[applies to any vf in $\mathbb{R}^3$]

$x \in \mathbb{R}^3$
$y = x + h$

$$\boxed{u(y) = u(x) + \underset{S}{\overset{S_D(x)h}{x^T Dh}} + \overset{?}{\tfrac{1}{2}} \underset{W}{\vec{\xi}_x \times h} + O(h^2)}$$

$h^2 = \|h\|^2$

Garrison pool?

Taylor Thm: $\overset{u(x+h)}{u(y)} = u(x) + Du_x h + O(h^2)$

Now define $S_D := \frac{1}{2}\left[Du + Du^T\right]$ and $W_S := \frac{1}{2}\left[Du - Du^T\right]$

so $Du = S_D + W_S$

co-ord expression for $W_S = \begin{bmatrix} 0 & -\xi^{(3)} & \xi^{(2)} \\ \xi^{(3)} & 0 & -\xi^{(1)} \\ -\xi^{(2)} & \xi^{(1)} & 0 \end{bmatrix}$ then $W_S h = \vec{\xi} \times h$

$$u(y) = u(x) + \underbrace{(S_D + W_S)h}_{S_D h + \xi \times h} + O(h^2)$$

$\exists$ real matrix Q $\ni$ $S_D = Q^T \Lambda_D Q$

$$\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$$

Not flow!

p. 20 Since S_D is a symm matrix $S_D = S_D^T$ bilinear

$$S_D h = \text{grad}_h(\varphi(x,h)) \quad \text{where} \quad \varphi(x,h) = \tfrac{1}{2}\langle S_D h, h\rangle$$

if ~~$b(x) = \langle Ah, h\rangle$~~
$b(x) = \langle Ax, x\rangle$
$Db_x = \langle A$

Avez bilinear
$b: E_1 \times E_2 \to F$
$b(a+h) - b(a) = b(a_1, h_2) + b(h_1, a_2) + b(h_1, h_2)$

In M&T VC he says

$$\frac{d}{dt}\langle v(t), w(t)\rangle\Big|_{t=0} = DF_0^T w_0 + v_0^T DF_0 w_0$$
$$= v^T \underbrace{\left[DF_0 + DF_0^T\right]}_{2S} W$$

~~Marsden~~ & Tromba
p. 186–189 ch 3.4

See Marsden & Chorin Fluid Mechanics

6/7/99

We have a vf $F: \mathbb{R}^3 \to \mathbb{R}^3$ and thus $\dot{x} = F(x)$
we want to show that $\dot{x}(0) = (S + W)\,x(0)$, where S is a matrix related to the rate of change of the inner product of 2 vectors under the flow, and W is related to the rotation around the vector $(\nabla \times F)(0)$. The discussion is suggestive, not rigorous.

1. wlog we take $x_0 = 0$ and we write

$$DF_0 = \underbrace{\tfrac{1}{2}\left(DF_0 + DF_0^T\right)}_{S} + \underbrace{\tfrac{1}{2}\left(DF_0 - DF_0^T\right)}_{W}$$

"deformation matrix" "rotation matrix".

2. First, lets work on the W piece. We need a Lemma, which is the book's Thm 4 p. 187 [I'm not going to phrase it as a lemma]

Given a fixed vector w,
The linear map $V: \mathbb{R}^3 \to \mathbb{R}^3$, $y \mapsto w \times y$ has the matrix $\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

and thus, the expression for V as a linear vf is

$\dot{y} = By$

If we choose the basis for $\mathbb{R}^3$ so that $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$ ← we can certainly accomplish this by a rotation i.e. a COB by an o.g. matrix Q $\det Q = +1$

A&P ODE p. 59

Then, magically, B is in ~~[scribbled]~~ block Jordan Normal form and thus we can explicitly write down the flow:

$$\dot{y} = By = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow y(t) = e^{Bt} y_0$$

$$= \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

and we see this flow is a circular rotation around the z axis.

It is the same circular flow! We just rotate all of $\mathbb{R}^3$ back so that $\vec{w}$ is in its orig direction. 6/8/2004 → Hence in the orig basis, the flow is some kind of a rotation about some axis.

Cont'd →

3 Now observe that $W = \frac{1}{2}\left(DF_0 - DF_0^T\right)$ has the form of a "B" for $\vec{w} = (\nabla \times F)(0)$.

$$\frac{1}{2}\left(DF_0 - DF_0^T\right) = \frac{1}{2}\begin{bmatrix} 0 & \left(\frac{\partial F^1}{\partial y} - \frac{\partial F^2}{\partial x}\right) & \left(\frac{\partial F^1}{\partial z} - \frac{\partial F^3}{\partial x}\right) \\ \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right) & 0 & \left(\frac{\partial F^2}{\partial z} - \frac{\partial F^3}{\partial y}\right) \\ \left(\frac{\partial F^3}{\partial x} - \frac{\partial F^1}{\partial z}\right) & \left(\frac{\partial F^3}{\partial y} - \frac{\partial F^2}{\partial z}\right) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(\nabla \times F)^{3} & (\nabla \times F)^{2} \\ (\nabla \times F)^{3} & 0 & -(\nabla \times F)^{1} \\ -(\nabla \times F)^{2} & (\nabla \times F)^{1} & 0 \end{bmatrix}$$

4 Now we show that the S piece falls out when we consider the rate of change of the inner product of 2 vectors carried by the flow. Note that we <u>don't</u> show that W is <u>wholly</u> resposible for rotation and S for deformation; that may be true but we don't show it.

See book for any missing details.

Let $\vec{v}(t) = D(\varphi^t)_0(v)$ $\quad \vec{w}(t) = D(\varphi^t)_0(w)$

$$\left.\frac{d}{dt}\left(v(t)\cdot w(t)\right)\right|_{t=0} = \dot{v}(0)^T w(0) + v(0)^T \dot{w}(0)$$

$$= \left(DF_0(v)\right)^T w + v^T DF_0(w)$$

$$= (Av)^T w + v^T A w$$

$$= v^T A^T w + v^T A w$$

$$= v^T \underbrace{\left[A^T + A\right]}_{\text{This is } 2S} w$$

$\square$