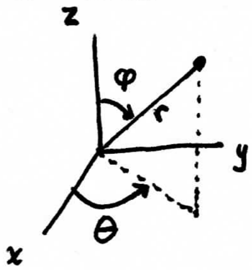
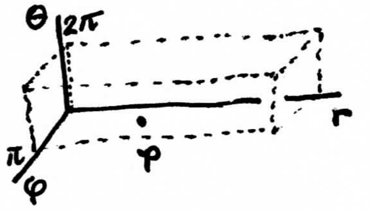
Now let us consider spherical co-ords



$\xleftarrow{h}$



$$h(r,\theta,\varphi) = \begin{bmatrix} r\cos\theta\sin\varphi \\ r\sin\theta\sin\varphi \\ r\cos\varphi \end{bmatrix}$$

$$Dh_p = \begin{bmatrix} c_\theta s_\varphi & -r s_\theta s_\varphi & r c_\theta c_\varphi \\ s_\theta s_\varphi & r c_\theta s_\varphi & r s_\theta c_\varphi \\ c_\varphi & 0 & -r s_\varphi \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r s_\varphi & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$Q = \begin{bmatrix} c_\theta s_\varphi & -s_\theta & c_\theta c_\varphi \\ s_\theta s_\varphi & c_\theta & s_\theta c_\varphi \\ c_\varphi & 0 & -s_\varphi \end{bmatrix}$$

$$Dh_p = Q\Lambda$$

Thm 6 Grad in Spherical

$$\begin{tikzcd} & \mathbb{R} & \\ \mathbb{R}^3_p \arrow[ur] \arrow[rr, "h"] & & \mathbb{R}^3_x \arrow[ul] \end{tikzcd}$$

$p = \langle r,\theta,\varphi\rangle \qquad \langle x,y,z\rangle$

$$\begin{tikzcd} T_p\mathbb{R}^3 \; \{e_r\, e_\theta\, e_\varphi\} \arrow[r, "{[Dh_p]^{-T} = Q\Lambda^{-1}}"] & T_x\mathbb{R}^3 \; \{e_x\, e_y\, e_z\} \\ \mathbb{R}^3_p \arrow[u, "\nabla g"] \arrow[r, "h"] & \mathbb{R}^3_x \arrow[u, "\nabla f"'] \end{tikzcd}$$

We seek a, b, c

$$\{e_x\ e_y\ e_z\}\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \{e_r\ e_\theta\ e_\varphi\}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Pf. Follow the same steps as for cylindrical [sheet ①]

$$f = g\circ h^{-1}$$
$$Df_x = Dg_{h^{-1}(x)}\, D(h^{-1})_x = Dg_p\,[Dh_p]^{-1} \Rightarrow \underbrace{[Df_x]^T}_{\nabla f(x)} = [Dh_p]^{-T}[Dg_p]^T = [Dh_p]^{-T}\,\nabla g(p)$$

step 2

$$\underbrace{\{e_x\ e_y\ e_z\}}_{\{e_r\ e_\theta\ e_\varphi\}Q^{-1}} \vec\nabla f(x)$$
$$\{e_r\ e_\theta\ e_\varphi\}\, Q^{-1}\,[Dh_p^{-1}]^T\,\nabla g(p)$$
$$Q^{-1}\,(Q^{-1})^T\,\Lambda^{-T}\,\nabla g$$
$$\underbrace{Q^T\,(Q^T)^T}_{I}\,\Lambda^{-1}\,\nabla g$$

$$= \{e_r\ e_\theta\ e_\varphi\}\begin{bmatrix} 1 & & \\ & 1/rs_\varphi & \\ & & \frac{1}{r} \end{bmatrix}\begin{bmatrix} g_r \\ g_\theta \\ g_\varphi \end{bmatrix} = \{\hat e_r\ \hat e_\theta\ \hat e_\varphi\}\begin{bmatrix} g_r \\ \frac{1}{rs_\varphi} g_\theta \\ \frac{1}{r} g_\varphi \end{bmatrix}$$

$\square$

Thm 6 Div in Spherical

Just like before:

$$\begin{tikzcd} T_p\mathbb{R}^3 \arrow[r, "Q"] & T_x\mathbb{R}^3 \\ \mathbb{R}^3_p \arrow[u, "\vec{g}"] \arrow[ur, "\underset{\sim}{f}"] \arrow[r, "h"'] & \mathbb{R}^3_x \arrow[u, "\vec{f}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R} & \\ \mathbb{R}^3_p \arrow[ur, "\mathrm{div}(\vec{g})"] \arrow[rr, "h"] & & \mathbb{R}^3_x \arrow[ul, "\mathrm{div}(\vec{f})"'] \end{tikzcd}$$

$$\mathrm{div}(\vec{f}) = \nabla\cdot\vec{f} = \mathrm{Tr}(Df_x)$$

We want to show:

$$\boxed{\mathrm{div}(\vec{g}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g^{r}) + \frac{1}{rS_\varphi} g^{\theta}_{\theta} + \frac{1}{rS_\varphi}\frac{\partial}{\partial\varphi}(S_\varphi g^{\varphi})}$$

pf

Following sheet 3

$$\nabla\cdot f = \mathrm{Tr}(Df_x) = \mathrm{Tr}(D\underset{\sim}{f}_p[Dh_p]^{-1})$$

$$Q^{-1}Df_x Q = \boxed{Q^{-1}M\Lambda^{-1} + Dg_p\Lambda^{-1}}$$

we need to compute this matrix. Then we take the Trace(.) for Div and χ(.) for curl

$$\underset{\sim}{f}(p) = Q(p)\vec{g}(p)$$

$$D\underset{\sim}{f}_p = DQ_p\vec{g} + QDg_p$$

$$= \underbrace{\begin{bmatrix} | & | & | \\ Q_r g & Q_\theta g & Q_\varphi g \\ | & | & | \end{bmatrix}}_{M} + QDg_p$$

$$Q(p) = \begin{bmatrix} C_\theta S_\varphi & -S_\theta & C_\theta C_\varphi \\ S_\theta S_\varphi & C_\theta & S_\theta C_\varphi \\ C_\varphi & 0 & -S_\varphi \end{bmatrix}$$

$$Q_\theta = \begin{bmatrix} -S_\theta S_\varphi & -C_\theta & -S_\theta C_\varphi \\ C_\theta S_\varphi & -S_\theta & C_\theta C_\varphi \\ 0 & 0 & 0 \end{bmatrix} \qquad Q_\varphi = \begin{bmatrix} C_\theta C_\varphi & 0 & -C_\theta S_\varphi \\ S_\theta C_\varphi & 0 & -S_\theta C_\varphi \\ -S_\varphi & 0 & -C_\varphi \end{bmatrix}$$

$$Q^{-1} M \Lambda^{-1} = Q^{-1}\left[Q_\theta \vec{g}\, e_2^T + Q_\varphi \vec{g}\, e_3^T \right] \Lambda^{-1}$$

$$= Q^{-1} Q_\theta \begin{bmatrix} 1 & g^r & 1 \\ 0 & g^\theta & 0 \\ 1 & g^\varphi & 1 \end{bmatrix} \Lambda^{-1} + Q^{-1} Q_\varphi \begin{bmatrix} 1 & 1 & g^r \\ 0 & 0 & g^\theta \\ 1 & 1 & g^\varphi \end{bmatrix} \Lambda^{-1}$$

$Q^{-1} = Q^T$

$$\underbrace{\begin{bmatrix} c_\theta s_\varphi & s_\theta s_\varphi & c_\varphi \\ -s_\theta & c_\theta & 0 \\ c_\theta c_\varphi & s_\theta c_\varphi & -s_\varphi \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} -s_\theta s_\varphi & -c_\theta & -s_\theta c_\varphi \\ c_\theta s_\varphi & -s_\theta & c_\theta c_\varphi \\ 0 & 0 & 0 \end{bmatrix}}_{Q_\theta} = \begin{bmatrix} 0 & -s_\varphi & 0 \\ s_\varphi & 0 & c_\varphi \\ 0 & -c_\varphi & 0 \end{bmatrix}$$

$$Q^T Q_\theta \begin{bmatrix} 0 & g^r & 0 \\ 0 & g^\theta & 0 \\ 0 & g^\varphi & 0 \end{bmatrix} \Lambda^{-1} = \underbrace{\begin{bmatrix} 0 & -s_\varphi & 0 \\ s_\varphi & 0 & c_\varphi \\ 0 & -c_\varphi & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & g^r & 0 \\ 0 & g^\theta & 0 \\ 0 & g^\varphi & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \frac{1}{r s_\varphi} & \\ & & \frac{1}{r} \end{bmatrix}}_{\begin{bmatrix} 0 & \frac{1}{r s_\varphi} g^r & 0 \\ 0 & \frac{1}{r s_\varphi} g^\theta & 0 \\ 0 & \frac{1}{r s_\varphi} g^\varphi & 0 \end{bmatrix}}}_{\begin{bmatrix} 0 & \frac{-1}{r} g^\theta & 0 \\ 0 & \left(\frac{1}{r} g^r + \frac{c_\varphi}{r s_\varphi} g^\varphi\right) & 0 \\ 0 & \frac{-c_\varphi}{r s_\varphi} g^\theta & 0 \end{bmatrix}}$$

$$Q^T Q_\varphi = \begin{bmatrix} c_\theta s_\varphi & s_\theta s_\varphi & c_\varphi \\ -s_\theta & c_\theta & 0 \\ c_\theta c_\varphi & s_\theta c_\varphi & -s_\varphi \end{bmatrix} \begin{bmatrix} c_\theta c_\varphi & 0 & -c_\theta s_\varphi \\ s_\theta c_\varphi & 0 & -s_\theta s_\varphi \\ -s_\varphi & 0 & -c_\varphi \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q^T Q_\varphi \begin{bmatrix} 1 & 1 & g^r \\ 0 & 0 & g^\theta \\ 1 & 1 & g^\varphi \end{bmatrix} \begin{bmatrix} 1 & & \\ & \frac{1}{r s_\varphi} & \\ & & \frac{1}{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{r} g^r \\ 0 & 0 & \frac{1}{r} g^\theta \\ 1 & 1 & \frac{1}{r} g^\varphi \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{-1}{r} g^\theta \\ 0 & 0 & 0 \\ 1 & 1 & \frac{1}{r} g^r \end{bmatrix}$$

$$\Rightarrow \quad Q^{-1} M \Lambda^{-1} = \begin{bmatrix} 0 & \frac{-1}{r} g^\theta & \frac{-1}{r} g^\theta \\ 0 & \left(\frac{1}{r} g^r + \frac{c_\varphi}{r s_\varphi} g^\varphi\right) & 0 \\ 0 & \frac{-c_\varphi}{r s_\varphi} g^\theta & \frac{1}{r} g^r \end{bmatrix}$$

Cont'd →

$$D g_p \Lambda^{-1} = \begin{bmatrix} g^r_r & \frac{1}{r s_\varphi} g^r_\theta & \frac{1}{r} g^r_\varphi \\ g^\theta_r & \frac{1}{r s_\varphi} g^\theta_\theta & \frac{1}{r} g^\theta_\varphi \\ g^\varphi_r & \frac{1}{r s_\varphi} g^\varphi_\theta & \frac{1}{r} g^\varphi_\varphi \end{bmatrix}$$

Then $Q^{-1} M \Lambda^{-1} + D g_p \Lambda^{-1} = \begin{bmatrix} g^r_r & \left(-\frac{1}{r} g^\theta + \frac{1}{r s_\varphi} g^r_\theta\right) & \left(-\frac{1}{r} g^\theta + \frac{1}{r} g^r_\varphi\right) \\ g^\theta_r & \left(\frac{1}{r} g^r + \frac{c_\varphi}{r s_\varphi} g^\varphi + \frac{1}{r s_\varphi} g^\theta_\theta\right) & \frac{1}{r} g^\theta_\varphi \\ g^\varphi_r & \frac{1}{r s_\varphi} g^\varphi_\theta - \frac{c_\varphi}{r s_\varphi} g^\theta & \frac{1}{r} g^r + \frac{1}{r} g^\varphi_\varphi \end{bmatrix}$

$$\operatorname{div}(\vec{g}) = \operatorname{Tr}\left(Q^{-1} M \Lambda^{-1} + D g_p \Lambda^{-1}\right) =$$

$$g^r_r + \frac{1}{r} g^r + \frac{c_\varphi}{r s_\varphi} g^\varphi + \frac{1}{r s_\varphi} g^\theta_\theta + \frac{1}{r} g^r + \frac{1}{r} g^\varphi_\varphi$$

$$= \underbrace{\left(\frac{2}{r} g^r + g^r_r\right)}_{\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 g^r)} + \frac{1}{r s_\varphi} g^\theta_\theta + \underbrace{\frac{c_\varphi}{r s_\varphi} g^\varphi + \frac{1}{r} g^\varphi_\varphi}_{\frac{1}{r s_\varphi} \frac{\partial}{\partial \varphi}(s_\varphi g^\varphi)}$$

$$\operatorname{div}(\vec{g}) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 g^r) + \frac{1}{r s_\varphi} g^\theta_\theta + \frac{1}{r s_\varphi} \frac{\partial}{\partial \varphi}(s_\varphi g^\varphi)$$

▷ Now for Curl in Spherical

$$\begin{tikzcd}
\{e_r, e_\theta, e_\varphi\} & \{e_x, e_y, e_z\} \\
T_p\mathbb{R}^3 \arrow[r, "Q"] & T_x\mathbb{R}^3 \\
\mathbb{R}^3_p \arrow[u, "\vec{g}"] \arrow[u, bend left=60, "\mathrm{curl}(\vec{g})"] \arrow[r, "h"'] \arrow[ur, "\vec{\underset{\sim}{f}}"] & \mathbb{R}^3_x \arrow[u, "\vec{f}"'] \arrow[u, bend right=60, "\mathrm{curl}(\vec{f})"']
\end{tikzcd}$$

Want to show:

$$\mathrm{curl}(\vec{g}) = \{e_r\ e_\theta\ e_\varphi\} \begin{bmatrix} \frac{1}{rs_\varphi}\left(\frac{\partial}{\partial\varphi}(s_\varphi g^\varphi) - g^\varphi_\theta\right) \\ \frac{1}{r}\left(\frac{\partial}{\partial r}(rg^\varphi) - g^r_\varphi\right) \\ \frac{1}{rs_\varphi} g^r_\theta - \frac{1}{r}\frac{\partial}{\partial r}(rg^\theta) \end{bmatrix}$$

pf.

$$\begin{aligned}
\mathrm{Curl}(\vec{f})_{(x)} &= \{e_x\ e_y\ e_z\}\, \chi(Df_x) \\
&= \{e_r\ e_\theta\ e_\varphi\}\, Q^{-1}\, \chi(Df_x) \\
&= \{e_r\ e_\theta\ e_\varphi\}\, \chi(Q^{-1} Df_x\, Q) \\
\mathrm{Curl}(\vec{g})_{(p)} &= \{e_r\ e_\theta\ e_\varphi\}\, \chi(Q^{-1} M \Lambda^{-1} + Dg_p \Lambda^{-1})
\end{aligned}$$

$$\chi\left(\begin{bmatrix} g^r_r & \frac{1}{rs_\varphi} g^r_\theta - \frac{1}{r} g^\theta & \frac{1}{r} g^r_\varphi - \frac{1}{r} g^\theta \\ g^\theta_r & (\ \cdots\ ) & \frac{1}{r} g^\theta_\varphi \\ g^\varphi_r & \left(\frac{1}{rs_\varphi} g^\varphi_\theta - \frac{c_\varphi}{rs_\varphi} g^\theta\right) & \frac{1}{r} g^r + \frac{1}{r} g^\varphi_\theta \end{bmatrix}\right)$$

$$\begin{bmatrix} \frac{1}{rs_\varphi} g^\varphi_\theta - \frac{c_\varphi}{rs_\varphi} g^\theta - \frac{1}{r} g^\theta_\varphi \\ \frac{1}{r} g^r_\varphi - \frac{1}{r} g^\theta - g^\varphi_r \\ g^\theta_r - \left(\frac{1}{rs_\varphi} g^r_\theta - \frac{1}{r} g^\theta\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{rs_\varphi}\left(g^\varphi_\theta - \frac{\partial}{\partial\varphi}(s_\varphi g^\theta)\right) \\ \frac{1}{r}\left(g^r_\varphi - \frac{\partial}{\partial r}(rg^\varphi)\right) \\ \frac{1}{r}\left(\frac{\partial}{\partial r}(rg^\theta) - \frac{1}{s_\varphi} g^r_\theta\right) \end{bmatrix}$$

This is negative of what I copied from M&T, but it agrees with Gemini's calculations. Std tables order the variables (r, φ, θ), and even though M&T don't do this order, they may have just copied the answer from tables.

□

Probs 11, 12 from p.196 reformulated a bit

we know $\mathrm{curl}(\mathrm{grad}(\cdot)) \equiv 0$ and $\mathrm{div}(\mathrm{curl}(\cdot)) \equiv 0$

Define Laplacian operator $\boxed{\mathrm{Lap}(f) = \mathrm{div}(\mathrm{grad}(f))}$ $\nabla \cdot \nabla f = \nabla^2 f$ in cartesian

Let's show this in spherical:

$$\begin{tikzcd} & \mathbb{R} & \\ \mathbb{R}^3_p \arrow[ur, "g"] \arrow[rr, "h"'] & & \mathbb{R}^3_x \arrow[ul, "f"'] \end{tikzcd}$$

$\nabla^2 f = \mathrm{div}(\mathrm{grad}(f))$

$\mathrm{Lap}(f)$

we want to compute $\mathrm{Lap}(g) = \mathrm{div}(\mathrm{grad}(g))$ in spherical

we found formula for grad: $\vec{G} = \mathrm{grad}(g) = \{e_r\ e_\theta\ e_\varphi\}\begin{bmatrix} g_r \\ \frac{1}{rS_\varphi} g_\theta \\ \frac{1}{r} g_\varphi \end{bmatrix} = \{\ \ \}\begin{bmatrix} G^{r} \\ G^{\theta} \\ G^{\varphi} \end{bmatrix}$

and the formula for div:

$$\mathrm{div}(\vec{G}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 G^{r}) + \frac{1}{rS_\varphi}\frac{\partial}{\partial\theta}(G^\theta) + \frac{1}{rS_\varphi}\frac{\partial}{\partial\varphi}(S_\varphi G^\varphi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_r) + \underbrace{\frac{1}{rS_\varphi}\frac{\partial}{\partial\theta}\left(\frac{1}{rS_\varphi} g_\theta\right)}_{\frac{1}{r^2 S_\varphi^2} g_{\theta\theta}} + \underbrace{\frac{1}{rS_\varphi}\frac{\partial}{\partial\varphi}\left(S_\varphi \frac{1}{r} g_\varphi\right)}_{\frac{1}{r^2 S_\varphi^2}\frac{\partial}{\partial\varphi}(S_\varphi g_\varphi)}$$

$$\mathrm{Lap}(g) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_r) + \frac{1}{r^2 S_\varphi^2} g_{\theta\theta} + \frac{1}{(rS_\varphi)^2}\frac{\partial}{\partial\varphi}(S_\varphi g_\varphi)$$

and if there was no dependence on φ (polar co-ords in plane)

$$\mathrm{Lap}(g) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_r) + \frac{1}{(rS_\varphi)^2} g_{\theta\theta} + 0$$

$\varphi = \pi/2$

$\sin = 1$

$$= \frac{2}{r} g_r + g_{rr} + g_{\theta\theta}$$

$\square$