Thm 5 (cylindrical) Given a fcn $f: \mathbb{R}^3 \to \mathbb{R}$ and a COV map h from ~~polar~~ cylindrical to Cartesian
$p = \langle r, \theta, z\rangle$ $\langle x, y, z\rangle$

Let $g(p) = f(x)$, that is $g = f \circ h$

g is "f in cylindrical co-ords"

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{R}^3_p \arrow[ur, "g"] \arrow[r, "h"] & \mathbb{R}^3_x \arrow[u, "f"'] \end{tikzcd}$$

$\langle r,\theta,z\rangle$ $\langle x,y,z\rangle$

$\{e_x\ e_y\ e_z\}$

we know $\operatorname{grad}(f)_x = \nabla f(x) = \{e_x\ e_y\ e_z\}\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$

What is $\operatorname{grad}(g)_p = \vec{\nabla} g(p)$?

$\{e_r, e_\theta, e_z\}$

$$\begin{tikzcd} T_p\mathbb{R}^3 \arrow[r, "{[Dh_p]^{-T}}", "= QS^{-1}"'] & T_x\mathbb{R}^3 \\ \mathbb{R}^3_p \arrow[u, "\nabla g"] \arrow[r, "h"'] & \mathbb{R}^3_x \arrow[u, "\nabla f"'] \end{tikzcd}$$

In other words

$$\{e_x\ e_y\ e_z\}\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \stackrel{!}{=} \{\hat{e}_r\ \hat{e}_\theta\ \hat{e}_z\}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we want to know what a, b, c are. we might think

$a = g_r$ (subscripts mean partials)
$b = g_\theta$
$c = g_z$

but this is <u>NOT</u> accurate.

$$h(r,\theta,z) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$Dh_p = \begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{e_r\ e_\theta\ e_z\} = \{e_x\ e_y\ e_z\}\underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}$$

$$Dh_p = \underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 1 & & \\ & r & \\ & & 1 \end{bmatrix}}_{\Lambda}$$

This factorization is essential — only happens for OG co-ord sys.

$$Dh_p^{-1} = \begin{bmatrix} c & s & 0 \\ -s/r & c/r & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Lambda^{-1}Q^{-1}$$

$\parallel$

$[Dh_p]^{-1}$

Derivation: $f = g \circ h^{-1}$

step 1

$$Df_x = Dg_{h^{-1}(x)} D(h^{-1})_x = Dg_p[Dh_p]^{-1}$$

$$[Df_x]^T = [Dh_p]^{-T}[Dg_p]^T$$

$$\vec{\nabla} f(x) = [Dh_p]^{-T}\,\vec{\nabla} g(p)$$

step 2

$$\underbrace{\{e_x\ e_y\ e_z\}}\ \underbrace{\vec{\nabla} f(x)}$$

$$\{\hat{e}_r\ \hat{e}_\theta\ \hat{e}_z\}Q^{-1}\,[Dh_p^{-1}]^T\,\nabla g_p$$

$$(Q^{-1})^T \Lambda^{-T}\,\nabla g$$

$$Q^T (Q^T)^T \Lambda^{-1}$$

Λ is diag

$$= \{\hat{e}_r\ \hat{e}_\theta\ \hat{e}_z\}\underbrace{\begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}\begin{bmatrix} g_r \\ g_\theta \\ g_z \end{bmatrix}} = g_r\hat{e}_r + \tfrac{1}{r}g_\theta\hat{e}_\theta + g_z\hat{e}_z$$

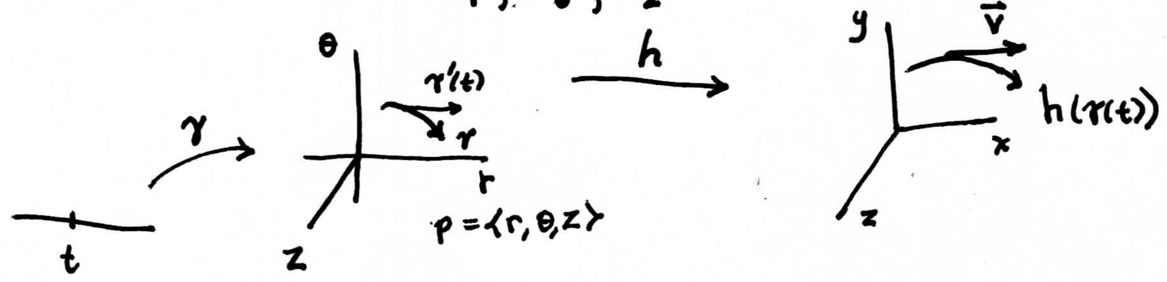
$\square$

$\Lambda^{-1}\vec{\nabla} g$

'Lamé' coeffs $\begin{bmatrix} g_r \\ \frac{1}{r}g_\theta \\ g_z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

How do we justify a vf $\vec{G} = G^{(r)}\hat{e}_r + G^{(\theta)}\hat{e}_\theta + G^{(z)}\hat{e}_z$?

How do we derive $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$?



$$\vec{v} = \frac{d}{dt}(h(\gamma(t))) = Dh_{\gamma(t)}(\vec{\gamma}'(t)) = Dh_p \begin{bmatrix} \dot{\gamma}^1 \\ \dot{\gamma}^2 \\ \dot{\gamma}^3 \end{bmatrix} = Dh_p \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{z} \end{bmatrix}$$

$$= Dh_p(\dot{\gamma}^1 \hat{e}_1 + \dot{\gamma}^2 \hat{e}_2 + \dot{\gamma}^3 e_3)$$

$$= \dot{\gamma}^1 Dh_p(e_1) + \dot{\gamma}^2 Dh_p(e_2) + \dot{\gamma}^3 Dh_p(e_3)$$

$$= \dot{\gamma}^1 \hat{e}_r + \dot{\gamma}^2 r \hat{e}_\theta + \dot{\gamma}^3 \hat{e}_z$$

because

$$Dh_p = \begin{array}{c} \begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \end{array} \end{array} \quad \underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

This explains $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$

For a vector field $\vec{G}$: $\dot{\vec{\gamma}}(t) = \vec{G}(\gamma(t))$

$$\dot{\gamma}^1 \hat{e}_r + \dot{\gamma}^2 r \hat{e}_\theta + \dot{\gamma}^3 \hat{e}_z = \vec{G}(p)$$

$$G^{(r)}\hat{e}_r + G^{(\theta)}\hat{e}_\theta + G^{(z)}\hat{e}_z = \vec{G}(p)$$

Thm 5 (Div in cylindrical)

Given $\{e_r e_\theta e_z\}\begin{bmatrix} g^r \\ g^\theta \\ g^z \end{bmatrix}$ $\quad\{e_x e_y e_z\}$

$$\begin{tikzcd} T_p\mathbb{R}^3 \arrow[r, "Q"] & T_x\mathbb{R}^3 \\ \mathbb{R}^3_p \arrow[u, "\vec{g}"] \arrow[ur, "\underset{\sim}{f}"] \arrow[r, "h"] & \mathbb{R}^3_x \arrow[u, "\vec{f}"'] \end{tikzcd}$$

$p = \langle r, \theta, z\rangle$

$$\begin{tikzcd} & \mathbb{R} & \\ \mathbb{R}^3_p \arrow[ur, "\mathrm{div}(\vec{g})"] \arrow[rr, "h"] & & \mathbb{R}^3_x \arrow[ul, "\mathrm{div}(\vec{F})"'] \end{tikzcd}$$

$$\mathrm{div}(\vec{f}) = \nabla\cdot\vec{f} = f^{(x)}_x + f^{(y)}_y + f^{(z)}_z = \mathrm{Trace}(D\vec{f}_x) = \mathrm{Tr}\begin{bmatrix} f^{(1)}_x & f^{(1)}_y & f^{(1)}_z \\ f^{(2)}_x & f^{(2)}_y & f^{(2)}_z \\ f^{(3)}_x & f^{(3)}_y & f^{(3)}_z \end{bmatrix}$$

$$\boxed{\text{We want to show } \mathrm{div}(\vec{g}) = \frac{1}{r}\frac{\partial}{\partial r}(r g^{(r)}) + \frac{1}{r} g^{(\theta)}_\theta + g^{(z)}_z}$$

From commutative diagram $\underbrace{\vec{f}(h(p))}_{\underset{\sim}{f}(p)} = Q(\vec{g}(p)) = Q(\theta)\vec{g}(p)$ Important: This is matrix multiplying vector

$\vec{g}(p) = \{e_r\ e_\theta\ e_z\}\begin{bmatrix} g^r \\ g^\theta \\ g^z \end{bmatrix}$

$\underset{\sim}{f} = \vec{f}\circ h \quad$ and $\quad \underset{\sim}{f} = Q\vec{g}$

$\{e_r\ e_\theta\ e_z\}\big|_p \begin{bmatrix} g^r \\ g^\theta \\ g^z \end{bmatrix} = \{e_x\ e_y\ e_z\}\underbrace{\begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix}\begin{bmatrix} g^r \\ g^\theta \\ g^z \end{bmatrix}}_{\begin{bmatrix} f^{(x)} \\ f^{(y)} \\ f^{(z)} \end{bmatrix}}$

$\vec{f}(x) = \underset{\sim}{f}\circ h^{-1}(x)$

$D\vec{f}_x = D\underset{\sim}{f}_p\, Dh^{-1}_x = D\underset{\sim}{f}_p\,[Dh_p]^{-1}$

$$\boxed{\nabla\cdot\vec{f} = \mathrm{Tr}(D\vec{f}_x) = \mathrm{Tr}\left(D\underset{\sim}{f}_p\,[Dh_p]^{-1}\right) \quad \text{Thus we have an expression for div completely in terms of } p \quad ★}$$

Let us compute $D\underset{\sim}{f}_p$ $\qquad Q_{(p)} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Does not depend on r, z

$G(p) = F(x)$

$$\begin{tikzcd} T_{G(p)}\mathbb{R}^3 \arrow[r, "Q\ \mathrm{cor}"] & T_{F(x)}\mathbb{R}^3 \\ T_p\mathbb{R}^3 \arrow[u, "Dg_p"] \arrow[u, shift right=2, "DQ"'] \arrow[ur, bend left, "M"] \arrow[ur, bend right, "D\underset{\sim}{f}"'] \arrow[r, "Dh_p"'] & T_x\mathbb{R}^3 \arrow[u, "Df_x(\cdot)"'] \end{tikzcd}$$

$\underset{\sim}{f}(p) = Q(\theta)\vec{g}(p)$

$$\begin{aligned} D\underset{\sim}{f}_p &= DQ_p\vec{g} + Q\,Dg_p \\ &= \begin{bmatrix} | & | & | \\ Q_r g & Q_\theta g & Q_z g \\ | & | & | \end{bmatrix} + Q\,D\vec{g}_p \\ &= \underbrace{\begin{bmatrix} | & | & | \\ 0 & Q_\theta\vec{g} & 0 \\ | & | & | \end{bmatrix}}_{M = Q_\theta\vec{g}\, e_2^T} + Q\,Dg_p \end{aligned}$$

This is justified in following Lemma Sheets 5 - 6

We are computing $D\underset{\sim}{f}_p\,[Dh_p]^{-1}$. we got $D\underset{\sim}{f}_p$

$$[Dh_p]^{-1} = \begin{bmatrix} c & s & 0 \\ -s/r & c/r & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}}_{\Lambda^{-1}} \underbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q^{-1}} \quad \text{where } \Lambda = \begin{bmatrix} 1 & & \\ & r & \\ & & 1 \end{bmatrix}$$

$$[Dh_p]^{-1} = \Lambda^{-1} Q^{-1}$$

▷ $$Df_x = D\underset{\sim}{f}_p\,[Dh_p]^{-1}$$
$$= [M + QDg_p][\Lambda^{-1}Q^{-1}]$$

we want to ~~Then~~ apply Trace(·) operator to both sides. But first let us derive a ~~(*) becomes:~~ specific form equation that will also reoccur in curl calculations

$$\cancel{\nabla\cdot F = Tr(D\underset{\sim}{f}_p[Dh_p]^{-1}) = Tr(D\underset{\sim}{f}_p[\Lambda^{-1}Q^{-1}])}$$

$$Df_x = M\Lambda^{-1}Q^{-1} + QDg_p\Lambda^{-1}Q^{-1}$$

$$Q^{-1}Df_x Q = Q^{-1}[M\Lambda^{-1}Q^{-1} + QDg_p\Lambda^{-1}Q^{-1}]Q$$
$$= Q^{-1}M\Lambda^{-1}\cancel{Q^{-1}Q} + \cancel{Q^{-1}Q}Dg_p\Lambda^{-1}\cancel{Q^{-1}Q}$$
$$= Q^{-1}M\Lambda^{-1} + Dg_p\Lambda^{-1}$$

This RHS is in fact a fcn of p: $[Q(p)]^{-1}M(p)[\Lambda(p)]^{-1} + Dg_p[\Lambda(p)]^{-1}$

Now we know $Tr(B^{-1}AB) = Tr(A)$ [EWs are preserved by similarity transforms]

So $$\underbrace{Tr(Df_x)}_{\nabla\cdot\vec{f}} = Tr(Q^{-1}Df_xQ) = Tr(Q^{-1}M\Lambda^{-1} + Dg_p\Lambda^{-1})$$
$$\overset{\text{linearity}}{=} Tr(Q^{-1}M\Lambda^{-1}) + Tr(Dg_p\Lambda^{-1})$$

▷ Lets take the 2nd term first:

$$Dg_p\Lambda^{-1} = \begin{bmatrix} g^r_r & g^r_\theta & g^r_z \\ g^\theta_r & g^\theta_\theta & g^\theta_z \\ g^z_r & g^z_\theta & g^z_z \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix} \Rightarrow Tr(Dg_p\Lambda^{-1}) = g^r_r + \frac{1}{r}g^\theta_\theta + g^z_z$$

▷ Now look at $Tr(Q^{-1}M\Lambda^{-1})$:

$$= Tr(Q^{-1}\,Q_\theta g e_2^T\,\Lambda^{-1})$$

mult this out

$$Q^{-1} Q_\theta g e_2^T \Lambda^{-1} = \underbrace{\underbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q^{-1}} \underbrace{\begin{bmatrix} -s & -c & 0 \\ c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Q_\theta}}_{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underbrace{\begin{bmatrix} 0 & g^r & 0 \\ 0 & g^\theta & 0 \\ 0 & g^z & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 1/r\, g^r & 0 \\ 0 & 1/r\, g^\theta & 0 \\ 0 & 1/r\, g^z & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & -1/r\, g^\theta & 0 \\ 0 & 1/r\, g^r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathrm{Tr} = 0 + \frac{1}{r} g^r + 0$$

$$\Rightarrow \nabla \cdot F = \mathrm{Tr}(Df_p [Dh_p]^{-1}) = \left(g^r_r + \frac{1}{r} g^\theta_\theta + g^z_z\right) + \frac{1}{r} g^r$$

$$= \left(\frac{1}{r} g^r + g^r_r\right) + \frac{1}{r} g^\theta_\theta + g^z_z$$

$$= \frac{1}{r} \frac{\partial}{\partial r}(r g^r) + \frac{1}{r}\frac{\partial}{\partial \theta} g^\theta + \frac{\partial}{\partial z} g^z$$

observe this trickery:

$$\frac{1}{r}\frac{\partial}{\partial r}(r g^r) = \frac{1}{r}\left[g^r + r g^r_r\right]$$

$$= \frac{1}{r} g^r + g^r_r$$

I guess we could write this $\begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial r}(r\,\cdot) & \frac{1}{r}\frac{\partial}{\partial \theta}(\cdot) & \frac{\partial}{\partial z}(\cdot) \end{bmatrix} \begin{bmatrix} g^r \\ g^\theta \\ g^z \end{bmatrix}$

Lemma Differentiating Qg

Want to show $D(Qg)_p = \begin{bmatrix} | & | & | \\ Q_r g & Q_\theta g & Q_z g \\ | & | & | \end{bmatrix} + Q \begin{bmatrix} | & | & | \\ g_r & g_\theta & g_z \\ | & | & | \end{bmatrix}$ $\square$

Pf. We will actually differentiate $A(x,y)\vec{g}(x,y)$

Case 1 single row $H(x,y) = A(x,y)\vec{g}(x,y) = \underbrace{\begin{bmatrix} a_1(x,y) & a_2(x,y) & a_3(x,y) \end{bmatrix}}_{A'} \begin{bmatrix} g^1(x,y) \\ g^2 \\ g^3 \end{bmatrix} = \sum a_i g^i$

$$DH_x = [H_x \; H_y] = \left[D_x\left(\sum a_i g^i\right) \quad D_y\left(\sum a_i g^i\right)\right]$$

$$= \left[\sum\left(D_x a_i\, g^i + a_i D_x g^i\right) \quad \sum\left(D_y a_i\, g^i + a_i D_y g^i\right)\right]$$

$$= \left[\sum D_x(a_i) g^i \quad \sum D_y(a_i) g^i\right] + \left[\sum a_i D_x(g^i) \quad \sum a_i D_y(g^i)\right]$$

cont'd →

$$A^1_x = \left[\tfrac{\partial}{\partial x}a_1 \;\; \tfrac{\partial}{\partial x}a_2 \;\; \tfrac{\partial}{\partial x}a_3\right]$$

$$= \left[A^1_x \vec{g} \quad A^1_y \vec{g}\right] + \left[A^1 \vec{g}_x \quad A^1 \vec{g}_y\right]$$

<u>Case 2</u>

$$H = \underbrace{\begin{bmatrix} a^{(1)}_1 & a^{(1)}_2 & a^{(1)}_3 \\ a^{(2)}_1 & a^{(2)}_2 & a^{(2)}_3 \end{bmatrix}}_{A} \begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix} = \begin{bmatrix} \sum a^{(1)}_i g^i \\ \sum a^{(2)}_i g^i \end{bmatrix}$$

$$DH_x = \begin{bmatrix} H^1_x & H^1_y \\ H^2_x & H^2_y \end{bmatrix} = \begin{bmatrix} \sum (a^1_{i,x} g^i + a^1_i g^i_x) & \sum (a^1_{i,y} g^i + a^1_i g^i_y) \\ \sum a^2_{i,x} g^i + a^2_i g^i_x & \sum a^2_{i,y} g^i + a^2_i g^i_y \end{bmatrix}$$

$$= \begin{bmatrix} A^1_x g & A^1_y g \\ A^2_x g & A^2_y g \end{bmatrix} + \begin{bmatrix} A^1 g_x & A^1 g_y \\ A^2 g_x & A^2 g_y \end{bmatrix}$$

$$= \begin{bmatrix} \vert & \vert \\ A_x g & A_y g \\ \vert & \vert \end{bmatrix} + \begin{bmatrix} \vert & \vert \\ A g_x & A g_y \\ \vert & \vert \end{bmatrix}$$

$$= \begin{bmatrix} \vert & \vert \\ A_x g & A_y g \\ \vert & \vert \end{bmatrix} + A \begin{bmatrix} \vert & \vert \\ g_x & g_y \\ \vert & \vert \end{bmatrix}$$

This gives the idea for the general case $\square$

Then $$D(Qg)_p = \begin{bmatrix} \vert & \vert & \vert \\ Q_r g & Q_\theta g & Q_z g \\ \vert & \vert & \vert \end{bmatrix} + Q\left[g_r \; g_\theta \; g_z\right]$$

but $Q = \begin{bmatrix} c(\theta) & -s(\theta) & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ This only depends on θ, so

$$D(Qg)_p = \underbrace{\begin{bmatrix} \vert & \vert & \vert \\ 0 & Q_\theta g & 0 \\ \vert & \vert & \vert \end{bmatrix}}_{M} + Q\left[g_r \; g_\theta \; g_z\right]$$

$\square$

▷ Now let us discuss curl $\vec{f}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

[I'm using lower case for vfs here because capital letters kept looking like matrices]

In Cartesian co-ords, we know

$$\mathrm{Curl}(\vec{f}) = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ f^{1} & f^{2} & f^{3} \end{vmatrix} = \begin{bmatrix} f^3_y - f^2_z \\ f^1_z - f^3_x \\ f^2_x - f^1_y \end{bmatrix}$$

But this form doesn't lend itself to the calculations we need to do. $\nabla \times \vec{f}$ is a vector, call it $\vec{u}$. We need to extract $\vec{u}$ from $D\vec{f}_x$. We can't do it directly, but we can construct the matrix Λ_u, that is $\vec{u} \times (\cdot)$. From matrix Λ_u, we extract the vector components that make $\vec{u}$.

How do we form a matrix that represents the cross prod by vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$?

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} i & j & k \\ a & b & c \\ x & y & z \end{vmatrix} = \begin{bmatrix} bz - cy \\ cx - az \\ ay - bx \end{bmatrix}$$

axial vector

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

How to make the axial vector from the skew-sym matrix

Now we can make a skew-sym operator out of any matrix A

$$(A - A^T) = \begin{bmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{bmatrix} - \begin{bmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_3 & a^2_3 & a^3_3 \end{bmatrix} = \begin{bmatrix} 0 & (a^1_2 - a^2_1) & \overset{b}{(a^1_3 - a^3_1)} \\ {}^{c}(a^2_1 - a^1_2) & 0 & (a^2_3 - a^3_2) \\ (a^3_1 - a^1_3) & \underset{a}{(a^3_2 - a^2_3)} & 0 \end{bmatrix}$$

So for any vector $v \in \mathbb{R}^3$
$(A - A^T)v = \vec{c}_A \times v$
axial vector $\vec{c}_A = \chi(A)$
see below and sheet 10

$$\Rightarrow \begin{bmatrix} a^3_2 - a^2_3 \\ a^1_3 - a^3_1 \\ a^2_1 - a^1_2 \end{bmatrix}$$

So when we go to $A = Df_x$
We can form this with a snake like pattern:

$$Df_x = \begin{bmatrix} f^1_x & f^1_y & f^1_z \\ f^2_x & f^2_y & f^2_z \\ f^3_x & f^3_y & f^3_z \end{bmatrix} \rightsquigarrow \begin{bmatrix} f^3_y - f^2_z \\ f^1_z - f^3_x \\ f^2_x - f^1_y \end{bmatrix} =: \chi(Df_x)$$

I define χ as the fcn that takes Df_x into this vector

→

Thm 5 (Curl in Cylindrical)

$$\det \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ f^1 & f^2 & f^3 \end{vmatrix}$$

$\{e_r, e_\theta, e_z\}$ $\qquad\qquad$ $\{e_x\ e_y\ e_z\}$

$$\begin{tikzcd}
T_p\mathbb{R}^3 \arrow[r, "Q"] & T_x\mathbb{R}^3 \\
\mathbb{R}^3_p \arrow[u, "\vec{g}"] \arrow[r, "h"] \arrow[ur, "\tilde{f}"] \arrow[u, bend left=60, "\mathrm{curl}(\vec{g})"] & \mathbb{R}^3_x \arrow[u, "\vec{f}"'] \arrow[u, bend right=60, "\mathrm{curl}(\vec{f})"']
\end{tikzcd}$$

$$\mathrm{curl}(\vec{f}) = \nabla \times \vec{f} = \begin{bmatrix} f^3_y - f^2_z \\ f^1_z - f^3_x \\ f^2_x - f^1_y \end{bmatrix}$$

We want to show

$$\mathrm{curl}(\vec{g}) = \frac{1}{r}\det \begin{vmatrix} e_r & r e_\theta & e_z \\ D_r & D_\theta & D_z \\ g^r & r g^\theta & g^z \end{vmatrix} = \frac{1}{r}(g^z_\theta - r g^\theta_z)\hat{e}_r + (g^r_z - g^z_r)\hat{e}_\theta + \frac{1}{r}(g^\theta + r g^\theta_r - g^r_\theta)\hat{e}_z$$

$$= \{e_r\ e_\theta\ e_z\} \begin{bmatrix} \frac{1}{r}\overset{(3)}{g^z_\theta} - \overset{(2)}{g^\theta_z} \\ \overset{(1)}{g^r_z} - \overset{(3)}{g^z_r} \\ \overset{(2)}{g^\theta_r} - \frac{1}{r}\overset{(1)}{g^r_\theta} + \frac{1}{r}g^\theta \end{bmatrix}$$

pf. $\mathrm{curl}(\vec{f}) = \{e_x\ e_y\ e_z\}\underbrace{X(Df_x)}_{\text{call this } V_e(x)}$

These are the co-ords of $\mathrm{curl}(\vec{f})$ wrt basis $\{e\} = \{e_x\ e_y\ e_z\}$ and as a fcn of x

COB: $\{f\} = \{e\}Q$

$$\underbrace{\{f\}}_{\{e\}Q}V_f = V = \{e\}V_e$$

$$\{e\}QV_f \Rightarrow QV_f = V_e$$
$$V_f = Q^{-1}V_e$$

BUT we want $V_f(p)$. That means

(1) Co-ords in terms of basis $\{f\} = \{e_r\ e_\theta\ e_z\}$

(2) fcn of p

So for $\mathrm{curl}(\vec{g})$ we must transform the basis $\{e_x\ e_y\ e_z\}$ and the co-ord vector $V_e(x) = X(Df_x)$

COB

V_f $\quad$ $V_e(x)$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "Q"] & \mathbb{R}^3 \\
\mathbb{R}^3_p \arrow[u, "B"] \arrow[r, "Q"] & \mathbb{R}^3_x \arrow[u, "A"']
\end{tikzcd}$$

This is an attempt to represent the co-ord vectors V_e and V_f in the same space. The COB gives us $V_f = Q^{-1}V_e$ but we still have to express V_f in terms of p, not x

V_e would be rotated by Q to V_f

$$B = Q^{-1}AQ$$

$$
\begin{aligned}
\mathrm{curl}(\vec{f}) &= [e_x\ e_y\ e_z]\, V_c(x) \\
&= [e_x\ e_y\ e_z]\, \mathcal{X}(Df_x) \\
&= [e_r\ e_\theta\ e_z]\, \underbrace{Q^{-1}\, \mathcal{X}(Df_x)}_{\mathcal{X}(Q^{-1} Df_x Q)} \quad \text{by } \underline{\mathcal{X} \text{ Lemma}} \text{ to follow} \\
&= [e_r\ e_\theta\ e_z]\, \mathcal{X}(\underbrace{Q^{-1} Df_x Q}_{Q^{-1} M \Lambda^{-1} + Dg_p \Lambda^{-1}}) \quad \text{from sheet } ④
\end{aligned}
$$

Note that this only depends on p: $[Q(p)]^{-1} M(p) [\Lambda(p)]^{-1}$

The matrix entries depend on p, and inversion is an algebraic operation on these elts.

$$
Q^{-1} M \Lambda^{-1} = Q^{-1}(Q_\theta g e_z^T)\Lambda^{-1} \quad \text{from sheet } ⑤
$$

$$
= \begin{bmatrix} 0 & -\frac{1}{r} g^\theta & 0 \\ 0 & \frac{1}{r} g^r & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
Dg_p \Lambda^{-1} = \begin{bmatrix} g^r_r & \frac{1}{r} g^r_\theta & g^r_z \\ g^\theta_r & \frac{1}{r} g^\theta_\theta & g^\theta_z \\ g^z_r & \frac{1}{r} g^z_\theta & g^z_z \end{bmatrix}
$$

$$
\mathcal{X}(Q^{-1} M \Lambda^{-1} + Dg_p \Lambda^{-1}) = \mathcal{X}\left(\begin{bmatrix} g^r_r & \frac{1}{r}(g^r_\theta - g^\theta) & g^r_z \\ g^\theta_r & \frac{1}{r}(g^\theta_\theta - g^r) & g^\theta_z \\ g^z_r & \frac{1}{r} g^z_\theta & g^z_z \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} \frac{1}{r} g^z_\theta - g^\theta_z \\ g^r_z - g^z_r \\ g^\theta_r - \frac{1}{r}(g^r_\theta - g^\theta) \end{bmatrix}
$$

$$
\Rightarrow \quad \mathrm{curl}(\vec{g}) = [e_r\ e_\theta\ e_z] \begin{bmatrix} \frac{1}{r} g^z_\theta - g^\theta_z \\ g^r_z - g^z_r \\ g^\theta_r - \frac{1}{r}(g^r_\theta - g^\theta) \end{bmatrix}
$$

This establishes $\mathrm{Curl}(\vec{f})(h(p)) = Q\, \mathrm{curl}(\vec{g})(p)$ with regards to diagram on top sheet ⑧ Gemini says Scale factor Λ remains internal to diff operation $\mathcal{X}$ and does not appear in top edge map, only Q. □

Now we need to prove the $\underline{\mathcal{X} \text{ Lemma}}$ $\longrightarrow$

$\mathcal{X}$ Lemma

Given $\mathcal{X}$ as defined on sheet 7

$$\left.\begin{array}{l}\text{Arb } 3\times 3 \text{ matrix } A \\ Q \text{ ON matrix } [Q^{-1} = Q^T]\end{array}\right\} \Rightarrow Q^{-1}\mathcal{X}(A) = \mathcal{X}(Q^{-1}AQ)$$

Pf. Recall for any $\vec{v} \in \mathbb{R}^3$ $(A - A^T)v = \vec{C}_A \times v$ This is the def of $\vec{C}_A = \mathcal{X}(A)$

Define $B := Q^TAQ$

Then $(B - B^T)y = \vec{C}_B \times y$ for any $\vec{y} \in \mathbb{R}^3$

$$\begin{array}{l} (B - B^T)y \\ \quad\ \ \| \\ (Q^TAQ - Q^TA^TQ)y \end{array}$$

$$= Q^T(A - A^T)Qy = C_B \times y$$

$$(A - A^T)Qy = Q(C_B \times y)$$

$$= QC_B \times Qy$$ by property of cross-prod with ON matrices See my sheets on Sym

$$\vec{C}_A \times Qy = QC_B \times Qy$$

$$\Rightarrow \vec{C}_A = Q\vec{C}_B$$ why is this true?

If $a \times q = b \times q$ for any q

$a \times q - b \times q = 0$

$(a - b) \times q = 0$

only true if $(a-b) = \lambda q$

But since q is arb, we must have $a = b$ □

$$Q^{-1}\vec{C}_A = \vec{C}_{Q^TAQ}$$

$$\boxed{Q^{-1}\mathcal{X}(A) = \mathcal{X}(Q^TAQ)}$$

QED