M&T ch 3.5 — Vector Differential Calculus — 5/30/2026

Table 3.1 Vector Identities

$f: \mathbb{R}^3 \to \mathbb{R}$ $\quad \vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

1. $\nabla(f+g) = \nabla f + \nabla g$ $\Big\}$ linearity of ∇
2. $\nabla(cf) = c\,\nabla f$
3. $\nabla(fg) = \vec{\nabla f}\, g + f\, \vec{\nabla g}$ — Prod Rule
4. $\nabla(f/g) = \frac{1}{g^2}(g\nabla f + f\nabla g)$ — special case of (3)
5. $\operatorname{div}(F+G) = \operatorname{div} F + \operatorname{div} G$
6. $\operatorname{Curl}(F+G) = \operatorname{curl}(F) + \operatorname{curl}(G)$
7. $\vec{\nabla}(\vec{F}\cdot\vec{G}) = (F\cdot\nabla)\vec{G} + (G\cdot\nabla)\vec{F} + \vec{F}\times \operatorname{Curl}(\vec{G}) + \vec{G}\times \operatorname{curl}(\vec{F})$
8. $\operatorname{div}(fF) = \nabla f\cdot F + f\operatorname{div}(F)$
9. $\operatorname{div}(F\times G) = G\cdot\operatorname{curl}(F) - F\cdot\operatorname{curl}(G)$
10. $\operatorname{div}(\operatorname{curl}(F)) = 0$ $\quad [d^2=0]$
11. $\operatorname{curl}(fF) = \nabla f\times F + f\operatorname{curl}(F)$
12. $\operatorname{curl}(F\times G) = F\operatorname{div} G - G\operatorname{div} F + (G\cdot\nabla)F - (F\cdot\nabla)G$
13. $\operatorname{curl}(\operatorname{curl} F) = \operatorname{grad}(\operatorname{div} F) - \nabla^2 F$
14. $\operatorname{curl}(\nabla f) = 0$ $\quad [d^2=0]$
15. $\nabla(F\cdot F) = 2(F\cdot\nabla)F + 2F\times\operatorname{Curl}(F)$ [special case of 7]
16. $\nabla^2(fg) = \nabla^2 f\, g + 2\nabla f\cdot\nabla g + f\nabla^2 g$
17. $\operatorname{div}(\nabla f\times\nabla g) = 0$ [special case of 9]
18. $\nabla(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
19. Cyclic Perms $F\cdot(G\times H) = G\cdot(H\times F) = H\cdot(F\times G) = \det\begin{bmatrix} - F - \\ - G - \\ - H - \end{bmatrix}$
20. $H\cdot\big((F\times\nabla)\times G\big) = \big((H\cdot\nabla)G\big)\cdot F - (\nabla\cdot G)(H\cdot F)$
21. bac-cab rule: $(\vec{a}\times(\vec{b}\times\vec{c})) = \vec{b}(a\cdot c) - \vec{c}(\vec{a}\cdot\vec{b})$

<u>Some things need explanation</u>:

$$(F\cdot\nabla)G := \begin{bmatrix} (\Sigma F^i D_i)G^1 \\ (\Sigma F^i D_i)G^2 \\ (\Sigma F^i D_i)G^3 \end{bmatrix} = \begin{bmatrix} F^1 D_1 G^1 + F^2 D_2 G^1 + F^3 D_3 G^1 \\ F^1 D_1 G^2 \quad F^2 D_2 G^2 \quad F^3 D_3 G^2 \\ F^1 D_1 G^3 \quad F^2 D_2 G^3 \quad F^3 D_3 G^3 \end{bmatrix} = \begin{bmatrix} D_1 G^1 & D_2 G^1 & D_3 G^1 \\ D_1 G^2 & D_2 G^2 & D_3 G^2 \\ D_1 G^3 & D_2 G^3 & D_3 G^3 \end{bmatrix}\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix}$$

$$= DG_x(F)$$

$$\nabla^2 F := \begin{bmatrix} \nabla^2 F^1 \\ \nabla^2 F^2 \\ \nabla^2 F^3 \end{bmatrix} = \begin{bmatrix} F^1_{xx} + F^1_{yy} + F^1_{zz} \\ F^2_{xx} + F^2_{yy} + F^2_{zz} \\ F^3_{xx} + F^3_{yy} + F^3_{zz} \end{bmatrix}$$

$(F\times\nabla)\times G$ is worked out when I prove #20.

#7 $\boxed{\vec{\nabla}(F\cdot G) = (F\cdot\nabla)\vec{G} + (G\cdot\nabla)\vec{F} + \vec{F}\times(\vec{\nabla}\times\vec{G}) + \vec{G}\times(\vec{\nabla}\times\vec{F})}$

This is the last identity I did. (#8) serves as an introduction to the techniques [also 9, 10, 11, 12]

(1) Let L_1 be just the terms touched by D_1: $L_1 = D_1(F^1G^1 + F^2G^2 + F^3G^3)$
(This is the 1st vector component) $= D_1F^1G^1 + F^1D_1G^1 + D_1F^2G^2 + F^2D_1G^2 + D_1F^3G^3 + F^3D_1G^3$

Now for RHS, take 1st vector component

Term 1: $(\sum F^iD_i)G^1 = F^1D_1G^1 + F^2D_2G^1 + F^3D_3G^1$

cancel

2: $G^1D_1F^1 + G^2D_2F^1 + G^3D_3F^1$

3: $(F\times(\nabla\times G))^{①} = F^2(\nabla\times G)^3 - F^3(\nabla\times G)^2 = F^2(D_1G^2 - D_2G^1) - F^3(D_3G^1 - D_1G^3)$

4: $G^2(D_1F^2 - D_2F^1) - G^3(D_3F^1 - D_1F^3)$

The curl terms exist to cancel certain parts of the first 2 terms "convective derivs"

$$\Rightarrow R_1 = F^1D_1G^1 + G^1D_1F^1 + F^2D_1G^2 + F^3D_1G^3 + G^2D_1F^2 + G^3D_1F^3$$

(2) Does $L_1 = R_1$? ie $L_1(\nabla, F, G) = R_1(\nabla, F, G)$

$$\begin{matrix} G^1D_1F^1 + F^1D_1G^1 + \\ G^2D_1F^2 + F^2D_1G^2 + \\ G^3D_1F^3 + F^3D_1G^3 \end{matrix} \overset{?}{=} \begin{matrix} F_1D_1G^1 + G^1D_1F^1 + \\ F^2D_1G^2 + G^2D_1F^2 + \\ F^3D_1G^3 + G^3D_1F^3 \end{matrix}$$ ⊛ ✓

(3) $L_1(S\nabla, SF, SG) \overset{?}{=} R_1(S\nabla, SF, SG)$

⊛ is a generic template for $D_i(fg + hk + pq)$ so it still holds if we plug in perm-ed index values.

(4) → should have been included in (1)

$\langle SF, SG\rangle = \langle F, S^TSG\rangle = \langle F, G\rangle$

$L_2(\nabla, F, G) = L_1(S\nabla, SF, SG)$ because $(S\nabla)\cdot(SF\cdot SG) = (S\nabla)(F\cdot G)$

$(S\nabla)^{①} = D_2$ Likewise S^2 and D_3

$R_2 = R_1(S\nabla, SF, SG)$ because for first 2 terms

$\langle SF, S\nabla\rangle = \langle F, \nabla\rangle$ thus $\left(\sum (SF)^i(SD)^i\right)(SG)^{①} = \sum F^iD_iG^2$

and for curl terms

$$S(F\times(\nabla\times G)) \text{ this brings 2}^{nd}\text{ elt to top position}$$
$$\|$$
$$SF\times(S\nabla\times SG)$$

□

#8 $\operatorname{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f \operatorname{div}(\vec{F})$

i.e $\nabla \cdot (fF) = \nabla f \cdot F + f\, \nabla \cdot F$

It's trivial to prove this directly, but we want to do all of these by perm sym and this easy one is a prototype for all div

Key we must break the LHS and RHS into pieces

(1) The pieces are defined such that $L_1 \overset{S}{\rightsquigarrow} L_2 \overset{S}{\rightsquigarrow} L_3$
$R_1 \overset{S}{\rightsquigarrow} R_2 \overset{S}{\rightsquigarrow} R_3$

That is to say $L_2(\nabla, f, F) = L_1(S\nabla, f, SF)$

and likewise for the other components (pieces)

cylic perm (1 3 2) $S = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$

See my Sym sheets

(2) We must show $L_1(\nabla, f, F) = R_1(\nabla, f, F)$ This gives the algebraic Relation ★

(3) we must show $L_2(S\nabla, f, SF) = R_1(S\nabla, f, SF)$ and likewise for S^2

we do this by showing the relation in (2) is generic

(1) write it out to find the pieces (basically this proves the identity!)

$L = D_1(fF^1) + D_2(fF^2) + D_3(fF^3)$

$R = (D_1 f)F^1 + (D_2 f)F^2 + D_3 f\, F^3 + f(D_1F^1) + f(D_2F^2) + f(D_3F^3)$

components of vector

$1 \to 3 \to 2$

$D_1(fF^1) \overset{S}{\rightsquigarrow} D_2(fF^2)$

$(D_1 f)F^1 + f(D_1F^1) \overset{S}{\rightsquigarrow} (D_2 f)F^2 + f(D_2F^2)$

indices in position at top

$1 \to 2 \to 3$

So

$L_1 := D_1(fF^1)$ $\qquad R_1 := (D_1 f)F^1 + f(D_1F^1)$

These are all terms touched by D_1 — this is important for a harder problem, like #9

(2) $L_1 \overset{?}{=} R_1$ This is a trivial exercise of prod rule

$$D_1(fF^1) = (D_1 f)F^1 + f(D_1F^1) \checkmark ★$$

(3) ★ is a generic relation that does not depend on indices D_1 or F^1

$$D_i(fg) = (D_i f)g + f(D_i g)$$

$S\nabla = S\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} D_2 \\ D_3 \\ D_1 \end{bmatrix}$ $\qquad SF = \begin{bmatrix} F^2 \\ F^3 \\ F^1 \end{bmatrix}$ $\qquad$ (S would have no effect on Scalar f)

$\Rightarrow L_1(S\nabla, f, SF) = R_1(S\nabla, f, SF)$ and the same for S^2

$\Rightarrow L_1 + L_1(S) + L_1(S^2) = R_1 + R_1(S) + R_1(S^2)$

$\Rightarrow D_1(fF^1) + D_2(fF^2) + D_3(fF^3) = (D_1 f)F^1 + fD_1F^1 + (D_2 f)F^2 + fD_2F^2 + (D_3 f)F^3 + fD_3F^3$

$\boxed{\text{QED}}$

#9 $\quad \mathrm{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot\mathrm{curl}(\vec{F}) - \vec{F}\cdot\mathrm{curl}(\vec{G})$

i.e. $\quad \nabla\cdot(F\times G) = G\cdot(\nabla\times F) - \vec{F}\cdot(\nabla\times G)$

Follow the plan established in #8

(1) write it out so we can establish the pieces

$$LHS = D_1(F\times G)^1 + D_2(F\times G)^2 + D_3(F\times G)^3$$

Take $L_1 = D_1(F\times G)^1$ this is the only term touched by D_1

$D_1(F\times G)^1 \xrightarrow{S} D_2(F\times G)^2$ we also know the perm works fine on the elts of $(F\times G)^1$ – see my Sym sheets

Really tricky!

$$RHS = [G^1\ G^2\ G^3]\left(\underbrace{\begin{bmatrix}0\\ -D_1F^3\\ D_1F^2\end{bmatrix}}_{D_1 \text{ piece}} + \begin{bmatrix}D_2F^3\\ 0\\ -D_2F^1\end{bmatrix} + \begin{bmatrix}-D_3F^2\\ D_3F^1\\ 0\end{bmatrix}\right) - [F^1F^2F^3]\left(\underbrace{\begin{bmatrix}0\\ -D_1G^3\\ D_1G^2\end{bmatrix}}_{D_1 \text{ piece}} + \begin{bmatrix}D_2G^3\\ 0\\ -D_2G^1\end{bmatrix} + \begin{bmatrix}\ \end{bmatrix}\right)$$

$$R_1 = -G^2D_1F^3 + G^3D_1F^2 - (-F^2D_1G^3 + F^3D_1G^2)$$

we can also directly compute R_2 as the D_2 piece and we confirm that it is in fact a rotation of the indices

$G^1D_2F^3 + -G^3D_2F^1 - (F^1D_2G^3 - F^3D_2G^1)$

(2) show $L_1(\nabla, F, G) = R_1(\nabla, F, G)$

$$D_1(F\times G)^1 \stackrel{?}{=} -G^2D_1F^3 + G^3D_1F^2 + F^2D_1G^3 - F^3D_1G^2 \quad ⭑$$

$D_1(F^2G^3 - F^3G^2)$

$G^3D_1F^2 + F^2D_1G^3 - G^2D_1F^3 - F^3D_1G^2$ ✓

(3) Algebraic relation ⭑ is a generic template; it holds for any D_i and any 4 fcns

$$D_i(fg - hk) = g\,D_if + f\,D_ig - k\,D_ih - h\,D_ik$$

This means it holds if we perm the components and also for S^2

$$L_1(S\nabla, SF, SG) = R_1(S\nabla, SF, SG)$$

$$\Rightarrow L_1 + L_1(S) + L_1(S^2) = R_1 + R_1(S) + R_1(S^2)$$

identity is established $\square$

#10 $\quad \mathrm{div}(\mathrm{curl}(F)) = 0$ Here the proof falls out before we can even employ the sym

$$D_1(\nabla\times F)^1 + D_2(\nabla\times F)^2 + D_3(\nabla\times F)^3$$

$$D_1(D_2F^3 - D_3F^2) + D_2(D_3F^1 - D_1F^3) + D_3(D_1F^2 - D_2F^1)$$

Collect all D_1 terms

$$D_1D_2F^3 - D_1D_3F^2 + D_2D_3F^1 - D_2D_1F^3 + D_3D_1F^2 - D_3D_2F^1 = 0$$

$\boxed{\text{QED}}$

#11 $\text{Curl}(fF) = \nabla f \times F + f\,\text{Curl}(F)$ #8

i.e. $\nabla \times (fF) = \nabla f \times F + f\,\nabla \times F$ just like $\nabla \cdot (fF) = \nabla f \cdot F + f\,\nabla \cdot F$

again we follow the basic ideas layed out in #8, but for these vector valued ids with cross-prod, we take the 1st vector component as L_1 and R_1

① Take $L_1 = D_2(fF^3) - D_3(fF^2)$ and $R_1 = D_2 f \cdot F^3 - D_3 f F^2 + f(D_2F^3 - D_3F^2)$

If we apply the cyclic index perm (1 3 2) [which results when S is applied to the vector components] do we know $L_1 \xrightarrow{S} L_2$ and $R_1 \xrightarrow{S} R_2$?

Refer to my Cross Prod and Sym Sheets 5/17/2026 stored with Sym papers.

Yes for LHS $S(\nabla \times fF) = (S\nabla) \times (f\,SF)$

so L_2 is achived from L_1 by rotating indices: $L_2 = D_3(fF^1) - D_1(fF^3)$

For RHS $S(\nabla f \times F) + S(f\,\nabla \times F) = (S\nabla f) \times (SF) + f\,(S\nabla \times SF)$

Thus $R_2 = D_3 f F^1 - D_1 f F^3 + f(D_3F^1 - D_1F^3)$ and similiarly for R_3

② $L_1(\nabla, f, F) \overset{?}{=} R_1(\nabla, f, F)$

Yes its just the prod rule:

$$D_2(fF^3) - D_3(fF^2) = \underbrace{D_2 f F^3 - D_3 f F^2 + f(D_2F^3 - D_3F^2)} \quad ★$$

③ Does it still hold (equality) for $\underbrace{L_1(S\nabla, f, SF)}_{L_2} = \underbrace{R_1(S\nabla, f, SF)}_{R_2}$

Yes because algebraic relation ★ is a statement about the prod rule, indep of particular partials or fcns:

$$D_i(fh) - D_j(fg) = D_i f\,h - D_j f g + f(D_i h - D_j h)$$

$$\Rightarrow \begin{bmatrix} L_1 \\ L_1(S) \\ L_1(S^2) \end{bmatrix} = \begin{bmatrix} R_1 \\ R_1(S) \\ R_1(S^2) \end{bmatrix}$$ and the identity is proven □

#12 $\nabla\times(F\times G) = (\nabla\cdot G)\vec{F} - (\nabla\cdot F)\vec{G} + (G\cdot\nabla)\vec{F} - (F\cdot\nabla)\vec{G}$

At the end, we shall discuss a heuristic method

1. Following the same plan, take the first component:

$L_1 = D_2(F\times G)^3 - D_3(F\times G)^2$ we know $S(\nabla\times(F\times G)) = S\nabla\times S(F\times G)$
$= S\nabla\times(SF\times SG)$

So we can get 2^{nd} comp by indices perm (also 3^{rd})

The RHS requires some explanation

$$RHS = (\nabla\cdot G)\begin{bmatrix}F^1\\F^2\\F^3\end{bmatrix} - (\nabla\cdot F)\begin{bmatrix}G^1\\G^2\\G^3\end{bmatrix} + \begin{bmatrix}G\cdot\nabla F^1\\G\cdot\nabla F^2\\G\cdot\nabla F^3\end{bmatrix} - \begin{bmatrix}F\cdot\nabla G^1\\F\cdot\nabla G^2\\F\cdot\nabla G^3\end{bmatrix}$$

$$R_1 = (\nabla\cdot G)F^1 - (\nabla\cdot F)G^1 + \left(\sum^3 G^iD_i\right)F^1 - \left(\sum F^iD_i\right)G^1$$

What happens after we apply the perm S? $(S\nabla)\cdot SG = \nabla\cdot G = D_1G^1+D_2G^2+D_3G^3$ Sums over all indices are invariant

Likewise $\left(\sum^3 G^iD_i\right)$ is invariant

$$\Rightarrow R_1 \overset{S}{\rightsquigarrow} R_2 = (\nabla\cdot G)F^2 - (\nabla\cdot F)G^2 + \left(\sum G^iD_i\right)F^2 - \left(\sum F^iD_i\right)G^2$$

2. Now show $L_1(\nabla, F, G) = R_1(\nabla, F, G)$

$$\underbrace{D_2(F\times G)^3 - D_3(F\times G)^2}_{D_2(F^1G^2-F^2G^1) + D_3(F^1G^3-F^3G^1)} \overset{?}{=} F^1\left(\sum D_iG^i\right) - G^1\left(\sum D_iF^i\right) + \sum G^iD_iF^1 - \sum F^iD_iG^1$$

$$= (D_2F^1)G^2 + F^1D_2G^2 - D_2F^2G^1 - F^2D_2G^1 + D_3F^1G^3 + F^1D_3G^3 - D_3F^3G^1 - F^3D_3G^1$$

$$= F^1(D_2G^2+D_3G^3) + F^1D_1G^1 - G^1(D_2F^2+D_3F^3) - G^1D_1F^1 + G^2D_2F^1 + G^3D_3F^1 + G^1D_1F^1 - F^2D_2G^1 - F^3D_3G^1 - F^1D_1G^1 \quad (\star)$$

$$= F^1(\nabla\cdot G) - G^1(\nabla\cdot F) + \sum G^iD_iF^1 - \sum F^iD_iG^1$$ and this is R_1 ✓

3. Does $L_1(S\nabla, SF, SG) = R_1(S\nabla, SF, SG)$?

Yes because $\star$ is a generic algebraic relation between 2 partial derivs D_i, D_j and a combo of 6 fcns (f, g, h) and (p, q, r)

$(\star) \Rightarrow D_2(fq-gp) + D_3(fr-hp) = f(D_2q+D_3r) - p(D_2g+D_3h) + qD_2f + rD_3f$
$- (gD_2p - gD_3p)$

Finally replace D_2, D_3 by D_i, D_j

Thus the same relation holds for a cyclic perm of the indices $\square$

cont'd →

Remark about **#12**: Let's try to expand $\nabla\times(F\times G)$ by bac-cab rule

$$a\times(b\times c) = \vec{b}(a\cdot c) - \vec{c}(a\cdot b)$$

$$\nabla\times(F\times G) = \vec{F}(\nabla\cdot G) - \vec{G}(\nabla\cdot F)$$ This is not the correct formula

But if we write $\nabla = \nabla_F + \nabla_G$ where ∇_F only operates on F, same for G

$$\nabla\times(F\times G) = \nabla_F\times(F\times G) + \nabla_G\times(\overset{F}{\cancel{\nabla}}\times G)$$

$$\vec{F}(\nabla_F\cdot G) - G(\nabla_F\cdot F) + \vec{F}(\nabla_G\cdot G) - \vec{G}(\nabla_G\cdot F)$$

flip around so ∇_F operates on F, same for G

$$(G\cdot\nabla_F)\vec{F} - (\nabla_F\cdot F)\vec{G} + (\nabla_G\cdot G)\vec{F} - (F\cdot\nabla_G)\vec{G}$$

Remove subscripts and this matches orig formula. □

#13 $$\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\nabla\cdot F) - \underbrace{\nabla^2\vec{F}}_{\begin{bmatrix}\nabla\cdot\nabla F^1\\ \nabla\cdot\nabla F^2\\ \nabla\cdot\nabla F^3\end{bmatrix}}$$

① Write out the 1st component

$$L_1 = D_2(\nabla\times F)^3 - D_3(\nabla\times F)^2 = D_2(D_1F^2 - D_2F^1) + D_3(D_1F^3 - D_3F^1)$$

We must expand RHS:

$$\vec{\nabla}(\nabla\cdot F) - \nabla^2\vec{F} = \nabla(D_1F^1 + D_2F^2 + D_3F^3) - \begin{bmatrix}\nabla\cdot\nabla F^1\\ \nabla\cdot\nabla F^2\\ \nabla\cdot\nabla F^3\end{bmatrix}$$

$$= \begin{bmatrix} D_1^2F^1 + D_1D_2F^2 + D_1D_3F^3\\ D_2D_1F^1 + D_2^2F^2 + D_2D_3F^3\\ D_3D_1F^1 + D_3D_2F^2 + D_3^2F^3\end{bmatrix} - \begin{bmatrix}\nabla\cdot\nabla F^1\\ \nabla\cdot\nabla F^2\\ \nabla\cdot\nabla F^3\end{bmatrix}$$

Thus $R_1 = \cancel{D_1^2F^1} + D_1D_2F^2 + D_1D_3F^3 - \cancel{D_1^2F^1} - D_2^2F^1 - D_3^2F^1$

$S \downarrow$

$R_2 = D_2D_3F^3 + D_2D_1F^1 - D_3^2F^3 - D_1^2F^1$ so rotating indices gives the 2nd row

② We must show $L_1 = R_1$

$$D_2(D_1F^2 - D_2F^1) + D_3(D_1F^3 - D_3F^1) \overset{?}{=} D_1D_2F^2 + D_1D_3F^3 - D_2^2F^1 - D_3^2F^1 \checkmark$$

$$D_2D_1F^2 - D_2^2F^1 + D_3D_1F^3 - D_3^2F^1$$

C^2 smoothness means we can interchange

③ This is a generic equation

$$D_jD_ig - D_j^2f + D_kD_ih - D_k^2f = D_iD_jg + D_iD_kh - D_j^2f - D_k^2f$$

Thus we can plug in the perm'd index values and get $L_1(S\nabla, S\nabla SF) = R_1(S\nabla, S\nabla, SF)$

Thus the identity is proven □

#14 Curl(grad f) = 0 $\quad \nabla \times \nabla f = 0 \quad$ again $d^2 = 0$

$$\begin{vmatrix} i & j & k \\ D_1 & D_2 & D_3 \\ D_1 f & D_2 f & D_3 f \end{vmatrix} = \begin{bmatrix} D_2 D_3 f - D_3 D_2 f \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \end{bmatrix}$$

← from interchange of partials for smooth fcns

The other terms are generated from perm sym of cross-prod which is discussed earlier, and is given in detail in my 5/17/2026 sheets 'Cross Prod and Sym' in Sym folder □

#15 special case of #7

#16 $\nabla^2(fg) = \nabla^2 f\, g + 2\vec{\nabla f} \cdot \vec{\nabla g} + f\, \nabla^2 g$

$\nabla \cdot \nabla(fg) = \nabla \cdot (\vec{\nabla f}\, g + f\, \vec{\nabla g})$ $\qquad$ (#8) $\nabla \cdot (f\vec{F}) = \nabla f \cdot F + f\, \nabla \cdot F$

$= \nabla g \cdot \nabla f + g\, \nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\, \nabla \cdot \nabla g$

$= g\, \nabla^2 f + 2\, \vec{\nabla f} \cdot \vec{\nabla g} + f\, \nabla^2 g$ □

#17 $\nabla \cdot (\nabla f \times \nabla g) = 0$

(#9) $\operatorname{div}(F \times G) = G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G$

$= \nabla g \cdot (\nabla \times \nabla f) - \nabla f \cdot (\nabla \times \nabla g) \underset{\#14}{=} \nabla g \cdot (\vec{0}) - \nabla f \cdot \vec{0} = 0$

#18 $\nabla \cdot (f \nabla g - g \nabla f) = f\, \nabla^2 g - g\, \nabla^2 f$ $\quad$ special case of #8

#19 $F \cdot G \times H = \det \begin{bmatrix} - F - \\ - G - \\ - H - \end{bmatrix} = (+1) \det \begin{bmatrix} - H - \\ - F - \\ - G - \end{bmatrix} = H \cdot (F \times G)$

$= (-1)^2 \det \begin{bmatrix} - G - \\ - H - \\ - F - \end{bmatrix} = G \cdot (H \times F)$

#20 will be on next sheet → → → → →

#21 bac-cab rule $\boxed{a \times (b \times c) = \vec{b}(a \cdot c) - \vec{c}(a \cdot b)}$

pf. Follow the perm sym programme: (just like we did in #12)

1. $L_1 = a^{2}(b \times c)^{3} - a^{3}(b \times c)^{2} = a^2(b^1c^2 - c^1b^2) + a^3(b^1c^3 - b^3c^1)$

$R_1 = b^1(a^1c^1 + a^2c^2 + a^3c^3) - c^1(a^1b^1 + a^2b^2 + a^3b^3)$

2. $a^2(b^1c^2 - c^1b^2) + a^3(b^1c^3 - b^3c^1) \overset{?}{=} b^1(\cancel{a^1c^1} + a^2c^2 + a^3c^3) - c^1(\cancel{a^1b^1} + a^2b^2 + a^3b^3)$

all terms match

3. $L_1(Sa, Sb, Sc) \overset{?}{=} R_1(Sa, Sb, Sc)$ $\qquad$ $a \cdot b = Sa \cdot Sb$ $\qquad$ $\langle Sa, Sb \rangle = \langle a, S^T S b \rangle$

we know this side is just rotating the indices
$a^3(b^2c^3 - c^2b^3) - a^1(b^2c^1 - b^1c^2)$

$\hookrightarrow b^2(\Sigma a^ic^i) - c^2(\Sigma a^ib^i)$ $\qquad$ $= \langle a, S^{-1} S b \rangle$

$= \langle a, b \rangle$ ✓

So they are, in fact, equal and bac-cab identity is established □

#20 $\boxed{\vec{H}\cdot\left((\vec{F}\times\vec{\nabla})\times\vec{G}\right) = \left[(H\cdot\nabla)\vec{G}\right]\cdot F - (H\cdot F)(\nabla\cdot G)}$

$$[F^1\ F^2\ F^3]\begin{bmatrix}(\sum H_i D_i)G^1\\ (\sum H^i D_i)G^2\\ (\sum H^i D_i)G^3\end{bmatrix} - \langle H, F\rangle\langle\nabla, G\rangle$$

I'm not going to do the perm scheme here because I want to explore the terms

First let's expand

∇ only operates on G

$$(F\times\nabla)\times G = \begin{bmatrix}F^2D_3 - F^3D_2\\ F^3D_1 - F^1D_3\\ F^1D_2 - F^2D_1\end{bmatrix}\times\begin{bmatrix}G^1\\ G^2\\ G^3\end{bmatrix} = \begin{bmatrix}(F^3D_1 - F^1D_3)G^3 - (F^1D_2 - F^2D_1)G^2\\ (F^1D_2 - F^2D_1)G^1 - (F^2D_3 - F^3D_2)G^3\\ (F^2D_3 - F^3D_2)G^2 - (F^3D_1 - F^1D_3)G^1\end{bmatrix}$$

note perm sym exists

$$= \begin{matrix}F^3G^3_x - F^1G^3_z - F^1G^2_y + F^2G^2_x\\ F^1G^1_y - F^2G^1_x - F^2G^3_z + F^3G^3_y\\ F^2G^2_z - F^3G^2_y - F^3G^1_x + F^1G^1_z\end{matrix} = \begin{bmatrix}(-G^3_z - G^2_y) & G^2_x & G^3_x\\ G^1_y & (-G^1_x - G^3_z) & G^3_y\\ G^1_z & G^2_z & (-G^2_y - G^1_x)\end{bmatrix}\begin{bmatrix}F^1\\ F^2\\ F^3\end{bmatrix}$$

$$= [DG_x]^T F - (\nabla\cdot G)F$$

Then

$$H^T\left[(F\times\nabla\times G\right] = H^T[DG_x]^T F - (\nabla\cdot G)H^T F$$

$$= [DG_x H]^T F -$$

$$= F^T[DG_x H] - (\nabla\cdot G)\, H\cdot F$$

see top $= F\cdot\left[(H\cdot\nabla)G\right] - (\nabla\cdot G)H\cdot F$

I wish I could make the 1st term $H\cdot\left[(F\cdot\nabla)G\right]$, that would really be having the H outside but that is not possible unless $DG_x = (DG_x)^T$
(This would happen if $\nabla\times G = 0$)

□

Let's do a few Ch 3.5 Exercises before establishing grad, curl, div in cylindrical and spherical co-ords

p.196

(7) Let F be arb vf. Is it true $\vec\nabla \times \vec F \perp \vec F$ always?

$F \cdot (\nabla \times F) = 0$? Take $F = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$ Then $\begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ z & x & y \end{vmatrix} = \begin{bmatrix} 1-0 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} z \\ x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = z + x \cdot 1 y \neq 0$ □

(8) Very common position vector calculations $\vec r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $r = \|r\|_2 = \sqrt{x^2+y^2+z^2} = [x^2+y^2+z^2]^{1/2}$

(a) Show $\nabla(r^n) = n r^{n-2} \vec r$ $\boxed{\text{Special case } n=-1 \quad \nabla(1/r) = \frac{-1}{r^3}\vec r}$

$\nabla(\ln r) = \frac{1}{r^2}\vec r$

We want to use Sym methods [see my sheets 'Sym and Derivs' 5/13/2026]

This is Lemma 1

Cyclic perm (1 3 2) rotates elts up in a col vector. $S = S_{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} y \\ z \\ x \end{bmatrix}$

If $f(x) = f(Sx)$ $Df_x = Df_{Sx} S \Rightarrow S \nabla f(x) = \nabla f(Sx)$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}}_{\begin{bmatrix} D_2 \\ D_3 \\ D_1 \end{bmatrix}} (f)(x) = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} (f)(Sx)$$

evaluated at pt Sx

so the recipe is $\pi_1 S^p \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} f(x) = \pi_1 \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} f(S^p x)$

For $p=0$, we must compute $D_1 f(x) = D_1 f(x)$, we get the fcn $D_1 f$ ($p=2$)

For $p=1$, we just plug into this fcn: $D_2 f(x) = (D_1 f)(Sx)$ and $D_3 f(x) = (D_1 f)(S^2 x)$

$r(Sx) = r(x)$

$f(x,y,z) = r^n$ $\frac{\partial f}{\partial x} = D_1 f = \frac{n}{2}[x^2+y^2+z^2]^{\frac{n}{2}-1}(2x) = n[x^2+y^2+z^2]^{\frac{n-2}{2}} x$

$= n r^{n-2} x$

Now by Sym $D_2 f(x,y,z) = D_1 f(Sx) = D_1 f(y,z,x) = n r^{n-2} y$

$D_3 f \qquad = n r^{n-2} z$

$\Rightarrow \boxed{\nabla(r^n) = n r^{n-2} \vec r}$

For $\nabla(\ln r)$ $f(x) = \ln r$ $\frac{\partial f}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\frac{1}{\cancel{2}}\frac{1}{r}\cancel{2}x = \frac{1}{r^2}x$

By Sym $D_2 f(x) = D_1 f(y,z,x) = \frac{1}{r^2} y$

$D_3 f \qquad = \frac{1}{r^2} z$

$\Rightarrow \nabla(\ln r) = \frac{1}{r^2}\vec r$

□

cont'd →

(b) $\nabla^2(r^n) = n(n+1)r^{n-2}$ special case $n=-1$ $\nabla^2(1/r) = 0$

$\nabla\cdot\nabla(r^n) = \nabla\cdot(\underbrace{n r^{n-2}\vec{r}}_{F})$ so here $\vec{F}(x) = n r^{n-2}\vec{r} = n r^{n-2}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

I want to use sym to help compute $D_1F^1 + D_2F^2 + D_3F^3$

$F^1(x,y,z) = n r^{n-2} x$ r is invariant under perms so we <u>could</u> say it is <u>obvious</u> that $D_y F^2(x,y,z) = (D_x F^1)\big|_{(y,z,x)}$ and be done.

Doing it in a careful and rigorous way:

since $r \circ S = r$ we see $F(Sx) = SF(x)$

<u>Lemma 2</u> $F(Sx) = SF(x)$ (equivariant) $\Rightarrow DF_{Sx}S = DS_{Fx}DF_x = S\,DF_x$

$\Rightarrow DF_{Sx} = S\,DF_x\,S^T$ and this gives $D_2F^2(x) = (D_1F^1)(Sx)$

$D_3F^3(x) = (D_1F^1)(S^2x)$

Pf. we have

$$\begin{bmatrix} F^1_x & F^1_y & F^1_z \\ F^2_x & F^2_y & F^2_z \\ F^3_x & F^3_y & F^3_z \end{bmatrix}\Bigg|_{Sx} = \underset{S}{\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}}\underset{DF_x}{\begin{bmatrix} F^1_x & F^1_y & F^1_z \\ F^2_x & F^2_y & F^2_z \\ F^3_x & F^3_y & F^3_z \end{bmatrix}}\underset{S^T}{\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}$$

when S acts from the back, it Perms <u>cols</u>. From front, it Perms <u>rows</u>

$$= \begin{bmatrix} F^2_y & F^2_z & F^2_x \\ F^3_y & F^3_z & F^3_x \\ F^1_y & F^1_z & F^1_x \end{bmatrix}\Bigg|_x$$

$\Rightarrow (D_2F^2)(x) = (D_1F^1)(Sx)$

$(D_3F^3)(x) = (D_2F^2)(Sx) = (D_1F^1)(S^2x)$ $\square$

Thus we can compute $D_1F^1(x)$ and then plug into it to get D_2F^2 and D_3F^3

$D_1F^1(x) = \frac{\partial}{\partial x}(n r^{n-2} x) = n r^{n-2} + n x (n-2) r^{n-3}\frac{\partial r}{\partial x}$ $\qquad \frac{\partial r}{\partial x} = \frac{x}{r}$

$= n r^{n-2} + n(n-2)x^2 \frac{r^{n-3}}{r} \rightarrow \frac{r^{n-2}}{r^2}$

$= n r^{n-2}\left[1 + \frac{(n-2)x^2}{r^2}\right]$

$\Rightarrow D_1F^1 + D_2F^2 + D_3F^3 = n r^{n-2}\left[1 + \frac{(n-2)x^2}{r^2}\right] + n r^{n-2}\left[1 + \frac{(n-2)y^2}{r^2}\right] + n r^{n-2}\left[1 + \frac{(n-2)z^2}{r^2}\right]$

$= n r^{n-2}\left[3 + \frac{(n-2)}{\cancel{r^2}}\cancel{(x^2+y^2+z^2)}\right]$

$= n(3 + (n-2))r^{n-2}$

$= n(n+1)r^{n-2}$

$\square$

cont'd →

(8) cont'd

(c) Show $\nabla\cdot(r^n\vec{r}) = (n+3)r^n$ Special case $n=-3$
$\nabla\cdot(\frac{1}{r^3}\vec{r}) = 0$

This is very similiar to what we did in (b) $\vec{F} = r^n \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$D_1F^1(x) = \frac{\partial}{\partial x}(r^n x) = r^n + n x \frac{\partial r}{\partial x} r^{n-1} = n r^{n-1}\frac{x^2}{r} = n r^n \frac{x^2}{r^2} + r^n$$

Then
$$D_1F + D_2F^2 + D_3F^3 = r^n\left[1 + \frac{nx^2}{r^2}\right] + r^n\left[1 + \frac{ny^2}{r^2}\right] + r^n\left[1 + \frac{nz^2}{r^2}\right]$$
$$= r^n\left[3 + \cancel{\frac{n}{r^2}(x^2+y^2+z^2)}\right] = (n+3)r^n \qquad \square$$

(d) $\nabla\times(r^n\vec{r}) = 0$ Special case $n=0$ $\vec{\nabla}\times\vec{r} = 0$

$\vec{F} = r^n \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ since $S(\nabla\times F) = S\nabla\times SF$ (*)

I can just compute the 1^{st} component and the others must follow by this perm of indices $1 \to 2 \to 3 \to 1$

Remember: the elts in the vector go to positions $1 \to 3 \to 2 \to 1$

Project 1^{st} comp from (*):

$$\pi_1(S(\nabla\times F)) = \pi_1(S\nabla\times SF)$$
$$\pi_1\begin{bmatrix} (\nabla\times F)^2 \\ (\nabla\times F)^3 \\ (\nabla\times F)^1 \end{bmatrix} = \pi_1\begin{bmatrix} (S\nabla\times SF)^1 \\ \vdots \\ \ \end{bmatrix} = \pi_1\begin{bmatrix} D_3F^1 - D_1F^3 \\ \vdots \\ \ \end{bmatrix}$$

Compute $(\nabla\times F)^1$

$$D_2F^3 - D_3F^2 = D_2(r^n z) - D_3(r^n y)$$
$$= \frac{\partial}{\partial y}r^n z - \frac{\partial}{\partial z}(r^n y) = n r^{n-1}\frac{\partial r}{\partial y} z - n r^{n-1}\frac{\partial r}{\partial z} y$$
$$= n r^{n-1}\frac{y}{r} z - n r^{n-1}\frac{z}{r} y$$
$$= \frac{n r^{n-1}}{r}\left[yz - zy\right] = 0$$

$(\nabla\times F)^2 = 0$ because if I perm the indices in $(\nabla\times F)^1 = 0$ the operation is vaccuous, the number 0 is unchanged.

$(\nabla\times F)^3 = 0$

QED