MBT 3/10/2022

Ch 3.5 $f: \mathbb{R}^3 \to \mathbb{R}$ DST tonight ①

Table 3.1

1. $\nabla(f+g) = \nabla f + \nabla g$ 2. $\nabla(cf) = c\vec{\nabla} f$ [$\nabla$ linear]
3. $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$ [Prod rule]
4. $\nabla(f/g) = \frac{1}{g^2}(g\vec{\nabla} f - f\vec{\nabla} g)$
5. $\text{div}(\vec{F}+\vec{g}) = \text{div}(\vec{F}) + \text{div}(\vec{G})$
6. $\text{Curl}(F+G) = \text{Curl}(F) + \text{Curl}(G)$
7. $\vec{\nabla}(\vec{F}\cdot\vec{G}) = (F\cdot\nabla)\vec{G} + (G\cdot\nabla)F + F\times\text{Curl}(G) + G\times\text{Curl}(F)$
8. $\text{div}(fF) = f\,\text{div}\,F + \vec{F}\cdot\vec{\nabla} f$ ($\nabla\cdot(f\vec{F}) = f(\nabla\cdot F) + F\cdot\nabla f$)
9. $\text{div}(F\times G) = G\cdot\text{curl}\,F - F\cdot\text{Curl}(G)$
10. $\text{div}(\text{Curl}\,F) = 0$ $[d^2=0]$ $\nabla\cdot(\nabla\times F)=0$
11. $\text{Curl}(fF) = f\,\text{Curl}\,F + \nabla f\times F$
12. $\text{curl}(F\times G) = F\,\text{div}(G) - G\,\text{div}\,F + (G\cdot\nabla)F - (F\cdot\nabla)G$
13. $\text{Curl}(\text{Curl}\,F) = \text{grad}(\text{div}\,F) - \nabla^2 F$
14. $\text{Curl}(\nabla f) = 0$ $[d^2=0]$ $\nabla\times\nabla f = 0$
15. $\nabla(F\cdot F) = 2(F\cdot\nabla)F + 2F\times(\text{Curl}\,F)$
16. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g)$
17. $\text{div}(\nabla f\times\nabla g) = 0$
18. $\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
19. (cyclic perms) $H\cdot(F\times G) = G\cdot(H\times F) = F\cdot(G\times H)$
20. $H\cdot((F\times\nabla)\times G) = ((H\cdot\nabla)G)\cdot F - (H\cdot F)(\nabla\cdot G)$
21. $A\times(B\times C) = \vec{B}(A\cdot C) - \vec{C}(A\cdot B)$ [BAC CAB]

<u>NOTE</u>:

$$V = (\vec{F}\cdot\vec{\nabla})\vec{G} = \begin{bmatrix} F^1 D_1 G^{①} + F^2 D_2 G^{①} + F^3 D_3 G^{①} \\ F^1 D_1 G^2 + F^2 D_2 G^2 + F^3 D_3 G^2 \\ F^1 D_1 G^3 + F^2 D_2 G^3 + F^3 D_3 G^3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} F^1 D_x & F^2 D_y & F^3 D_z \end{bmatrix}}_{(F\cdot\nabla)} \begin{bmatrix} G^1 \\ G^2 \\ G^3 \end{bmatrix}$$

(ex 2) $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ scalar $r = \|r\|_2 = (x^2+y^2+z^2)^{1/2}$

Compute ⓐ $\vec{\nabla} r$ and ⓑ $\nabla\cdot(r\vec{r})$

$\vec{\nabla} r = [r_x \;\; r_y \;\; r_z]^T =$

let $\rho = x^2+y^2+z^2$

$r = \rho^{1/2}$ $r_x = \frac{1}{2}\rho^{-1/2} 2x = \frac{x}{r}$

$r_y = \frac{y}{r}$

$r_z = z/r$

$\Rightarrow \vec{\nabla} r = \begin{bmatrix} x/r \\ y/r \\ z/r \end{bmatrix} = \frac{1}{r}\vec{r} = \hat{r}.$

// ↳ this pen

$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ch 3.5 3/11/2022

ⓑ $\nabla \cdot (r\vec{r}) = r(\nabla \cdot \vec{r}) + \vec{r} \cdot \vec{\nabla r}$

$= r(1+1+1) + \vec{r} \cdot \frac{1}{r}\vec{r} = 3r + \frac{1}{r} r^2 = 4r$ □

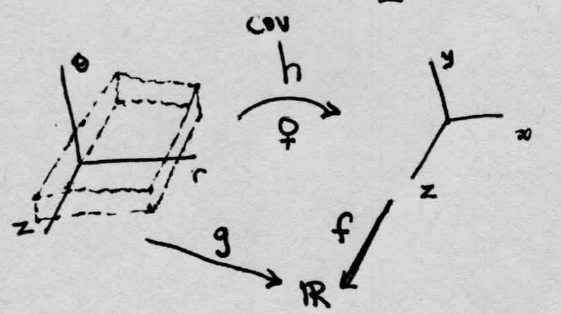
**Thm 5′ Cylindrical Co-ords**

(i) $\nabla f = f_r \hat{e}_r + \frac{1}{r} f_\theta \hat{e}_\theta + f_z \hat{e}_z$

(ii) $\nabla \cdot F = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF^{(r)}) + \frac{\partial}{\partial \theta}(F^{(\theta)}) + \frac{\partial}{\partial z}(rF^{(z)})\right]$

(iii) $\nabla \times F = \frac{1}{r} \det \begin{bmatrix} \hat{e}_r & re_\theta & e_z \\ D_r & D_\theta & D_z \\ F^{(r)} & rF^{(\theta)} & F^{(z)} \end{bmatrix}$

Pf




$g = f \circ h$ want to compute:

$\nabla g = g_r \hat{e}_r + \frac{1}{r} g_\theta \hat{e}_\theta + g_z \hat{e}_z$

we know $e_r = \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix}$ $e_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

we specify $\hat{e}_\theta$ by $e_r \cdot e_\theta = 0$ is this right?
$e_\theta \cdot e_z = 0$
$e_r \times e_\theta = e_z$

Schaums LA p.153

$\{e_r\ e_\theta\ e_z\} = \{e_x\ e_y\ e_z\} \begin{bmatrix} c_\theta & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$Q$

$Q^T$ since O.N.

$\{e_r\ e_\theta\ e_z\} Q^{-1} = \{e_x\ e_y\ e_z\}$

**Here is the plan:**

$g = f \circ h \longrightarrow f = g \circ h^{-1}$ we started with $\nabla f = \{e_x\ e_y\ e_z\} \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}$

$Dg_p = Df_{h(p)} \cdot Dh_p$

$\{e_r\ e_\theta\ e_z\} Q^{-1}$ $(Dh_p^{-1})^T Dg_p^T$

$Dg_p (Dh_p)^{-1} = Df_{h(p)}$

transpose

$\nabla f = [Df_{h(p)}]^T = (Dh_p^{-1})^T [Dg_p]^T$

$h(\overbrace{r, \theta, z}^{p}) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$

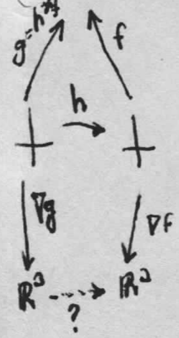
$A = Dh_p^{-1}$

$Dh_p = \begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -\frac{s}{r} & \frac{c}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$Q^{-1} = Q^T = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so we can see this is $Dh_p^{-1}$

$(Dh_p^{-1})^T = \begin{bmatrix} c & -\frac{s}{r} & 0 \\ s & c/r & 0 \\ 0 & 0 & 1 \end{bmatrix}$




$\nabla f = \{e_r\ e_\theta\ e_z\} Q^{-1} (Dh_p^{-1})^T Dg_p^T$

$= \{e_r\ e_\theta\ e_z\} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & -s/r & 0 \\ s & c/r & 0 \\ 0 & 0 & 1 \end{bmatrix} Dg_p^T$

$g : \mathbb{R}^3 \to \mathbb{R}$
$(r, \theta, z)$

$\begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}$ $\begin{bmatrix} D_1 g \\ D_2 g \\ D_3 g \end{bmatrix}$

$= \{e_r\ e_\theta\ e_z\} \begin{bmatrix} g_r \\ \frac{1}{r} g_\theta \\ g_z \end{bmatrix}$ QED

$\nabla g = h^*(\nabla f)$

$\stackrel{?}{=} \nabla(h^* f)$

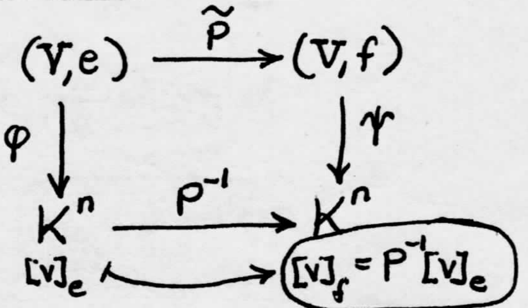
5/31/2016

## Summary of Linear Change of Basis from Schaum's LA p. 150

$\tilde{P} : \{\text{Bases for } V\} \longrightarrow \{\text{Bases for } V\}$

$\{e\} \longmapsto \{e\}P = \{f\}$

Then in Co-ords

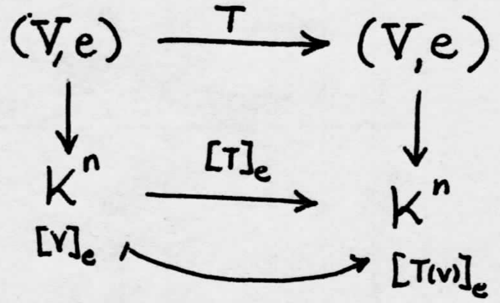



and $P v_f = v_e$

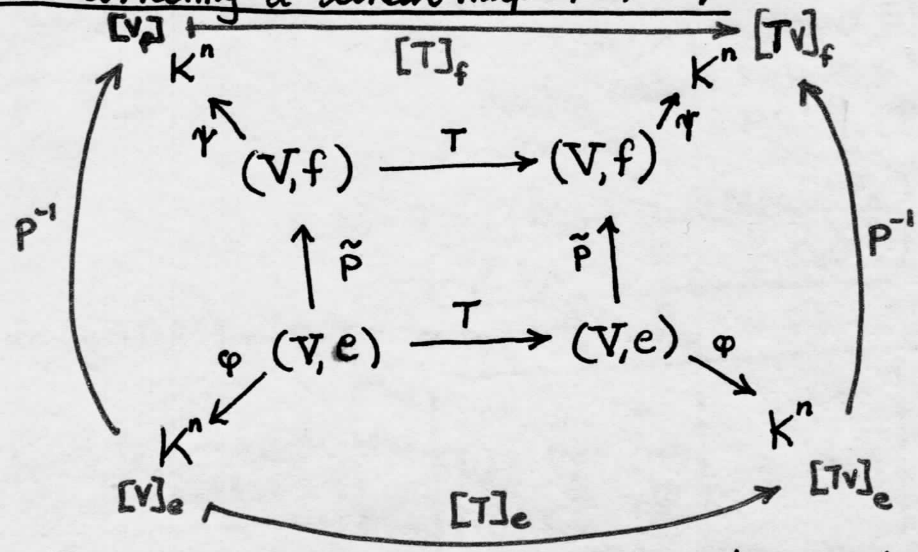
Also since $[f_i]_f = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow P_i = [f_i]_e$

$P = \begin{bmatrix} [f_1]_e & \cdots & [f_n]_e \end{bmatrix}$

For a *linear operator*:




**Change of basis affecting a linear map $T: V \to V$**




▷ Now following Taylor & Mann AC ch 10.4 we show how vector operations change with a change of basis O.N Change of Basis, specifically. $P^{-1} = P^T$

**Invariance of dot prod:**

$$[a]_f \cdot [b]_f = [a]_f^T [b]_f = (P^T a_e)^T P^T b_e = a_e^T \underbrace{P P^T}_{I} b_e = a_e^T b_e = a_e \cdot b_e$$

**Invariance of Cross Prod:** $\{i' j' k'\}_f = \{i j k\}_e \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$ $\quad v_f = Q^{-1} v_e = Q^T v_e$ $\quad$ Require $\det Q = +1$

$$a_e \times b_e = \det\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \det\begin{vmatrix} i & j & k \\ - & a & - \\ - & b & - \end{vmatrix} \underbrace{\det\begin{vmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{vmatrix}}_{\text{since } \det Q = +1} = \det\begin{vmatrix} e\cdot q_1 & (e\cdot q_2) & e\cdot q_3 \\ a\cdot q_1 & (a\cdot q_2) & a\cdot q_3 \\ b\cdot q_1 & (b\cdot q_2) & b\cdot q_3 \end{vmatrix} = \det\begin{vmatrix} i' & j' & k' \\ - & a' & - \\ - & b' & - \end{vmatrix} = a_f \times b_f$$

[More generally $\vec{Ma} \times \vec{Mb} = (\det M)(M^{-1})^T(\vec{a} \times \vec{b})$ as I show in my 4/25/2011 sheet]

**Remark on terminology:** Invariance ~ the value remains the same after transformation

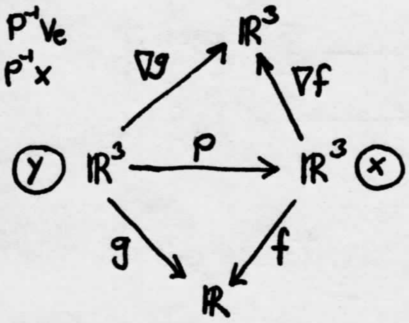
Covariance ~ the form of the expression remains the same. (I think)

5/31/2016

## Gradient under linear Change of Basis (does not have to be O.N.):

$V_f = P^{-1} V_e$

$y = P^{-1} x$




let $g = f \circ P$

$$Dg_y = Df_{P(y)} DP_y = Df_x \cdot P \quad \text{by linearity of } P$$

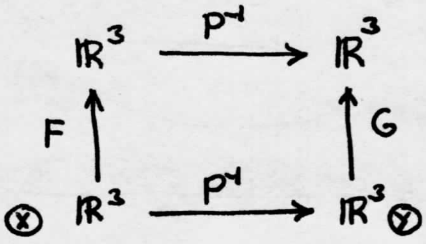
Transpose to get grad:

$$(Dg_y)^T = P^T (Df_x)^T \Rightarrow \boxed{\nabla g_y = P^T \nabla f_x}$$

Taylor & Mann AC p. 301

## Divergence of Vector Field

$y = P^{-1} x$




$$G = P^{-1} \circ F \circ P$$

$$DG_y = P^{-1} \cdot DF_x \cdot P$$

Thus $DG_y$ and $DF_x$ are similiar matrices, and similiar matrices have the same trace:

$$\text{Trace}(A) = \text{Sum of EWs of } A$$

Similiar matrices have same EWs:

$$Ax = \lambda x \text{ and } A = MBM^{-1} \Rightarrow \begin{matrix} MBM^{-1}x = \lambda x \\ B(M^{-1}x) = \lambda (M^{-1}x) \end{matrix}$$

$$\Rightarrow \text{Trace}(DG_y) = \text{Trace}(DF_x)$$

$$\boxed{\sum D_i G^{i} = \sum D_i F^{i}} \Rightarrow \text{div}(G) = \text{div}(F)$$

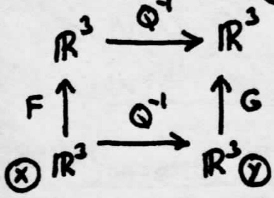
## Laplacian

$$\text{div}(\text{grad}(f)) = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz} = \nabla^2 f$$

## Curl

we want to establish: $\boxed{\text{Curl}(F) = \text{Curl}(G)}$

$$\det \begin{vmatrix} i & j & k \\ D_x(\cdot) & D_y(\cdot) & D_z(\cdot) \\ F^{1}(x) & F^{2}(x) & F^{3}(x) \end{vmatrix} = \det \begin{vmatrix} i' & j' & k' \\ D_{x'}(\cdot) & D_{y'}(\cdot) & D_{z'}(\cdot) \\ G^{1}(y) & G^{2}(y) & G^{3}(y) \end{vmatrix}$$




we do this by showing

$$\begin{bmatrix} i & j & k \\ D_x & D_y & D_z \\ F^1 & F^2 & F^3 \end{bmatrix} \underbrace{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}}_{\det Q = +1} = \begin{bmatrix} i' & j' & k' \\ D_{x'} & D_{y'} & D_{z'} \\ G^1 & G^2 & G^3 \end{bmatrix}$$

$$\vec{F}(x) = Q \circ G \circ Q^{-1}(x) = Q \vec{G}(y)$$

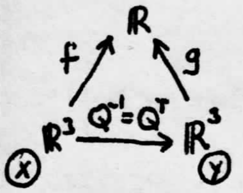
we already established

$$\{i' \ j' \ k'\} = \{i \ j \ k\} Q$$

$$\xrightarrow{Q^T F = G} [G^1(y) \ G^2(y) \ G^3(y)] = [F^1(x) \ F^2(x) \ F^3(x)] Q$$

Now show $\left[\frac{\partial}{\partial x'} \ \frac{\partial}{\partial y'} \ \frac{\partial}{\partial z'}\right] = \left[\frac{\partial}{\partial x} \ \frac{\partial}{\partial y} \ \frac{\partial}{\partial z}\right] Q$

Consider




$$g(y) = f \circ Q(y)$$

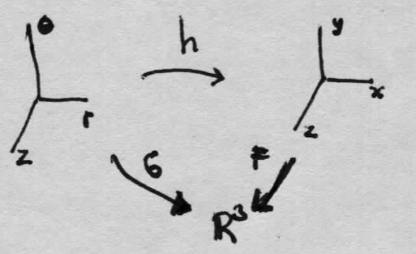
$$Dg_y = Df_x Q$$

$$\left[\frac{\partial}{\partial y_1} g \ \frac{\partial}{\partial y_2} g \ \frac{\partial}{\partial y_3} g\right] = \left[\frac{\partial}{\partial x} f \ \frac{\partial}{\partial y} f \ \frac{\partial}{\partial z} f\right] \left[ \ Q \ \right] \Rightarrow \frac{\partial}{\partial y_i}(g) = \left( \left[\frac{\partial}{\partial x}(\cdot) \ \frac{\partial}{\partial y}(\cdot) \ \frac{\partial}{\partial z}(\cdot)\right] \begin{bmatrix} | \\ q_i \\ | \end{bmatrix} \right)(f)$$

□

3/11/2022

Now show (cylindrical) $\nabla \cdot F = \frac{1}{r}\left[ D_r(rF^r) + D_\theta(F^\theta) + D_z(rF^z)\right]$




$G = F \circ h \qquad \vec{G}(r,\theta,z) = \vec{F}(h(r,\theta,z))$

$$[D_r\ D_\theta\ D_z]\begin{bmatrix} G^1 \\ G^2 \\ G^3 \end{bmatrix} = [D_r\ D_\theta\ D_z]\begin{bmatrix} F^1(h(r,\theta,z)) \\ F^2(h(r,\theta,z)) \\ F^3(h(r,\theta,z)) \end{bmatrix}$$

$$= D_r(F^1(h(p))) + D_\theta(F^2(h(p))) + D_z(F^3 \ldots$$

$$\frac{\partial}{\partial x}(f(\vec{h}(x,y,z)) = DF^{1}_{h(p)}\vec{h}_r + DF^2_{h(p)}\vec{h}_\theta + DF^3_{h(p)}\vec{h}_z$$

$$= Df_{h(p)} \cdot \begin{bmatrix} h^1_x \\ h^2_x \\ h^3_x \end{bmatrix}$$

$$\{e_r\ e_\theta\ e_z\}\begin{bmatrix} G^r \\ G^\theta \\ G^z \end{bmatrix} \qquad Dh_p = \begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans p. 552

$F = G \circ h^{-1}$

$L_x(F) = L_x(G \circ h^{-1})$

$$\nabla \cdot F = [D_x\ D_y\ D_z] \cdot [F^r e_r + F^\theta e_\theta + F^z e_z]$$

(with $L_x$, $G^r$, $G^\theta$, $G^z$ labelled above)

But regarding $\vec{F}$ as a fcn of $r, \theta, z$:

$$= \left[c_\theta D_r - \frac{s}{r}D_\theta,\ s_\theta D_r + \frac{c}{r}D_\theta,\ D_z\right] \cdot \left(F^r\begin{bmatrix} c \\ s \\ 0 \end{bmatrix} + F^\theta\begin{bmatrix} -s \\ c \\ 0 \end{bmatrix} + F^z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= \left(c_\theta D_r - \frac{s}{r}D_\theta\right)(F^r c - F^\theta s) + \left(sD_r + \frac{c}{r}D_\theta\right)(F^r s + F^\theta c) + D_z(F^z 1)$$

$$= D_r F^r + \frac{1}{r}F^r + \frac{1}{r}D_\theta F^\theta + D_z F^z = \left[\left(\frac{1}{r} + D_r\right)\ \frac{1}{r}D_\theta\ \ D_z\right]\begin{bmatrix} G^r \\ G^\theta \\ G^z \end{bmatrix}$$

($h$, $h^{-1}$ between the two coordinate systems)

$$= \frac{1}{r}\left[D_r(rF^r) + D_\theta F^\theta + D_z(rF^z)\right]$$

$G = F \circ h$

Here is (ex 5) doing this for spherical

$$h = \begin{bmatrix} \rho s_\varphi c_\theta \\ \rho s_\varphi s_\theta \\ \rho c_\varphi \end{bmatrix} \qquad \boxed{G \circ h^{-1} = F}$$

$$\nabla \cdot F = D_x F^1 + D_y F^2 + D_z F^3$$

but $F^1 = F^1(h($

I think this is more like what I have above at top)

(★) $D_x F^1 = D_\rho F^1 \rho_x + D_\theta F^1 \theta_x + D_\varphi F^1 \varphi_x \left( = [D_1 G^1\ D_2 G^2\ D_3 G^3]\begin{bmatrix} h_x \\ \ldots \end{bmatrix}\right.$

$$\frac{\partial}{\partial x}\ h(\rho,\theta,\varphi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Differentiate implicitly (to find $Dh^{-1}$?)

$$1 = \rho_x s_\varphi c_\theta + \rho c_\varphi c_\theta \varphi_x + -\rho s_\varphi s_\theta \theta_x$$
$$0 = \rho_x s_\varphi s_\theta + \rho c_\varphi s_\theta \varphi_x + \rho s_\varphi c_\theta \theta_x$$
$$0 = \rho_x c_\varphi - \rho s_\varphi \varphi_x$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s_\varphi c_\theta & \rho c_\varphi c_\theta & -\rho s_\varphi s_\theta \\ s_\varphi s_\theta & \rho c_\varphi s_\theta & \rho s_\varphi c_\theta \\ c_\varphi & -\rho s_\varphi & 0 \end{bmatrix}\begin{bmatrix} \rho_x \\ \varphi_x \\ \theta_x \end{bmatrix}$$

Solve

$$D_1(h^{-1})^{③} = \varphi_x = c_\theta s_\varphi$$
$$D_1(h^{-1})^{①} = \rho_x = \frac{c_\theta c_\varphi}{\rho}$$
$$D_1(h^{-1})^{②} = \theta_x = \frac{-s_\theta}{\rho s_\varphi}$$

Subs into (★)

$$D_x F^1 = D_\rho F^1 \frac{c_\theta s_\varphi}{\rho} + D_\theta F^1\left(\frac{-s_\theta}{\rho s_\varphi}\right) + D_\varphi F^1 (c_\theta s_\varphi)$$

$D_x h$? $Dh_p(\vec{\psi}_x)$?

also

$$F^r = \vec{F} \cdot e_r = \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} \cdot \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix} = F^1\cos\theta + F^2\sin\theta$$

$$F^\theta = -F^1\sin\theta + F^2\cos\theta$$

could this be $Q\Lambda$ for some O.N. and some Diag $\Lambda$?

Then $(Q\Lambda)^{-1} = \Lambda^{-1}Q^T$

$a \wedge b = \wedge_a b$

3/11/2022

$$\{e_r \ e_\theta \ e_z\} = \{e_x \ e_y \ e_z\} \ Q$$

$$Q = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

whereas $Dh_p = \begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix}$

~~$Dh_p = AQ$ for some $A$?~~

~~$AQA^T$?~~

This matrix of scale factors normalizes $Dh_p$

$$Q^T (Dh_p^{-1})^T = \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}$$

$$\Rightarrow Dh_p^{-1} \cdot Q = \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}$$

$$Q \overset{?}{=} Dh_p \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix} \overset{?}{=} \underbrace{\begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S$

$$\boxed{QS^{-1} = Dh_p}$$

$$\begin{bmatrix} 1 & & \\ & r & \\ & & 1 \end{bmatrix}$$

Key relation

$$Q = Dh_p \begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix}$$

This O.Ns $Dh_p$

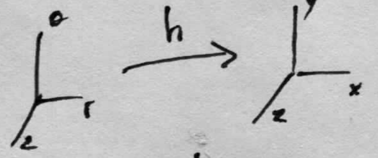
This means I can look at Taylor & Mann

$$\nabla \cdot F = [D_x \ D_y \ D_z] \cdot \{e_r \ e_\theta \ e_z\} \begin{bmatrix} G^r \\ G^\theta \\ G^z \end{bmatrix}$$

$$= \left( \{e_x \ e_y \ e_z\} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \right)^T$$

$$\{e_x \ e_y \ e_z\} \ Q \begin{bmatrix} G^r \\ G^\theta \\ G^z \end{bmatrix}$$

$$Dh_p = \begin{bmatrix} | & | & | \\ e_r & re_\theta & e_z \\ | & | & | \end{bmatrix}$$




$$Dh_p(e_1) = e_r$$

$$Dh_p(e_2) = r e_\theta$$

$$Dh_p(e_3) = e_z$$

$$\left[ c D_r - \frac{s}{r} D_\theta, \ s D_r + \frac{c}{r} D_\theta, \ D_z \right]$$

$$= [D_r \ D_\theta \ D_z] \underbrace{\begin{bmatrix} c & s & 0 \\ -\frac{s}{r} & \frac{c}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(Dh_p^{-1})^{?}\,?}$$

Taylor & Mann AC  **ch 10.4 Invariants**

$\{f\} = \{e\}P$

**Dot prod invariant**

Schuans LA

$$[a]_f \cdot [b]_f = a_f^T b_f = (P^T a_e)^T P^T b_e = a_e^T \underbrace{P P^T}_{I} b_e = a_e^T b_e$$

($P^T = P^{-1}$)

- value in $\mathbb{R}$ is the same
- form is the same

$$a_f \cdot b_f = (P^T a_e)^T P^T b_e \checkmark$$

$\vec{V} = \{f\} v_f = \{e\} P v_f$

$\vec{V} = \{e\} v_e \Rightarrow v_e = P v_f$

but $\{f\} = \{e\} P$ or $P^{-1} v_e = v_f$

p.288

**Cross prod**

$$a_e \times b_e = \det \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \det \begin{vmatrix} i & j & k \\ — & a & — \\ — & b & — \end{vmatrix} \det \begin{vmatrix} | & | & | \\ Q_1 & Q_2 & Q_3 \\ | & | & | \end{vmatrix}$$

($\det Q = +1$)

$\{i' \ j' \ k'\}_f = \{i \ j \ k\}_e Q$

$$= \det \begin{vmatrix} e \cdot q_1 & e \cdot q_2 & e \cdot q_3 \\ a \cdot q_1 & a \cdot q_2 & a \cdot q_3 \\ b \cdot q_1 & b \cdot q_2 & b \cdot q_3 \end{vmatrix}$$

$$= \det \begin{vmatrix} i' & j' & k' \\ — & a' & — \\ — & b' & — \end{vmatrix} = a_f \times b_f \quad \square$$

**p.296 Gradient**

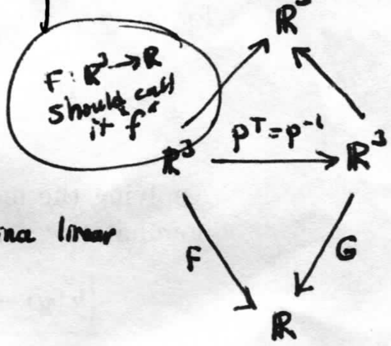
$\{i' j' k'\} = \{i j k\} P$  $v_f = P^{-1} v_e$ so

$P^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

Let $G = F \circ P$

$DG_{x'} = DF_{Px'} \, DP_{x'}$ ($Px' = x$; $DP_{x'} = P$ since linear)

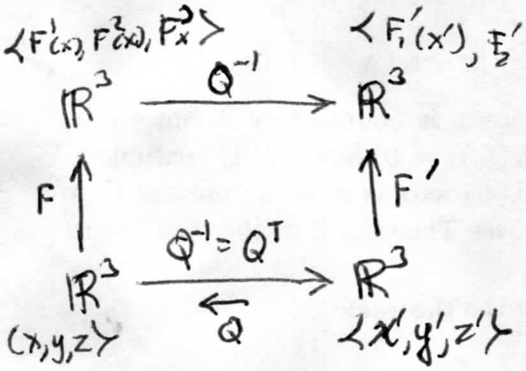



Transpose to get gradient

$$DG_{x'}^T = P^T DF_x^T$$

$$\nabla G(x') = P^{-1} \nabla F(x)$$

p.301 **Div**




$$F' = Q^{-1} \circ F \circ Q$$

$$DF'_{x'} = Q^{-1} DF_x Q \quad Q \text{ linear}$$

$$\text{Trace}(DF'_{x'}) = \text{Trace}(DF_x)$$

since matrices are similar (same EVs ⟹ same EVs is same)

We did not need Q O.G. just linear and invertible.

Taylor & Mann

p. 306

Curl

want to establish: $\det\begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ F^1_{(x)} & F^2_x & F^3_{(x)} \end{vmatrix} = \det\begin{vmatrix} i & j & k \\ D_{x'} & D_{y'} & D_{z'} \\ F_1'(x') & F_2' & F_3 \end{vmatrix}$

$\mathbb{R}^3 \xrightarrow{Q^{-1}} \mathbb{R}^3$, $\uparrow F$, $\uparrow F'$, $\mathbb{R}^3 \xrightarrow{Q^{-1}} \mathbb{R}^3$, $x$, $x'$

$\{i' j' k'\} = \{i\ j\ k\} Q$

we must establish ① $[D_{x'}\ D_{y'}\ D_{z'}] = [D_x\ D_y\ D_z] Q$

② $[F_1'\ F_2'\ F_3'(x')] = [F_1\ F_2\ F_3(x)] Q$

~~we know $V_f = Q^T V_e$~~

~~$V_f^T = V_e^T Q$~~

But what about $x, x'$?

$(F'(x'))^T \overset{?}{=} F^T_{(x)} Q$

$F'(x') = Q^{-1} \circ F \circ Q(x')$

$= Q^T F(x)$

$(F'(x))^T = F^T_{(x)} Q$

~~About~~

$\mathbb{R}^3 \xrightarrow{Q^{-1}} \mathbb{R}^3$, $f$, $f'$ to $\mathbb{R}$, $x$, $x'$, $h'$

$f'(x') = f \circ Q(x')$

$\frac{\partial}{\partial x'} f' = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x'}$

$Df'_{x'}(h') = Df_{Q(x)}\, \underbrace{DQ_{x'}}_{Q}(h')$

This makes sense assuming $Q$ is linear in $x'$

$[D_1(f')\ D_2(f')\ D_3(f')] = [D_1(f)\ D_2(f)\ D_3(f)]\ Q$

$\mathbb{R}$, $f$, $f'$, $[D_1(\cdot)$

$\mathbb{R}^3 \xleftarrow{Q} \mathbb{R}^3(x')$

$Df \downarrow$, $\downarrow Df'$

$(\mathbb{R}^3)^* \xleftarrow{Q^*} (\mathbb{R}^3)^*$

$[D_1 f\ D_2 f\ D_3 f]$, $[D_1 f'\ D_2 f'\ D_3 f']$

$Df'_{(\cdot)} = Df_{Q(\cdot)} \cdot Q(\cdot)$

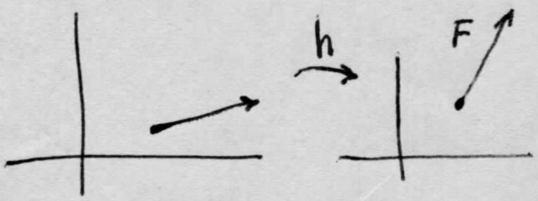
$Df'_{(\cdot)} = Q^{-1} Df . Q(\cdot)$

$[\ ][\ ][\ ]$

$[\ ] = [\ ][\ ]$ This cannot multiply

$[\ ][\ ] = [\ ]$

3/12/2022




$$F = F^1 e_x + F^2 e_y + F^3 e_z$$
$$= a\hat{e}_r + b\hat{e}_\theta + c\hat{e}_z$$
$$= a\begin{bmatrix} c \\ s \\ 0 \end{bmatrix} + b\begin{bmatrix} -s \\ c \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$a(r,\theta,z)$

$$= \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \\ c \end{bmatrix}$$

$$\frac{\partial}{\partial x}\left(a\cos\theta - b\sin\theta\right)$$

pull back a vector field? G&P p. 134

$v : X \longrightarrow \mathbb{R}^N$

$x \longmapsto \vec{v}(x) \in T_x X$

$(\varphi^* v) : \mathcal{U} \longrightarrow \mathbb{R}^k$

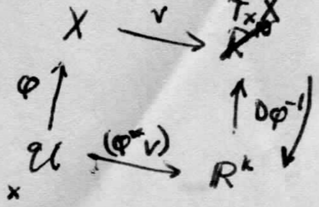
$x \longmapsto D\varphi^{-1}_x(v(x))$

$D(\varphi^{-1}) = (D\varphi)^{-1}$

$\nabla \cdot \vec{F} = \Gamma(\vec{G})$ what is $\Gamma$?

Some Differential operator in $(r,\theta,z)$ space

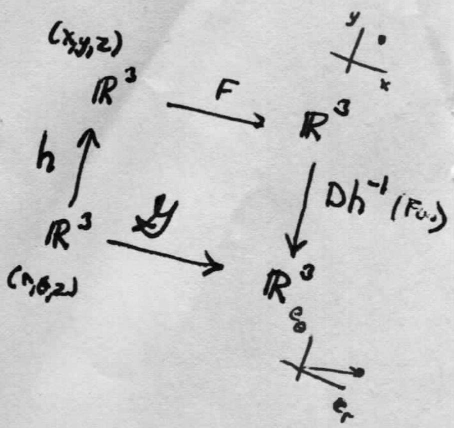
What is its form?



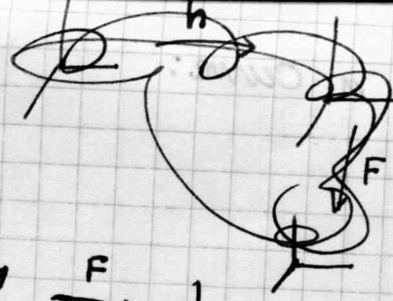

Do Carmo

$D\varphi_{uv} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(a,b) \longmapsto [\vec{\varphi}_u \ \vec{\varphi}_v]\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{\varphi}_u + b\vec{\varphi}_v$

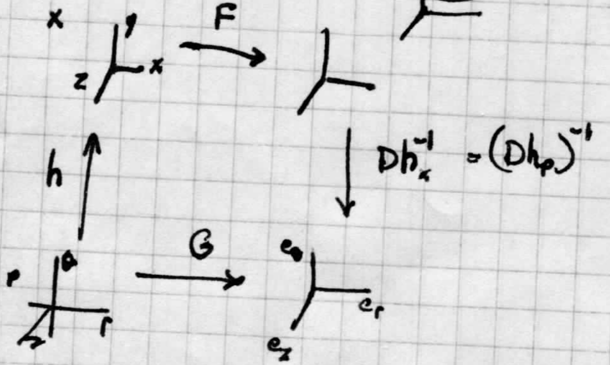


3/12/2022




$G^1 = F \cdot \hat{e}_r$

$G^2 = F \cdot \hat{e}_\theta$




$$F(x) = Dh_p \cdot G \cdot h^{-1}(x)$$

$$= \begin{bmatrix} e_r & re_\theta & e_z \end{bmatrix} \begin{bmatrix} G^1(p) \\ G^2(p) \\ G^3(p) \end{bmatrix} = \begin{bmatrix} e_r & e_\theta & e_z \end{bmatrix} \begin{bmatrix} G^1(p) \\ rG^2(p) \\ G^3(p) \end{bmatrix}$$

$Q$

[illegible]

---

$$\nabla \cdot F(x) = D_x F^1(x) + D_y F^2(x) + D_3 F^3(x)$$ but if $x = h(p) = h(r,\theta,z)$

$$\text{If } \vec{F}(x) = \overset{a}{G^r} e_r + \overset{b}{G^\theta} e_\theta + \overset{c}{G^z} e_z$$

then
$$\frac{\partial}{\partial x} F(x) = \frac{\partial}{\partial x}(G^r e_r) + \frac{\partial}{\partial x}(G^\theta e_\theta)$$

$G^r(r,\theta,z)\, e_r(r,\theta,z)$    $Dh_p(e_1)$

$$= \left(D_1 a \frac{\partial r}{\partial x} + D_2 a\, \theta_x + D_3 a\, z_x\right) e_r + a \frac{\partial}{\partial x}(\hat{e}_r)$$

$$= Da_p \begin{bmatrix} r_x \\ \theta_x \\ z_x \end{bmatrix} + a \frac{\partial}{\partial x}\hat{e}_r + Db_p \begin{bmatrix} r_x \\ \theta_x \\ z_x \end{bmatrix} + b \frac{\partial}{\partial x} e_\theta + Dc_p \begin{bmatrix} r_x \\ \theta_x \\ z_x \end{bmatrix} + c \frac{\partial}{\partial x} e_z$$

$\vec{\psi}_x$    $\vec{\psi}_x$    $\vec{\psi}_x$

$$h(r,\theta,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} h^1 \\ h^2 \\ h^3 \end{bmatrix}$$

$$\begin{bmatrix} r \\ \theta \\ z \end{bmatrix} = h^{-1}(x,y,z) = \psi(x,y,z)$$

$r = \psi^1$

$\theta = \psi^2$

$z = \psi^3$

$$\begin{bmatrix} x \\ y \end{bmatrix} = h(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

look at this r

reduce to polar

$$F^1(x,y) = f(x,y) = \begin{bmatrix} e_r & e_\theta \end{bmatrix} \begin{bmatrix} a \\ rb \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + [\text{illegible}]$$

$$D_1 f = \frac{\partial f}{\partial x}(x,y) = \begin{bmatrix} -s\theta\,\theta_x & -c\theta\,\theta_x \\ c\theta\,\theta_x & -s\theta\,\theta_x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} a_r r_x + a_\theta \theta_x \\ b_r r_x + b_\theta \theta_x \end{bmatrix} + Q \begin{bmatrix} Da(h_x) \\ Db(h_x) \end{bmatrix}$$

$$+ Q \begin{bmatrix} Da_\theta \\ Db_\theta \end{bmatrix} \begin{bmatrix} [\text{illegible}] \end{bmatrix} \psi_x$$

$$\theta_x \begin{bmatrix} -s & -c \\ c & -s \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \theta_x \begin{bmatrix} DQ_\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

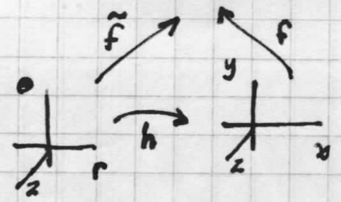
$f(x) = A\cos v(x)$

$D^1 f(x) = \frac{\partial}{\partial x} f(x) =$

$s \to c$

$c \to -s$

3/13/2022




$\tilde{f}(r,\theta,z) = f \circ h\,(r,\theta,z)$

$D_r \tilde{f} = D_r\big(f(h(r,\theta,z))\big)$ $\qquad h(r,\theta,z) = \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} h^1 \\ h^2 \\ h^3 \end{matrix}$

$= Df_{x} \cdot D_r \vec{h}$ ($x = h(p)$)

also $D_\theta \tilde{f} = Df_x(\vec{h}_\theta)$

$D_z \tilde{f} = Df_x(\vec{h}_z)$

$= Df_x \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = Df_x \begin{bmatrix} \vec{h}_r \end{bmatrix}$

we can write the whole thing:

$D\tilde{f}_{r\theta z} = Df_x \, Dh_{r\theta z}$ $\qquad = Df_x \left[\frac{\partial}{\partial r}\vec{h}_p\right]$

$\left[D_1\tilde{f} \;\; D_2\tilde{f} \;\; D_3\tilde{f}\right] = \left[D_1 f \;\; D_2 f \;\; D_3 f\right]\begin{bmatrix} | & | & | \\ h_r & h_\theta & h_z \\ | & | & | \end{bmatrix}$

This is what I had in Taylor & Mann

$\left[D_r(\tilde{f}) \;\; D_\theta(\tilde{f}) \;\; D_z(\tilde{f})\right]$

$Q\Lambda = \Lambda Q$

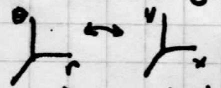
$(\Lambda Q)^{-1} = Q^{-1}\Lambda^{-1} = Q^T\Lambda^{-1}$

$\left[D_r(\cdot) \;\; D_\theta(\cdot) \;\; D_z(\cdot)\right]\begin{bmatrix} -e_r^T- \\ -e_\theta^T- \\ -e_z^T- \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1/r & \\ & & 1 \end{bmatrix} = \left[D_x(\cdot) \;\; D_y(\cdot) \;\; D_z(\cdot)\right]$ $\qquad \underset{Dh_p}{\begin{bmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

$e_r^T D_r(\cdot) + e_\theta^T D_\theta(\cdot)$

$\overset{\tilde{f}}{\left[D_r \;\; D_\theta \;\; D_z\right]} \overset{Dh_p^{-1}}{\begin{bmatrix} c & s & 0 \\ -\frac{s}{r} & \frac{c}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \overset{f}{\left[D_x \;\; D_y \;\; D_z\right]}$

If we "Identify"



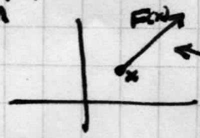

"we pretend the map h doesn't exist!" Alec Norton

So this explains what Marsden & T has for answer.

$f(x(r,\theta), y(r,\theta))$

$\frac{\partial}{\partial x} f = \cos\theta \frac{\partial}{\partial r} f + \sin\theta \frac{\partial}{\partial \theta} f$

'Slang interpretation', physical vector




we can regard this as

$F^r = F(x) \cdot \hat{e}_r$ then regard $x = x(r,\theta,z)$

$F^1(x)e_x + F^2 e_y + F^3 e_z$

AND

$F^r_{(r,\theta,z)}\hat{e}_r + F^\theta_{(r,\theta,z)}\hat{e}_\theta + F^z_{(r,\theta,z)}\hat{e}_z$

We pretend 2 co-ord sys are overlaid over same physical space

3/13/2022

$$\vec{\nabla}\cdot\vec{F} = [D_x\ D_y\ D_z]\begin{bmatrix}F^x\\F^y\\F^z\end{bmatrix}$$

But $\begin{bmatrix}F^x\\F^y\\F^z\end{bmatrix} = [e_r\ e_\theta\ e_z]\begin{bmatrix}F^r\\F^\theta\\F^z\end{bmatrix}$ ($Q$)

$$= \begin{bmatrix}c & -s & 0\\ s & c & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}F^r\\F^\theta\\F^z\end{bmatrix} = \begin{bmatrix}cF^r - sF^\theta\\ sF^r + cF^\theta\\ F^z\end{bmatrix}$$

and $[D_x\ D_y\ D_z] = [D_r\ D_\theta\ D_z][D\phi^{-1}]$

$$\left[cD_r - \frac{s}{r}D_\theta,\ sD_r + \frac{c}{r}D_\theta,\ D_z\right]\begin{bmatrix}cF^r - sF^\theta\\ sF^r + cF^\theta\\ F^z\end{bmatrix}$$

$$= \left(cD_r - \frac{s}{r}D_\theta\right)\left(cF^r - sF^\theta\right) + \left(sD_r + \frac{c}{r}D_\theta\right)\left(sF^r + cF^\theta\right) + D_zF^z$$

$s \to c$

$c \to -s$

$$= c^2D_rF^r - csD_rF^\theta - \frac{s}{r}D_\theta(cF^r) + \frac{s}{r}D_\theta(sF^\theta)$$

$$-\frac{s}{r}\left(-sF^r + cD_\theta F^r\right) + \frac{s}{r}cF^\theta + \frac{s^2}{r}D_\theta F^\theta$$

$$+ s^2D_rF^r + scD_rF^\theta + \frac{s^2}{r}F^r - \frac{sc}{r}D_\theta F^r + \frac{sc}{r}F^\theta + \frac{s^2}{r}D_\theta F^\theta$$

$$\frac{c}{r}\left(cF^r + sD_\theta F^r\right) + \frac{c}{r}\left(-sF^\theta + cD_\theta F^\theta\right)$$

$$\frac{c^2}{r}F^r + \frac{cs}{r}D_\theta F^r + -\frac{cs}{r}F^\theta + \frac{c^2}{r}D_\theta F^\theta$$

$$D_rF^r + \frac{1}{r}F^r + \frac{1}{r}D_\theta F^\theta + D_zF^z$$

$$= \left(\frac{1}{r} + D_r\right)(F^r) + \frac{1}{r}D_\theta(F^\theta) + D_z(F^z)$$

Cylindrical

$$= \left[\frac{1}{r} + D_r,\ \frac{1}{r}D_\theta\ \ D_z\right]\begin{bmatrix}F^r\\F^\theta\\F^z\end{bmatrix}$$

But they want to rewrite it as: