4/19/2023

15

$$\begin{tikzcd}
& \overset{\underset{\sim}{x}}{U\cap W} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[r] & \mathbb{R} \\
\underset{x}{\mathbb{R}^n} \arrow[rr, "h=\beta\alpha^{-1}"] \arrow[urr, bend left=60, "\omega_\alpha = h^*\omega_\beta"] & & \underset{y}{\mathbb{R}^n} \arrow[u, bend right, "\omega_\beta"']
\end{tikzcd}$$

$y=h(x)$

See end of ch 9.8 problems sheet 17 for examples

Ⓐ

If Ⓐ is unclear, it is done again and expanded in Ⓒ

Given 1-form ω, we can express it upstars using chart (U,α) $\omega = \sum a_i \underset{\sim}{d} x^i_\alpha =: \underset{\sim}{\omega}^\alpha$

and in chart (W,β) $\omega = \sum b_i \underset{\sim}{d} y^i_\beta =: \underset{\sim}{\omega}^\beta$

In chart (U,α) and $f: U \longrightarrow \mathbb{R}$ we have

$$\underset{\sim}{d} f_{\underset{\sim}{x}}(\cdot) = \sum^n D_i f_\alpha \, dx^i_\alpha = \sum \frac{\partial f_\alpha}{\partial x^j_\alpha} \underset{\sim}{d} x^i_\alpha$$

$$\left.\begin{aligned} &\text{Thus taking } f=\beta^i, \quad \underset{\sim}{d}\beta^i_{\underset{\sim}{x}} = \sum_j \frac{\partial \beta^i_\alpha}{\partial x^j_\alpha} dx^j_\alpha \\ &\text{This is the same thing as } \underset{\sim}{d} y^i_\beta = \sum_j \frac{\partial y^i}{\partial x_j} \underset{\sim}{d} x^j \end{aligned}\right\} \quad (8.12)$$

Then

Subs →

$$\underset{\sim}{\omega}^\beta = \sum_i b_i \underset{\sim}{d} y^i = \sum_i b_i \left(\sum_j \frac{\partial y^i}{\partial x_j} dx^j\right) = \sum_j \Big(\underbrace{\sum b_i \frac{\partial y^i}{\partial x_j}}_{a_i}\Big) dx^j$$

$y = h(x)$ $y^i = h^i(x)$ Take $n=2$ for illustration

This is example of L&S subs and expand method. other method is better Ⓒ

$$\omega^\beta = \sum_i b_i dy^i = \sum_i^2 b_i \Big(\sum^2 \underbrace{\frac{\partial h^i}{\partial x_j}}_{D_j h^i} dx^j\Big) = \sum^2 b_i \left(h^i_1 dx^1 + h^i_2 dx^2\right)$$

$$= b_1\left(h^1_1 dx^1 + h^1_2 dx^2\right) + b_2\left(h^2_1 dx^1 + h^2_2 dx^2\right) = [b_1 \;\; b_2]\begin{bmatrix} h^1_1 & h^1_2 \\ h^2_1 & h^2_2 \end{bmatrix}\begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix}$$

$$= [b_1 \;\; b_2]\begin{bmatrix} D_1h^1 & D_2h^1 \\ D_1h^2 & D_2h^2 \end{bmatrix}\begin{bmatrix} \pi_1(\cdot) \\ \pi_2(\cdot) \end{bmatrix}$$

Book quotes (7.4) but they really mean $\omega_\alpha = h^*\omega_\beta = \omega_\beta(Dh_x(\cdot))$

▷ Now we basically repeat and expand the same argument for $\psi: M \longrightarrow N$ and the pull back of <u>fcns</u> and then <u>1-forms</u> :

Ⓑ

$$\begin{tikzcd}
& \mathbb{R} & \\
U \arrow[ur, "\psi^*f"] \arrow[rr, "\psi"] \arrow[d, "\alpha"] & & W \arrow[ul, "f"'] \arrow[d, "\beta"'] \\
\underset{x}{\mathbb{R}^m} \arrow[rr, "\overset{\beta\circ\psi\circ\alpha^{-1}}{h}"] \arrow[uur, bend left=60, "(\psi^*f)_\alpha"] \arrow[uur, bend right=90, "h^*(f_\beta)"'] & & \underset{y}{\mathbb{R}^n} \arrow[uul, bend right=50, "f_\beta"']
\end{tikzcd}$$

$y = h(x)$

$$h := \beta\circ\psi\circ\alpha^{-1}$$

$$h: \alpha(U) \longrightarrow \beta(W)$$

Book has

(8.13) $y^i(\psi(x)) = y^i(x^1, \dots, x^m)$ $i = 1, \dots, n$

This means $\beta^{ⓘ}(\psi(\underset{\sim}{x})) = h^{ⓘ}(\alpha(\underset{\sim}{x}))$

and (8.14) $f_\alpha(x^1, \dots, x^m) = f_\beta(y^1(x), \dots, y^n(x))$

BUT there is <u>no</u> "f_α" only $(\psi^*f)_\alpha$

so (8.14) means $\boxed{(\psi^*f)_\alpha(x) = f_\beta(h(x))}$

p.396

4/19/2023

Ⓒ Pullback of 1-form ω (expressed in chart β $\underset{\sim}{\omega}^{\beta}$) into 1-form (expressed in chart α $\underset{\sim}{\omega}^{\alpha}$)

What we want to show is

$$\underset{\sim}{\omega}^{\beta}_{y} := \sum b_i\, \underset{\sim}{d}y^i \Rightarrow \psi^*(\omega^\beta)_x = \sum_{j=1}^{n}\sum_{i=1}^{m} b_i \frac{\partial y^i}{\partial x_j}\underset{\sim}{d}x^j$$

$$\begin{tikzcd}
 & \mathbb{R} & \\
\underset{\sim}{x}\ U \arrow[ur, "\psi^*\omega"] \arrow[rr, "\psi"] \arrow[d, "\alpha"'] & & W \arrow[ul, "\omega"'] \arrow[d, "\beta"] \quad \psi(\underset{\sim}{x})=y \\
\mathbb{R}^m \arrow[rr, "\overset{(\beta\circ\psi\circ\alpha^{-1})}{h}"] & & \mathbb{R}^n \\
x & & y
\end{tikzcd}$$

$y=h(x)$

$$\begin{tikzcd}
 & \mathbb{R} & \\
\xi\ T_xU \arrow[ur, "(\psi^*\omega)_{\underset{\sim}{x}}"] \arrow[rr, "\psi_{*x}"] \arrow[d, "\kappa_\alpha"'] & & T_{\psi(x)}W \arrow[ul, "\omega_{\underset{\sim}{y}} = \text{“}\underset{\sim}{\omega}^{\beta}_{y}\text{”}"'] \arrow[d, "\kappa_\beta"] \quad \psi_*(\xi)=\eta \\
\underset{\xi_\alpha}{\mathbb{R}^m} \arrow[rr, "Dh_x"] \arrow[uur, bend left=70, "(\psi^*\omega)_\alpha"] & & \underset{\eta_\beta}{\mathbb{R}^n} \arrow[uul, bend right=60, "(\omega_\beta)_y(\cdot)"'] \arrow[uul, bend right=100, "h^*(\omega_\beta)_x"']
\end{tikzcd}$$

This reduces to the discussion in Ⓐ if we take ψ = Id, so it is just change of variables in overlapping charts

$\psi_{*x}(\xi) = \eta \Rightarrow$
$Dh_x(\xi_\alpha) = \eta_\beta$

See next sheet, where I also give an example from Munkres AOM
$\varphi: \mathbb{R}^2 \to \mathbb{R}^3 \quad \varphi^* dy^i = d\varphi^i$

Key

We know $\boxed{(\psi^*\omega)_{\underset{\sim}{x}}(\xi) = \omega_{\psi(x)}(\psi_*\xi)} = \omega_y(\eta)$

that means $(\psi^*\omega)_{\alpha,x}(\xi_\alpha) \overset{!}{=} \omega_{\beta,y}(\eta_\beta) = \omega_{\beta,\underset{h(x)}{\underset{\downarrow}{y}}}(Dh_x(\xi_\alpha)) = (h^*\omega_\beta)_x(\xi_\alpha)$

$\underset{\sim}{\omega}^{\beta}_{\underset{\sim}{y}} = \sum b^i(\underset{\sim}{y})\,\underset{\sim}{d}y^i_\beta(\cdot)$ then downstairs $\omega_\beta(\cdot) = [b^1(y), \dots, b^n(y)]\begin{bmatrix}\pi_1(\cdot)\\ \vdots \\ \pi_n(\cdot)\end{bmatrix}$

Key idea

$$\boxed{(h^*\omega_\beta)(\xi_\alpha) = \omega_\beta(Dh_x(\xi_\alpha))} = [b^1 \cdots b^n]\begin{bmatrix} | & & | \\ D_1h & \cdots & D_nh \\ | & & | \end{bmatrix}\begin{bmatrix} \\ \xi \\ \\ \end{bmatrix}$$

Just lik x^TAy
$b^T H \xi$

$\begin{bmatrix}\pi_1(\xi)\\ \vdots \\ \pi_n(\xi)\end{bmatrix}$

so in the case $\underset{\sim}{\omega}^{\beta} = dy^i$ ★
$(h^*\omega_\beta)(\xi_\alpha) = [0..\overset{\frown}{1}..0]\begin{bmatrix} Dh_x \end{bmatrix}\begin{bmatrix}\xi_\alpha\end{bmatrix}$
$= \sum_{j=1}^{n} D_j h^{ⓘ}\, dx^j(\xi_\alpha)$ No i sum

$$= \sum_i b_i \sum_j D_j h^i\, dx^j(\xi)$$

$\longrightarrow y=h(x)$ so we can say $\dfrac{\partial y^i}{\partial x_j}$

Thus upstairs

$$(\psi^*\omega)^{(\cdot)}_x = \sum_i\sum_j b_i(\psi(x))\frac{\partial y^i}{\partial x_j}(x)\,\underset{\sim}{d}x^j(\cdot)$$

Basis for $T_xU = \left\{\frac{\partial}{\partial x_i}\right\} = \{[e_i]\}$
$T_yW = \left\{\frac{\partial}{\partial y_j}\right\} = \{[f_j]\}$

Ⓓ Now we want to show $\boxed{\psi_{*x}\left(\frac{\partial}{\partial x_i}\right) = \sum_j \frac{\partial y^j}{\partial x_i}(x)\frac{\partial}{\partial y_j}\Big|_{\psi(x)}}$

$\boxed{\psi_{*x}([e_i]_x) = \sum \frac{\partial y^j}{\partial x_i}[f_j]_y}$

$\psi_*([e_i])$ downstairs:

std basis in β chart

$$Dh_x(\hat{e}_i) = \{\hat{f}_1 \cdots \hat{f}_n\}\begin{bmatrix} D_ih^{①} \\ \vdots \\ D_ih^{ⓝ}\end{bmatrix} = \sum_{j=1}^{n} D_i h^{ⓙ}\hat{f}_j \xrightarrow{\text{upstairs}} \sum \frac{\partial y^j}{\partial x_i}[f_j]_{\underset{\sim}{y}} \quad \square$$

cf Schuams LA ch 7 $T: E \longrightarrow F$

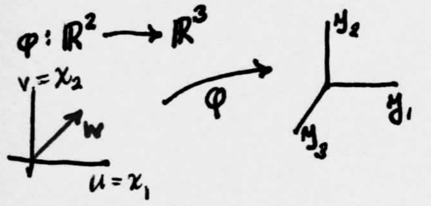
$T(e_i) = \{f_1 .. f_n\}\begin{bmatrix} a_i^{①} \\ \vdots \\ a_i^{ⓝ}\end{bmatrix}$ $\quad T(\{e_1 \dots e_m\}) = \{f_1 \dots f_n\}\begin{array}{c|c} & m \\ \hline n & \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\end{array}$ $\quad T(v) = \{f\}[A][v]_e$

L&S ch 9.8

6/1/2023

⑰

ASIDE: Let's also show how Ⓐ and Ⓒ are related to the discussion in Spivak COM and Munkres AOM Thm 32.2 p.269



Let $\vec{w} \in T_x(\mathbb{R}^2) = \mathbb{R}^2$

$$(\varphi^* dy^i)_x(w) = \underset{\uparrow \text{ d commutes with } *}{dy^i_{\varphi(x)}}(D\varphi_x(w)) = \pi_i\left(\begin{bmatrix} \varphi^1_u & \varphi^1_v \\ \varphi^2_u & \varphi^2_v \\ \varphi^3_u & \varphi^3_v \end{bmatrix}\begin{bmatrix} w^1 \\ w^2 \end{bmatrix}\right)$$

$$= \varphi^{ⓘ}_u w^{①} + \varphi^{ⓘ}_v w^{②} = \frac{\partial \varphi^i}{\partial x_1} dx^1(w) + \frac{\partial \varphi^i}{\partial x_2} dx^2(w)$$

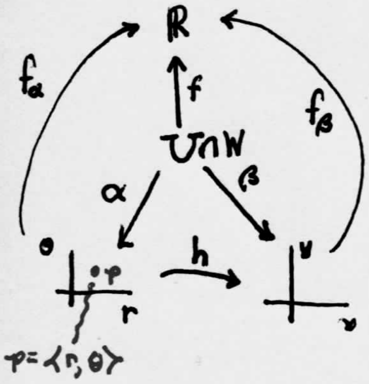
$$= d\varphi^i(w)$$

$$\Rightarrow \boxed{\varphi^* dy^i(\cdot) = d\varphi(\cdot)}$$

□

Variations of chapter prob 8.1

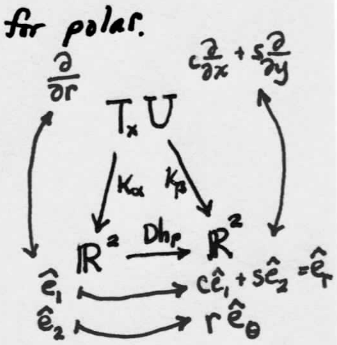
How do $\{\frac{\partial}{\partial x_i}\}$ change in co-ord transform? Show it for polar.



$$f_\alpha = h^* f_\beta = f_\beta \circ h$$

$$D(f_\alpha)_p = D(f_\beta \circ h)_p = D(f_\beta)_x Dh_p$$

$$\boxed{\begin{bmatrix} D_1 f_\alpha & D_2 f_\alpha \end{bmatrix} = \begin{bmatrix} D_1 f_\beta & D_2 f_\beta \end{bmatrix}\begin{bmatrix} \\ Dh_p \\ \\ \end{bmatrix}}$$



We can write this as

$$\begin{bmatrix} \frac{\partial}{\partial r} f_\alpha & \frac{\partial}{\partial \theta} f_\alpha \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} f_\beta & \frac{\partial}{\partial y} f_\beta \end{bmatrix}\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix}$$

If we identify f_β and f_α as the "same fcn" — we <u>identify the domain and range of h</u> as the <u>same</u> copy of $\mathbb{R}^2$, then $\begin{bmatrix} \frac{\partial}{\partial r}(\cdot) & \frac{\partial}{\partial \theta}(\cdot) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix}$

In terms of Schuams LA, if we regard this as a COB $\{f\} = \{e\} P$ upstairs we'd have $\boxed{\left\{\frac{\partial}{\partial r} \ \frac{\partial}{\partial \theta}\right\} = \left\{\frac{\partial}{\partial x} \ \frac{\partial}{\partial y}\right\}\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix}}$

And then for $\{e\} = \{f\} P^{-1}$ $\left\{\frac{\partial}{\partial x} \ \frac{\partial}{\partial y}\right\} = \left\{\frac{\partial}{\partial r} \ \frac{\partial}{\partial \theta}\right\} \nearrow$ $\frac{1}{r}\begin{bmatrix} rc & rs \\ -s & c \end{bmatrix}$

▷ Now let's show COV for $\{dx, dy\}$

From sheet ⑯ the $\boxed{\text{Key Idea for Pullback of forms:}}$

$$\underset{\sim}{\omega}^\beta := \sum b_i dy^i$$

$$\boxed{(h^* \omega_\beta)(\xi_\alpha) = \omega_\beta(Dh_x(\xi_\alpha))}$$

$$= [b_1 \dots b_n]\begin{bmatrix} Dh_x \\ \cdots\cdots \end{bmatrix}\begin{bmatrix} \vec{\xi}_\alpha \end{bmatrix} \begin{matrix} \pi_1(\xi) \\ \vdots \\ \pi_n(\xi) \end{matrix}$$

row $D_j h^i \rightarrow$

$$= \sum_i b_i \sum_j D_j h^i \, dx^j(\xi_\alpha)$$

$\nwarrow$ really π_j

Now for $\omega^\beta = dy^i$ singleton

$$(h^* \omega_\beta)(\xi_\alpha) = [0 \dots 1 \dots 0]\begin{bmatrix} Dh_x \end{bmatrix}\begin{bmatrix} \xi_\alpha \end{bmatrix} = \sum_{j=1}^n D_j h^i dx^j(\xi_\alpha) \quad \text{No } i \text{ sum.}$$

so for us here $\omega^\beta = dy^1$

$$(h^* \omega_\beta)(\xi_\alpha) = [1 \ 0]\begin{bmatrix} Dh_p \end{bmatrix}\begin{bmatrix} dr(\xi) \\ d\theta(\xi) \end{bmatrix}$$

$h^* dy^1 \longrightarrow$

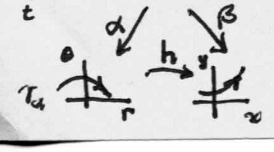
$$= \sum_{j=1}^2 D_j h^{①} dx^j = [1 \ 0]\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix}\begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

We identify the <u>domain and range</u> of h

Thus "h^*" = Id and we rename y_1, y_2 as x, y

$$h^* dx = dx \longrightarrow \boxed{\begin{aligned} dx &= \cos\theta \, dr - r\sin\theta \, d\theta \\ dy &= \sin\theta \, dr + r\cos\theta \, d\theta \end{aligned}}$$

Compare this with another way, in terms of curves



$$\gamma_\beta(t) = h(\gamma_\alpha(t))$$

$$\dot{\gamma}_\beta(t) = Dh_p(\dot{\gamma}_\alpha)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} c & -rs \\ s & rc \end{bmatrix}\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix}$$

$$\frac{dx}{\cancel{dt}} = c\frac{dr}{\cancel{dt}} - rs\frac{d\theta}{\cancel{dt}},$$

$$\frac{dy}{\cancel{dt}} = s\frac{dr}{\cancel{dt}} + rc\frac{d\theta}{\cancel{dt}}$$

L&S haven't mentioned this yet, but a 1-form is what we integrate over a curve

$$\int \vec{F}(\sigma(t)) \cdot \vec{\sigma}'(t) \, dt$$

$$F^{①}\dot{x} + F^{②}\dot{y} + F^{③}\dot{z}$$

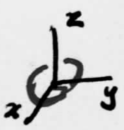
□

L&S ch 9.8
p. 396

5/20/2023

Problem 8.2 x, y, z co-ords on $\mathbb{R}^3$. Consider vfs

$$X := y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$
$$Y := z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$
$$Z := x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$



(a) Show Z is the inf generator for circular flows around the z-axis (rotations) (Likewise for X, Y)

(b) Compute Lie Bracket $[X, Y]$ (likewise $[X,Z]$, $[Y,Z]$)

(a) The partial deriv vf notation looks weird, but it is just $Z = x\,[e_2]_x - y\,[e_1]_x$ or even better here, since $\mathbb{R}^3$ has its own natural chart $Z = x\hat{e}_2 - y\hat{e}_1 = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$

This gives rise to $\dot{x} = Z(x)$ i.e. $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} = \left[\begin{array}{cc:c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \dot{x} = Ax$

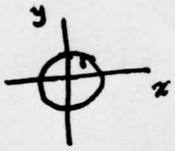
Thus we see $z(t) = \text{const}$ and the flow of any pt takes place in an x-y plate at z_0. We can reduce to the familiar 2-dim case studied in A&P ODE ch 2 [see my sheets "Linear Sys of ODEs" $\dot{x} = Ax$ 3/12/2016]. From that other paper, $\dot{x} = Jx$ (J Jordan form of matrix) we see the soln is $x(t) = e^{tJ}x_0$ and it is shown on sheet ⑤ there:

$$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \Rightarrow e^{tJ} = \begin{bmatrix} e^{\alpha t} & \\ & e^{\alpha t} \end{bmatrix} \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}$$

But we have $\alpha = 0$ and $\beta = 1$

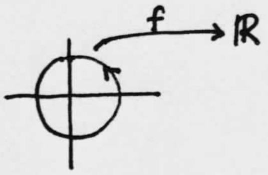
To demonstrate the circular flow, we can solve $\begin{array}{l} \dot{x} = -y \\ \dot{y} = x \end{array}$ directly with polar COV $\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$

and get $\boxed{\begin{array}{l} x(t) = r_0 \cos\theta(t) \\ y(t) = r_0 \sin\theta(t) \end{array}}$ $\begin{array}{l} \theta(t) = \beta t + \theta_0 = t \text{ here} \\ x^2(t) + y^2(t) = r_0^2 \end{array}$



▷ Insight into notation

Consider this vf Z and its flow φ



The Lie deriv $\mathcal{L}_Z f = \frac{d}{dt} f(\varphi(t)) = Df_{\varphi(t)}\,\dot{\vec{\varphi}}(t) = [D_1 f \;\; D_2 f]\begin{bmatrix} \dot{\varphi}^{①} \\ \dot{\varphi}^{②} \end{bmatrix}$

But $\dot{\varphi} = Z(\varphi)$ that is $\begin{bmatrix} \dot{\varphi}^{①} \\ \dot{\varphi}^{②} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \varphi^{①} \\ \varphi^{②} \end{bmatrix}$ a.k.a. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

So we can re-write: $\mathcal{L}_Z f = [D_1 f \;\; D_2 f]\begin{bmatrix} \dot{\varphi}^{①} \\ \dot{\varphi}^{②} \end{bmatrix} = \left[\frac{\partial}{\partial x}f \;\; \frac{\partial}{\partial y}f\right]\begin{bmatrix} -y \\ x \end{bmatrix}$

$$= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = \boxed{\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(f)}$$

$\boxed{\text{or } Zf}$ of p. 384 [sheet ⑤]

cont'd →

5/20/2023

⑲

8.2 cont'd

ord calculus Frechet derivs

ⓑ Lets compute Lie Bracket $[X,Y]_x = (\mathcal{L}_X Y)_{\alpha=\mathrm{Id}} = D(Y_\alpha)_x(X_\alpha(x)) - D(X_\alpha)_x(Y_\alpha(x))$

$$X(x) = y\overset{\hat e_3}{\frac{\partial}{\partial z}} - z\overset{\hat e_2}{\frac{\partial}{\partial y}} = \begin{bmatrix}0\\-z\\y\end{bmatrix}$$

$$Y(x) = z\underset{\hat e_1}{\frac{\partial}{\partial x}} - x\underset{\hat e_3}{\frac{\partial}{\partial z}} = \begin{bmatrix}z\\0\\-x\end{bmatrix}$$

$$\begin{tikzcd} & X \arrow[dr] & \\ Z \arrow[ur] & & Y \arrow[ll] \end{tikzcd}$$

Then

$$DY_x = \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix} \qquad \mathcal{L}_X Y = \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}\begin{bmatrix}0\\-z\\y\end{bmatrix} - \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}z\\0\\-x\end{bmatrix}$$

$$DX_x = \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix} \qquad = \begin{bmatrix}y\\0\\0\end{bmatrix} - \begin{bmatrix}0\\-x\\0\end{bmatrix} = \begin{bmatrix}y\\-x\\0\end{bmatrix} = -Z$$

This makes me think I have a sign err (or L&S does).

▷ Alternate method - From sheet ⑧ p.388 (6.10)

SAME

$$(\mathcal{L}_X Y)_\alpha(x) = D(Y_\alpha)_x(X_\alpha(x)) - D(X_\alpha)_x(Y_\alpha(x))$$

$$= \mathcal{L}_{X_\alpha}\vec{Y}_\alpha - \mathcal{L}_{Y_\alpha}\vec{X}_\alpha \qquad \mathcal{L}_X f = Xf$$

$$= \mathcal{L}_{(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})}\vec{Y}_\alpha - \mathcal{L}_{(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})}\vec{X}_\alpha \qquad \mathcal{L}_{aX+bY} f = aXf + bYf$$

$$= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\begin{bmatrix}z\\0\\-x\end{bmatrix} - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\begin{bmatrix}0\\-z\\y\end{bmatrix}$$

$$= y\begin{bmatrix}1\\0\\0\end{bmatrix} - z\begin{bmatrix}0\\0\\0\end{bmatrix} - z\begin{bmatrix}0\\0\\0\end{bmatrix} + x\begin{bmatrix}0\\-1\\0\end{bmatrix} = \begin{bmatrix}y\\-x\\0\end{bmatrix}$$

▷ Lets compute $[Y,Z] = \mathcal{L}_Y Z$

$$Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \begin{bmatrix}-y\\x\\0\end{bmatrix} \qquad \mathcal{L}_Y Z = \mathcal{L}_{Y_\alpha} Z_\alpha - \mathcal{L}_{Z_\alpha} Y_\alpha$$

$$= \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\begin{bmatrix}-y\\x\\0\end{bmatrix} - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\begin{bmatrix}z\\0\\-x\end{bmatrix}$$

$$= z\begin{bmatrix}0\\1\\0\end{bmatrix} - x\begin{bmatrix}0\\0\\0\end{bmatrix} - x\begin{bmatrix}0\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\z\\-y\end{bmatrix} = -X$$

L&S ch 9.8

5/24/2023
⑳

Problem 8.3 Let vf $A := y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y} = \begin{bmatrix} 0 \\ z \\ y \end{bmatrix}$

$$B := x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} = \begin{bmatrix} z \\ 0 \\ x \end{bmatrix}$$

chart $\alpha = \text{Id}$

$$C := x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

ⓐ Compute Lie Bracket:

$$\begin{aligned}[A, B] = \mathcal{L}_A B &= \mathcal{L}_{A_\alpha} B_\alpha - \mathcal{L}_{B_\alpha} A_\alpha \\ &= \left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)\begin{bmatrix} z \\ 0 \\ x \end{bmatrix} - \left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right)\begin{bmatrix} 0 \\ z \\ y \end{bmatrix} \\ &= y\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - x\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} = -C\end{aligned}$$

ⓑ Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x,y,z) \longmapsto x^2 + y^2 - z^2$ hyperboloid $f^{-1}(c)$

show $Af = 0$ $Bf = 0$ $Cf = 0$

I think this means f is a <u>conserved quantity</u> of each of these flows — a <u>Const of Motion</u>

First Integral

$$\begin{aligned} Af = \mathcal{L}_A f &= \left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)f \\ &= \left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)(x^2+y^2-z^2) \\ &= y(-2z) + z(2y) = 0 \ ✓ \end{aligned}$$

ⓒ sketch the integral curves for A, B, C

$$\dot{x} = A(x) \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ y \end{bmatrix} \Rightarrow \begin{aligned} x &= x_0 \text{ const} \\ \dot{y} &= z \\ \dot{z} &= y \end{aligned} \Rightarrow \ddot{y} = \dot{z} = y$$

harmonic oscillator

$$\begin{aligned} y(t) &= K_1 \cos t + K_2 \sin t \\ &= K \cos(\omega t + t_0) \end{aligned}$$

L&S ch9.9 Riemann Metrics 6/25/2023

a <u>Riemann metric</u> is really an <u>Inner Prod on T_xM</u> (so it should be called a 'Riemann IP' – But we can use it to find the norm of vectors, and thus, by integration, the length of curves on M).

In <u>my notation</u> $\mathfrak{m} : M \longrightarrow \left\{ \begin{matrix} \text{bilinear forms} \\ \text{on } TM \end{matrix} \right\}$

$x \longmapsto \langle\!\langle [\cdot]_x , [\cdot]_x \rangle\!\rangle$ where $\langle\!\langle \cdot , \cdot \rangle\!\rangle : T_xM \times T_xM \longrightarrow \mathbb{R}$

in a chart $\qquad \xi, \eta \longmapsto \sum_i \sum_j g_{ij}(x)\, dx^i(\xi)\, dx^j(\eta)$ $\qquad dx^j(\eta)$

In a chart (U, α) $\qquad$ In T_xU $\quad \xi = \sum \xi^i_\alpha [e_i]_x = \sum \xi^i \frac{\partial}{\partial x_i}$

$$\begin{tikzcd} U \arrow[r, "\mathfrak{m}^\alpha"] \arrow[d, "\alpha"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "m_\alpha"'] & \end{tikzcd}$$

$$\underset{\sim}{m}^\alpha_x(\xi, \eta) := \langle\!\langle \xi, \eta \rangle\!\rangle_x \overset{\text{linearity}}{=} \sum_i^n \sum_j^n \xi^{ⓘ}_\alpha \eta^{ⓙ}_\alpha \underbrace{\langle\!\langle [e_i], [e_j] \rangle\!\rangle}_{g_{ij}(x)} \quad \left[\text{eg } (9.2) \text{ and } (9.1)\right]$$

Following convention back end term becomes fcnl.

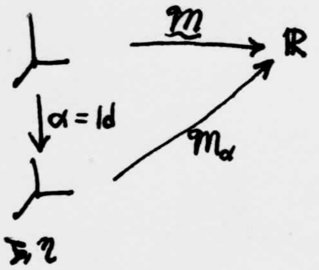
$$= \vec{\xi}_\alpha^{\,T} G \vec{\eta}_\alpha \quad \left[\text{probably should be } \eta_\alpha^T G \xi_\alpha\right]$$

G must be symm pos def matrix.

If M is embedded in $\mathbb{R}^N$, then T_xM just inherits the IP from $\mathbb{R}^N$, so we can write $\langle\!\langle \cdot , \cdot \rangle\!\rangle = \langle \cdot , \cdot \rangle$ ordinary IP

Let's do some examples:

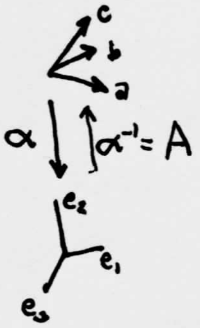
① Let's consider $M = \mathbb{R}^n$ (or even a solid ball in $\mathbb{R}^n$, say)



Here we identify $\vec{\xi}_\alpha = \xi = [\varphi]_x$ in $\mathbb{R}^N$

$m_\alpha(\xi, \eta) = \eta^T I \xi = \eta^T \xi$ std IP of vectors.

② A more illustrative example is when the mfd $M = \mathbb{R}^n$ has a non-O.N. basis.



we know $g_{ij}(x) = \langle\!\langle [e_i], [e_j] \rangle\!\rangle$ and $K_\alpha([e_i]) = \hat{e}_i$

i.e. $\alpha_{*x}([e_i]) = \hat{e}_i$

That means $\alpha_*^{-1}(\hat{e}_i) = [e_i]$

In this linear case $\alpha_*^{-1} = A$ $\quad [D\alpha_x^{-1} = A]$

$A e_1 = \vec{a}$
$A e_2 = \vec{b}$ $\quad \Rightarrow A = \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix}$
$A e_3 = \vec{c}$

$$\begin{aligned} g_{ij} &= \langle\!\langle [e_i], [e_j] \rangle\!\rangle \\ &= \langle\!\langle \alpha_*^{-1} e_i , \alpha_*^{-1} e_j \rangle\!\rangle \\ &= \langle A e_i , A e_j \rangle \\ &= e_j^T \underbrace{A^T A}_{G} e_j \end{aligned}$$

$$G = \begin{bmatrix} a^Ta & a^Tb & a^Tc \\ b^Ta & b^Tb & b^Tc \\ c^Ta & c^Tb & c^Tc \end{bmatrix}$$

KEY IDEAS: $G = A^TA$

NOTE: Strang LAAIA ch 6 my write up sheet ②

G symm pos def $\Leftrightarrow$ $\exists$ matrix A with LI cols (equivalently A one-to-one) such that $G = A^TA$

6/27/2023

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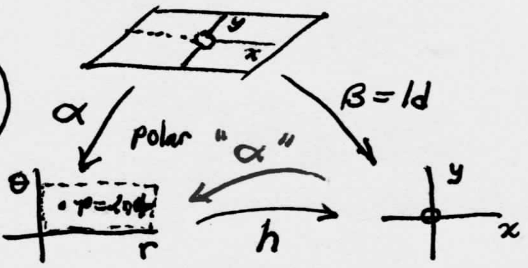
③ Now consider the plane $\mathbb{R}^2$



L&S give $\boxed{\text{eq. (9.7)}: dx^2 + dy^2 = dr^2 + r^2 d\theta^2}$

I will derive this 2 ways:

ⓐ We can regard β as an identification and then "α" $= h^{-1}$ is a chart

$$\varphi = h(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

ⓑ We can regard h as a COV (transform between charts) and compute the pullback of Riemann IP : $m_\alpha = h^* m_\beta$

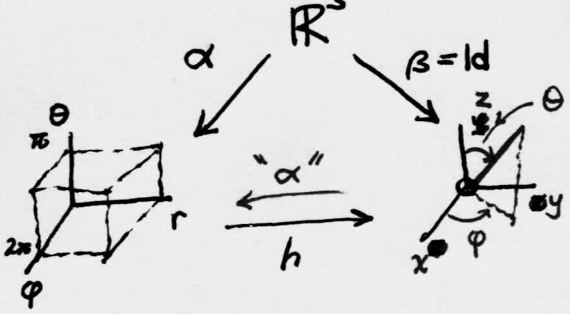
First we will show (a) and then use the same method for spherical.

"α" $:= h^{-1}$ so $h = \alpha^{-1}$ and $Dh_p = D(\alpha^{-1})_p =: A$

Then $G = A^T A = Dh_p^T Dh_p = \begin{bmatrix} c & s \\ -rs & rc \end{bmatrix}\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$

$$\underset{\text{polar}}{m_\alpha}(\xi,\eta) = \sum^2\sum^2 dx^i(\xi)\,dx^j(\eta)\,g_{ij}(x) = [dr(\cdot)\ d\theta(\omega)]\begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}\begin{bmatrix} dr \\ d\theta \end{bmatrix} = \boxed{dr^2 + r^2 d\theta^2}$$

▷ Now let's do it for spherical



[L&S switch φ, θ from M&T]

$$h(r, \overset{\varphi}{\cancel{\theta}}, \overset{\theta}{\cancel{\varphi}}) = \begin{bmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = Dh_p = \begin{bmatrix} C_\varphi S_\theta & -rS_\varphi S_\theta & rC_\varphi C_\theta \\ S_\varphi S_\theta & rC_\varphi S_\theta & rS_\varphi C_\theta \\ C_\theta & 0 & -rS_\theta \end{bmatrix}$$

$$A^T A = \begin{bmatrix} C_\varphi S_\theta & S_\varphi S_\theta & C_\theta \\ -rS_\varphi S_\theta & rC_\varphi S_\theta & 0 \\ rC_\varphi C_\theta & rS_\varphi C_\theta & -rS_\theta \end{bmatrix}\begin{bmatrix} C_\varphi S_\theta & -rS_\varphi S_\theta & rC_\varphi C_\theta \\ S_\varphi S_\theta & rC_\varphi S_\theta & rS_\varphi C_\theta \\ C_\theta & 0 & -rS_\theta \end{bmatrix} = \begin{bmatrix} 1 & & \\ & r^2 S_\theta^2 & \\ & & r^2 \end{bmatrix}$$

$$\Rightarrow [dr\ d\varphi\ d\theta]\begin{bmatrix} 1 & & \\ & r^2 S_\theta^2 & \\ & & r^2 \end{bmatrix}\begin{bmatrix} dr \\ d\varphi \\ d\theta \end{bmatrix} = \boxed{dr^2 + r^2 S_\theta^2\, d\varphi^2 + r^2 d\theta^2}$$

This is eq (9.9)

6/28/2023

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▷ Before Specifically doing the COV version ③ⓑ I want to give a general discussion of COV (not just for polar)

Just as in my ch 9.8 writeup with parts Ⓐ and Ⓒ, we can do the same thing here. The same pattern works if $\psi : M \to N$ is a map between charts on different mfds, or just a chart COV (in which case $\psi = Id$)

$$\begin{tikzcd}
 & \mathbb{R} & \\
U \arrow[ur, "\underset{\sim}{m}^{\alpha}"] \arrow[rr, "\psi"] \arrow[d, "\alpha"'] & & W \arrow[ul, "\underset{\sim}{m}^{\beta}"'] \arrow[d, "\beta"] \\
\mathbb{R}^m \arrow[rr, "h"] \arrow[uur, bend left, "m_\alpha = h^* m_\beta"] & & \mathbb{R}^n \arrow[uul, bend right, "m_\beta"']
\end{tikzcd}$$

$\textcircled{\cdot p}$

(i) My way (thinking in terms of matrices)

$$\underset{\sim}{m}^{\alpha}_{x}(\xi, \eta) = \sum_i \sum_j \xi_\alpha^{\textcircled{i}} \eta_\alpha^{\textcircled{j}} \underbrace{\langle\!\langle \overset{\text{"}\frac{\partial}{\partial x_i}\text{"}}{[e_i]}, \overset{\text{"}\frac{\partial}{\partial x_j}\text{"}}{[e_j]} \rangle\!\rangle}_{g_{ij}} = \xi_\alpha^T G \eta_\alpha$$
$$= \eta_\alpha^T G \xi_\alpha$$

$$[e_i] = \alpha^{-1}_{*}(e_i) = \underbrace{D(\alpha^{-1})_x}_{A}(e_i)$$

At least in the case $\alpha^{-1} : \mathbb{R}^n \hookrightarrow \mathbb{R}^N$

$$g_{ij} = \langle Ae_i, Ae_j \rangle = \langle \vec{a}_i, \vec{a}_j \rangle \text{ cols of } A$$

$$m_\alpha(\xi_\alpha, \eta_\alpha) = \langle A\xi_\alpha, A\eta_\alpha \rangle = \eta_\alpha^T \underbrace{A^T A}_{G} \xi_\alpha$$

Likewise, a priori, $m_\beta(\xi_\beta, \eta_\beta) = \eta_\beta^T \underbrace{B^T B}_{P} \xi_\beta$

But we have $m_\alpha \overset{!}{=} h^* m_\beta$

That is to say $m_\alpha(\xi_\alpha, \eta_\alpha) = m_\beta(Dh_p(\xi_\alpha), Dh_p(\eta_\alpha))$

$$\eta_\alpha^T \underbrace{A^T A}_{G} \xi_\alpha \qquad \eta_\alpha^T Dh_p^T \underbrace{B^T B}_{P} Dh_p \xi_\alpha$$

pt "p" no relation to matrix P

$$\Rightarrow \boxed{G = Dh_p^T P Dh_p \quad \text{eq}(9.5)}$$

From this picture:

$\beta = h \circ \alpha$

$h^{-1} \circ \beta = \alpha$

Invert both sides $\beta^{-1} \circ h = \alpha^{-1}$

Differentiate:

$$\underbrace{D\beta^{-1}}_{B} Dh_p = \underbrace{D\alpha^{-1}}_{A}$$

Thus directly from this:

$$(B\,Dh_p)^T B\,Dh_p = A^T A$$

$$\boxed{Dh_p^T B^T B\,Dh_p = G}$$

(ii) Now let's attempt L&S version (sums of components) although still using some of my notation

Let $q(x) = Q(y(x))$

Then $$\frac{\partial q}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial Q}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \Big(\sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j} \Big) Q \Rightarrow$$

Renaming fcn y as h; take n=3

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial h^{\textcircled{j}}}{\partial x_i} \frac{\partial}{\partial y_j} = \begin{bmatrix} D_i h^{\textcircled{1}} & D_i h^{\textcircled{2}} & D_i h^{\textcircled{3}} \end{bmatrix} \begin{bmatrix} \partial_{y_1} \\ \partial_{y_2} \\ \partial_{y_3} \end{bmatrix}$$
$$= (D_i \vec{h})^T \vec{\partial}_y$$

Thus $$g_{ij} = \langle\!\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle\!\rangle = \langle\!\langle D_i h^T \vec{\partial}_y, D_j h^T \vec{\partial}_y \rangle\!\rangle$$
$$= \langle\!\langle \sum_k D_i h^{\textcircled{k}} \frac{\partial}{\partial y_k}, \sum_l D_j h^{\textcircled{l}} \frac{\partial}{\partial y_l} \rangle\!\rangle$$
$$= \sum_k \sum_l D_i h^{\textcircled{k}} D_j h^{\textcircled{l}} \langle\!\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \rangle\!\rangle \quad \boxed{\text{eq } 9.5}$$

Thus $$g_{ij} = \begin{bmatrix} D_j h^{\textcircled{1}} & D_j h^{\textcircled{2}} & D_j h^{\textcircled{3}} \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix} \begin{bmatrix} D_i h^1 \\ D_i h^2 \\ D_i h^3 \end{bmatrix}$$

$$\boxed{G = Dh_p^T P Dh_p}$$

SAME!

6/30/2023

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At last, we can close out example ③ⓑ:

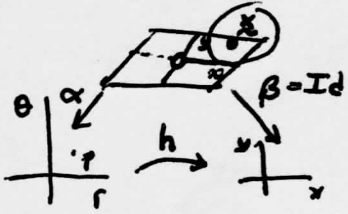
In the polar co-ord example $B = Id$ (Remark: we could also have B be any O.N. matrix).

$$m_\beta(\cdot,\cdot) = [dx\ dy]\, I^T I \begin{bmatrix} dx \\ dy \end{bmatrix} = dx^2 + dy^2$$

$$m_\alpha(\cdot,\cdot) = h^* m_\beta(\cdot,\cdot) = m_\beta(Dh_p(\cdot), Dh_p(\cdot)) = [dr\ d\theta] \begin{bmatrix} Dh_p^T \end{bmatrix} I \begin{bmatrix} Dh_p \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

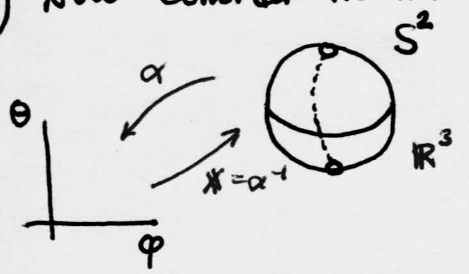
$$= [dr\ d\theta] \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix} = dr^2 + r^2 d\theta^2$$

Now if we equate the representations for the same pt $\underset{\sim}{x}$ in the mfd:



$$\boxed{dx^2 + dy^2 = dr^2 + r^2 d\theta}$$ This is eq 9.7

$\textcircled{4}$ Now consider the more general case where n-mfd embedded in $\mathbb{R}^N$ $n < N$.



L&S do it like this:

On Sheet ㉒ we computed m for spherical co-ord map.

$$(9.9)\quad dx^2 + dy^2 + dz^2 = dr^2 + r^2 S_\theta^2 d\varphi^2 + r^2 d\theta^2$$

we have the inclusion map $i : S^2_{(0,1)} \hookrightarrow \mathbb{R}^3$ $\quad i = Id\big|_{S^2}$

So $r \equiv 1$ const.

$$i^*(dx^2 + dy^2 + dz^2) = i^*(\overset{dr = 0}{dr^2} + r^2 \sin^2\theta\, d\varphi^2 + r^2 d\theta^2)$$

$$= \underset{0}{d(r \circ i)^2} + 1^2 \sin^2\theta \underbrace{(i^* d\varphi)^2}_{(d\bar\varphi)^2} + \underbrace{(i^* d\theta)^2}_{(d\bar\theta)^2}$$

$$\boxed{= \sin^2\theta\, (d\bar\varphi)^2 + (d\bar\theta)^2} \quad (9.10)$$

p. 400

They seem to be defining

$i^*(f\, dx \cdot dx)_m [\gamma]_m =$

$(f \circ i)_{(m)} \cdot dx_{i(m)} [i \circ \gamma]_m \cdot dx_{i(m)} [i \circ \gamma]_m$

of course $i(m) = m$

Now lets give doCarmo DGOCAS ch 2.5

1st Fundamental Form (Norm on T_pS)

Let T_pS inherit the inner prod $\langle \cdot,\cdot \rangle$ from $\mathbb{R}^3$. Define 1st Fund. Form $I_p : T_pS \longrightarrow [0,\infty)$, $v \longmapsto \langle v,v \rangle = \|v\|^2$

Express I_p in terms of basis $\{X_u, X_v\}$:

For any $w \in T_pS$, $\exists\ \vec{a} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \ni DX_u(\vec{a}) = \vec{w} \in \mathbb{R}^3$

Then
$$I_p(w) = \langle w,w \rangle = w^T w = a^T DX_u^T DX_u a = [a,\ b] \begin{bmatrix} \overset{E}{X_u^T X_u} & \overset{F}{X_u^T X_v} \\ \underset{F}{X_v^T X_u} & \underset{G}{X_v^T X_v} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= Ea^2 + 2Fab + Gb^2$$

Then for the sphere S^2 (Fix $r = R_0 = 1$)

$$X : (0,\pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^3$$
$$(\theta,\varphi) \longmapsto \begin{bmatrix} R_0 S_\theta C_\varphi \\ R_0 S_\theta S_\varphi \\ R\ C_\theta \end{bmatrix}$$

$$DX_{(\theta,\varphi)} = \begin{bmatrix} R_0 C_\theta C_\varphi & -R_0 S_\theta S_\varphi \\ R_0 C_\theta S_\varphi & R S_\theta \cancel{S_\varphi} C_\varphi \\ \underset{\vec{X}_\theta}{R_0 S_\theta} & \underset{\vec{X}_\varphi}{0} \end{bmatrix}$$

$E = \langle X_\theta, X_\theta \rangle = R_0^2$

$F = \langle X_\theta, X_\varphi \rangle = 0$

$G = \langle X_\varphi, X_\varphi \rangle = R_0^2 \sin^2\theta$

$$\vec{w} = a\vec{X}_\theta + b\vec{X}_\varphi$$

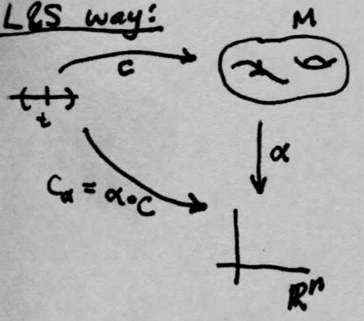
$$I_p(w) = \|w\|^2 = Ea^2 + 2Fab + Gb^2 = a^2 + \sin^2\theta\, b^2$$
$$= R_0 a^2 + 0 + R_0^2 \sin^2\theta\, b^2 \quad \boxed{= d\theta^2 + \sin^2\theta\, d\varphi^2}$$

□

6/30/2023

▷ Now we want to finally start using this to measure the lengths of curves on M:

L&S way:



$$\begin{tikzcd} T_t\mathbb{R}^1 \arrow[r, "c'"] \arrow[d, "K_{id}"'] & T_{c(t)}M \arrow[d, "K_\alpha"] \\ \mathbb{R}_t \arrow[r, "c'_\alpha"] & \mathbb{R}^n \end{tikzcd}$$

we define $c'(s) = c_*\left(\frac{\partial}{\partial t}\right)_{(s)}$

$$c_\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \begin{bmatrix} x^{\textcircled{1}} \circ c(t) \\ \vdots \\ x^{\textcircled{n}} \circ c(t) \end{bmatrix}$$

For c_α, by ordinary calculus,

$$c'_\alpha(t) = \begin{bmatrix} \dot{c}_\alpha^{\textcircled{1}}(t) \\ \vdots \\ \dot{c}_\alpha^{\textcircled{n}}(t) \end{bmatrix}$$

$$\|c'(t)\| = \sqrt{\langle\!\langle c'(t), c'(t) \rangle\!\rangle}$$

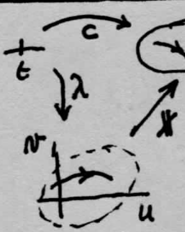
Given chart (U,α) $\quad c'(t) = \xi \in T_{c(t)}M$ but $\xi = \sum \dot{c}_\alpha^{\textcircled{i}}(t)[e_i]_{c(t)}$

$$\|c'(t)\| = \left[\sum\sum g_{ij}(c(t))\,\dot{c}_\alpha^{\textcircled{i}}\,\dot{c}_\alpha^{\textcircled{j}}\right]^{1/2}$$

$\int_c \|c'(t)\|\,dt$ is arc length of curve c

Arc Len is indep of parameterization (allowable params) cf my writeup M&T VC ch 6

DoCarmo DGOCAS p.95 (changing some names)



$$\lambda(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

$$\text{length}(\lambda) = s(t) = \int_0^t \|\lambda'(t)\|\,dt$$

$$\|\lambda'\| = \sqrt{\langle \lambda', \lambda' \rangle}$$

$$c = \mathbf{X} \circ \lambda$$
$$\dot{c} = D\mathbf{X}_{\lambda(t)}(\dot{\lambda}(t))$$
$$\langle \dot{c}, \dot{c} \rangle = \dot{\lambda}^T D\mathbf{X}^T D\mathbf{X}\,\dot{\lambda} = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

Then $\underset{\text{length}(c)}{s(t)} = \int_0^t \|\dot{c}\|\,dt = \int_0^t \sqrt{E\dot{u} + 2F\dot{u}\dot{v} + G\dot{v}^2}\,dt$

Thus people like to say

$$\text{“}ds^2\text{”} - E\,du^2 + 2F\,du\,dv + G\,dv^2\text{”}$$

what is the angle between 2 intersecting curves?

we know
$\vec{a} \cdot \vec{b} = \|a\|\,\|b\| \cos\theta$

$$\Rightarrow \cos\theta = \frac{\dot{\mu} \cdot \dot{\lambda}}{\|\dot{\mu}\|\,\|\dot{\lambda}\|}$$

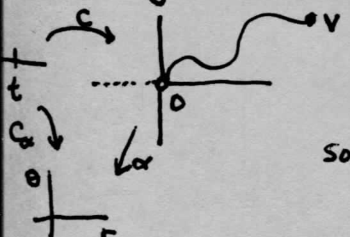
what is angle between $\vec{\mathbf{X}}_u$ and $\vec{\mathbf{X}}_v$

$$\cos\theta = \frac{\langle \mathbf{X}_u, \mathbf{X}_v \rangle}{\|X_u\|\,\|X_v\|} = \frac{F}{\sqrt{E}\sqrt{G}}$$

&S p.402

xample) Show the straight line has the shortest len of all curves in $\mathbb{R}^2$ joining 2 pts.

y translation and rotation invariance, we may take the 1st pt s the origin and 2nd pt not on neg x axis — so we are n the image of the std polar co-ord chart (never mind that 0 is technically excluded).



$c : [0,1] \longrightarrow \mathbb{R}^2$
$c(0) = 0$
$c(1) = v$

$$c_\alpha(t) = \begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} \qquad \mathfrak{m}_\alpha(\dot{c}_\alpha, \dot{c}_\alpha) = \dot{r}^2 + r^2\dot{\theta}^2$$

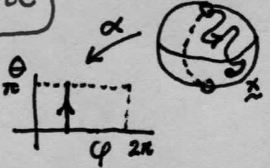
$$\text{so } \operatorname{len}(c) = \int_0^1 \|c'(t)\|\,dt = \int_0^1 \sqrt{\dot{r}^2 + \underbrace{r^2\dot{\theta}^2}_{\text{drop this term}}}\,dt \geq \int_0^1 |\dot{r}|\,dt \geq \int_0^1 \dot{r}\,dt = \underset{\|v\|}{r(1)} - \underset{0}{r(0)} = \|v\|$$

And we have equality iff $\dot{\theta} \equiv 0$ & $\dot{r} \geq 0$ ⇒ Straight line has shortest length ☐

xample 2) Show (portions of) great circles are geodesics on S^2

Cf. Troutman VCWEC ch 1 p.16

$C : [0,1] \longrightarrow S^2$
$C(0) = \langle 0,0,1 \rangle$ North pole
$C(1) = \underset{\sim}{x} \neq \langle 0,0,-1 \rangle$ So not South pole
$\theta_1 := \theta(\underset{\sim}{x}) = \theta(c(1))$



$\mathfrak{m} = \sin^2\theta\,d\varphi^2 + d\theta^2$

$$\text{ngth}(c) = \int_0^1 \|\dot{c}\|\,dt = \int_0^1 \sqrt{\dot{\theta}^2 + \underbrace{\sin^2\theta\,\dot{\varphi}^2}_{\text{drop this term}}}\,dt \geq \int_0^1 |\dot{\theta}|\,dt$$

Let t_1 denote the 1st pt in $(0,1]$ where $\theta = \theta_1$

$$\text{Then } \int \|c'\|\,dt \geq \int |\dot{\theta}|\,dt \geq \int_0^{t_1} \dot{\theta}\,dt = \theta(t_1) - \theta(0) = \theta_1 - 0 = \theta_1$$

All ineqs become equality iff $\dot{\varphi} \equiv 0$ and $t_1 = 1$

Then the entire curve is just varying θ ⇒ tracing a portion of a great circle ☐

both of these examples, we had a lucky choice of co-ords that let us minimize the length. Ch 13 section 11 we shall see how we can always do this, at least locally.