L&S ch 9 Differentiable Mfds - Summary and Extensions

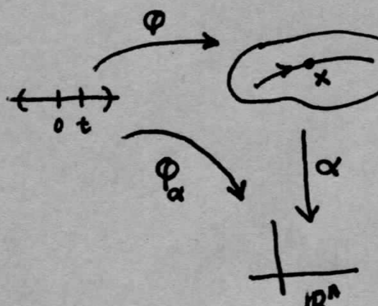
12/25/2022
Christmas ①

I'm not writing up some introductory material. For that see my write ups for G&P and doCarmo. I put the material about a vector ξ acting on a fcn f in ch 9.6 Lie Deriv section sheet ⑤

Ch 9.4 The Tangent Space p. 373

For mfds not embedded in $\mathbb{R}^N$, T_xM is comprised of equivalence classes of curves.

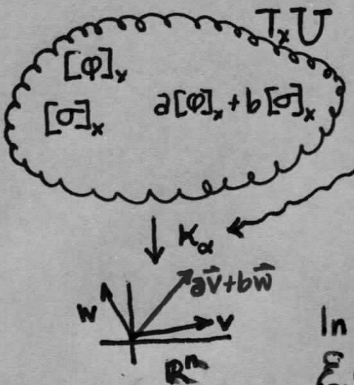
Let (U,α) be a chart, fix $x \in U$ and we know $T_xM = T_xU$



A 'vector' in T_xU

$$\xi = [\varphi]_x = \left\{ \begin{matrix} \text{all curves} \\ \sigma : I \to M \end{matrix} \;\middle|\; \begin{matrix} \sigma(0) = x \\ (\alpha\circ\sigma)'(0) = \vec{v} = (\alpha\circ\varphi)'(0) \end{matrix} \right\}$$

Fixed vector in $\mathbb{R}^n$



This map is an <u>iso by design</u> to make T_xM a vector sp.

$$K_\alpha([\varphi]_x) := D(\alpha\circ\varphi)_0(1) \quad \text{ordinary calculus deriv}$$
$$= (\alpha\circ\varphi)'(0) = \vec{v} \in \mathbb{R}^n$$
$$= \vec{\varphi}_\alpha'(0)$$

We <u>define</u> $a[\varphi]_x + b[\sigma]_x := K_\alpha^{-1}\big(a\underset{v}{\vec{\varphi}_\alpha'(0)} + b\underset{w}{\vec{\sigma}_\alpha'(0)}\big)$

In other words, downstairs in $\mathbb{R}^n$ we have a curve

$$\xi_\alpha(t) = a\,\varphi_\alpha(t) + b\,\sigma_\alpha(t) \quad \text{then}$$
$$\xi_\alpha'(0) = a\,\varphi_\alpha'(0) + b\,\sigma_\alpha'(0) = a\,d(\alpha\circ\varphi)_0 + b\,d(\alpha\circ\sigma)_0$$

deriv is linear
$$= d\big(a(\alpha\circ\varphi) + b(\alpha\circ\sigma)\big)_0(1)$$
$$= \vec{u} \in \mathbb{R}^n$$

p. 374-375

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "\varphi"] & U \arrow[r, "\Psi"] \arrow[d, "\alpha"] & W \arrow[d, "\beta"] \\ & \mathcal{U} \subseteq \mathbb{R}^m \arrow[r, "h = \beta\circ\Psi\circ\alpha^{-1}"] & \mathcal{W} \subseteq \mathbb{R}^n \end{tikzcd}$$

mfd M   mfd N

$x \mapsto v = \alpha(x)$

$$\begin{tikzcd} T_xU \arrow[r, "\Psi_{*x} \text{ or } T\Psi_x"] \arrow[d, "K_\alpha"] & T_{\Psi(x)}W \arrow[d, "K_\beta"] \\ \mathbb{R}^m_v \arrow[r, "Dh_v", "dh_v = d(\beta\circ\Psi\circ\alpha^{-1})"'] & \mathbb{R}^n_{h(v)} \end{tikzcd}$$

$\xi = [\varphi]_x$ ← equiv class of curves thru x

$[\Psi\circ\varphi]_{\Psi(x)}$

"almost" α_* but $\alpha_*([\varphi]_x) = [\alpha\circ\varphi]_{\alpha(x)}$

$$\xi = [\varphi]_x \longmapsto \xi_\alpha = d(\alpha\circ\varphi)_0(1)$$

$$\eta_\beta = d(\beta\circ\Psi\circ\varphi)_0(1)$$

$$\xi_\alpha = d(\alpha\circ\varphi)_0(1)$$
$$= \vec{w} \in \mathbb{R}^m \text{ vector}$$
$$= (\alpha\circ\varphi)'(0) \text{ because } (\alpha\circ\varphi) : \mathbb{R} \to \mathbb{R}^m$$

This 'd' means the std calculus Frechet deriv (matrix of partials) in Euclidean space. Many authors call this 'D'.

Observe:

$$Dh_v(\xi_\alpha) = D(\beta\circ\Psi\circ\alpha^{-1})_v\,(D(\alpha\circ\varphi)_0)$$

I invented the symbol K_α for the map $K_\alpha : T_xM \longrightarrow \mathbb{R}^m$

$$[\varphi]_x \longmapsto d(\alpha\circ\varphi)_0(1) = (\alpha\circ\varphi)'(0)$$

But this is implicitly relying on the fixed x

Better to use $T\alpha : TU \longrightarrow T\mathcal{U} \cong \mathcal{U}\times\mathbb{R}^m$

$$[\varphi]_x \longmapsto \{\alpha(x), (\alpha\circ\varphi)'(0)\}$$

A&M use this often, L&S introduce it on p. 511

One more remark about K_α:

$$K_\alpha^{-1} : \vec{w} \longmapsto \big[\overset{\text{curve}}{\varphi_?} : t \longmapsto \alpha^{-1}(\alpha(x) + t\vec{w})\big]$$

L&S ch 9.5 Flows and Vector Fields

Let M be a smooth mfd

Consider the map $\varphi : M \times \textcircled{\mathbb{R}} \longrightarrow M$

Satisfying (i) φ is smooth

(ii) $\varphi(x,0) = x$ $[\varphi_0 = Id]$

(iii) $\varphi(\varphi(x,s),t) = \varphi(x, s+t)$ $[\varphi_s \circ \varphi_t = \varphi_{s+t}]$

So we call $\{\varphi_t\}_{t\in\mathbb{R}}$ a <u>1-Param Group</u> of Diffeos or a <u>Complete Flow</u> (time domain is all of $\mathbb{R}$)

This gives rise to 2 families of maps.

For any fixed x, we get

$\varphi_x : \mathbb{R} \longrightarrow M$ $\overset{\text{streamline}}{\text{flow curve}}$ of pt x

For fixed t:

$\varphi_t : M \longrightarrow M$ diffeo

Each pt flows for time t

NOTE: φ_t being a diffeo means pts cannot converge — streams don't cross [related to autonomous ODE]

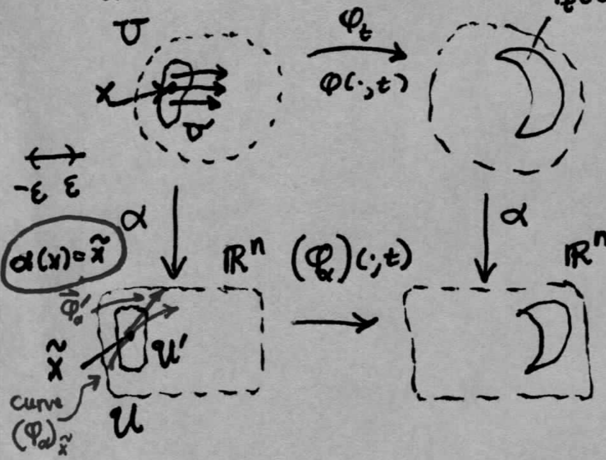
If each pt $x \in M$ only has its streamline defined for some interval I_x (depending on x) then we just have a $\boxed{\text{Flow}}$ and $\varphi_s \circ \varphi_t$ may not be defined if $s+t$ exceeds this time interval.

Note that if we take $t=-s$ then $\varphi_s \circ \varphi_{-s} = \varphi_0 = Id$. Thus $\varphi_{-s} = \varphi_s^{-1}$

For a given $x \in M$, $\varphi_x : (-\varepsilon, \varepsilon) \longrightarrow M$ is a curve and its tangent vector we denote $X(x)$.

▷ Given a flow φ, we want to generate the associated vf (they call φ the <u>infinitessimal generator</u> of X) and show it is smooth (by looking at downstairs route)

Given a chart (U,α) choose a subset U' that still stays in U if it only flows for time $|t| < \varepsilon$



By the downstairs route, we have $\overset{\text{well, we are defining}}{}$ the downstairs route

$$(\varphi_\alpha) : \mathcal{U}' \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}$$

$$(\tilde{x}, t) \longmapsto \alpha \circ \varphi(\alpha^{-1}(\tilde{x}), t)$$

This is a nice map in $\mathbb{R}^n \times I \to \mathbb{R}^n$ we can use ordinary calculus.

Fix $\tilde{x}$ $(\varphi_\alpha)(\tilde{x},t) = \alpha \circ \varphi(x,t) = \alpha \circ \varphi_x(t)$

$= (\alpha \circ \varphi_x)(t)$

$$\frac{\partial}{\partial t}(\varphi_\alpha)(\tilde{x},t)\bigg|_{t=0} = (\alpha \circ \varphi_x)'(0) =: \underset{\text{smooth}}{\vec{X}_\alpha(\tilde{x})}$$

Let's draw the associated picture from A&M 1st Ed p.37 (excuse the slightly differing notation) and the do some examples.

$$\vec{X}_\alpha(\tilde{x}) := (\varphi_\alpha)_{\tilde{x}}'(0)$$

i.e.

$$\boxed{X_\alpha(\varphi_{\tilde{x}}^\alpha(t)) = \dot{\varphi}_{\tilde{x}}^\alpha(t)}$$

p.379

$$\begin{tikzcd}
 & x \arrow[rr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & {[\gamma^X]_x} \arrow[ddd, mapsto, bend left] \\
(-\varepsilon, \varepsilon) \arrow[urrr, bend left=40, "\text{“}\gamma'\text{”}"] \arrow[r, bend left, "\gamma"] \arrow[dr, bend right, "\gamma_\alpha"'] \arrow[drr, bend right=40, "\gamma_\alpha'"'] & U \arrow[r, "X"] \arrow[d, "\alpha"'] & TU \arrow[d, "T\alpha"] & \\
 & \mathcal{U} \arrow[r, "X_\alpha = T\alpha \circ X \circ \alpha^{-1}", "X_\alpha"'] & \mathcal{U} \times \mathbb{R}^n & \\
 & \alpha(x) \arrow[rr, mapsto] & & \langle \alpha(x), (\alpha \circ \gamma^X)'(0) \rangle
\end{tikzcd}$$

γ^X is a flow line of vf X thru pt x

VECTOR FIELDS IN A CHART
The arrows in red show my write up of diff eqs upstairs and downstairs from A&M 1st Ed p.37

By def, a curve γ on a mfd M is an integral curve of X if $\underbrace{T\gamma_{(t,1)}}_{\gamma'(t)} = \underbrace{(X \circ \gamma)_{(t)}}_{X(\gamma(t))}$ $\forall\, t \in (-\varepsilon, \varepsilon)$

Using a co-ord chart (U, α), we can pull this down to $\mathbb{R}^n$ and get a system of ODEs. Apply $T\alpha$ to both sides: →

$$T\alpha \circ T\gamma(t,1) = T\alpha \circ (X \circ \gamma)(t)$$

$$= T(\alpha \circ \gamma)(t,1) = \underbrace{T\alpha \circ X \circ \alpha^{-1}}_{X_\alpha} \circ \underbrace{\alpha \circ \gamma}_{\gamma_\alpha(t)} \Rightarrow \gamma_\alpha'(t) = X_\alpha(\gamma_\alpha(t)) \text{ nice ODE in } \mathbb{R}^n$$

$$= \gamma_\alpha'(t) =$$

note $T(\gamma_\alpha)_{(t,1)} = \langle \gamma_\alpha(t), \underset{\overset{\parallel}{\gamma_\alpha'(t)}}{D(\gamma_\alpha)_t(1)} \rangle$ □

$M = V$ vectr sp.

ⓔⓧ ex 1 p.373 p.380

$$\varphi : V \times \mathbb{R} \longrightarrow V$$
$$(x, t) \longmapsto x + tW$$

Straight line flow
$w \in V$ const $\forall x, t$

Fix base pt x. Then $\varphi_x(t) = x + tW$

$$\begin{cases} \varphi_x(0) = x & \text{IC} \\ \varphi_x'(t) = W \end{cases}$$

Now lets look at it as ODE on mfd:

$$\begin{tikzcd}
 & x \arrow[r, mapsto, bend left] & \text{const } W & \varphi_x'(t) = X(\varphi_x(t)) \arrow[ddl, mapsto, bend left] \\
(-\varepsilon, \varepsilon) \arrow[urr, bend left=40, "\varphi_x'(t)"] \arrow[r, bend left, "\varphi_x"] \arrow[ddrr, bend right=40, "(\varphi_\alpha)_x'"'] & V \arrow[r, "X"] \arrow[d] & TV \cong V \times V \arrow[d, "T\alpha"] & \\
 & \mathbb{R}^n \arrow[r, "X_\alpha"] & \mathbb{R}^n \times \mathbb{R}^n & \\
 & \tilde{x} \arrow[r, mapsto] & \text{const } \vec{W}_\alpha & \\
 & & (\varphi_x)_\alpha'(t) = \vec{W}_\alpha = \begin{bmatrix} w_\alpha^1 \\ \vdots \\ w_\alpha^n \end{bmatrix} &
\end{tikzcd}$$

Solve ODE $(\varphi_x)_\alpha(t) = \tilde{x} + t\vec{W}_\alpha$

(ex 2) V fin dim v.s. Linear vf $A: V \longrightarrow V$

Here I am maybe cheating and not following the previous framework (basically I am identifying $TV \leftrightarrow V$)

$$\begin{tikzcd} \underset{\sim}{x} = x_1\vec{b}_1 + x_2\vec{b}_2 & Ax \\ V \arrow[r, "A"] \arrow[d, "\alpha"'] & V \arrow[d] \\ \mathbb{R}^n \arrow[r, "A_\alpha"] & \mathbb{R}^n \\ \langle x_1, x_2 \rangle \arrow[r, mapsto] & A_\alpha \vec{x} \end{tikzcd}$$

Flow $\varphi(x,t) = e^{tA}x$

Here we have the ODE $\dot{x} = Ax$

x is fixed here

Soln $\varphi_x(t) = e^{tA}x$

Then $\varphi_x'(t) = Ae^{tA}x$

Plug this in and see it is soln:

$$\dot{x} = Ax$$

$$\varphi_x'(t) = A(e^{tA}x) = A\varphi_x(t) \checkmark$$

[See my A&P ODE write up sheets for lengthy discussion]

I won't draw here all the cases of the flow which depends on Jordan normal form of A

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k$$

$$\frac{d}{dt}e^{tA} = A\left[\sum \cdot A^k\right] = \left[\sum \cdot A^k\right]A$$

(ex 3) p.378 Flow on circle S^1

I'm not going to write up how they used charts on S^1

The flow was just $(r, \theta_0) \longmapsto (r, ta)$ for $\theta_0 = a$ fixed angle

(ex 4) for $T = S^1 \times S^1$ and 2 fixed Real numbers a, b and angle θ_1 θ_2 flow $\theta_1(t) = at$, $\theta_2(t) = bt$

If a/b is rational, flow curve winds around T^2 and closes on itself.

" " " irration " " winds around T^2 densely.

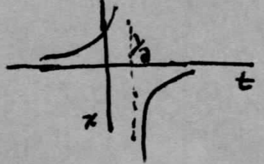
Avez & Arnold p.115-116

Thirring p.38

▷ Is every flow defined for all $t \in \mathbb{R}$? i.e. always a <u>complete flow</u> aka <u>1-Param Grp</u>.

No! <u>Many</u> examples, but just take $\dot{x} = x^2$ with IC $x(0) = a$

Soln $x(t) = \dfrac{a}{1 - at}$ which has singularity at $t = 1/a$



A&M denote the set of vfs as $\mathfrak{X}(M)$ and it itself is a v.s.

<u>Prop 5.1</u> Existence of Local flows

Given $X: M \longrightarrow TM$ smooth vf $\Rightarrow$ Every pt $x \in M$ has a stream line φ_x defined for some time interval depending on x : $I_x = (-\varepsilon_x, \lambda_x)$

Boyce & DiPrima

Avez DC p.82

Thirring

<u>Pf.</u> First they construct $\varphi_x(t)$ for any $x \in M$

Then verify $\varphi : (x,t) \longmapsto \varphi(x,t)$ is a flow.

Down in a chart, use fundamental existence thm for ODEs

Now show existence does not depend on particular chart.

<end summary>

L&S ch 9.6 <u>Lie Derivatives</u>

12/13/2022

⑤

First, back up to p. 373. No flows yet, just a curve φ and we want the notion of a vector ξ <u>acting on a fcn f</u>:

$$\begin{tikzcd}
(-\varepsilon, \varepsilon) \arrow[r, bend left, "\text{curve } \varphi"] \arrow[dr, bend right, "\varphi_\alpha = \alpha\circ\varphi"'] & U \ni x \arrow[r, "f"] \arrow[d, "\alpha"'] & \overset{(f\circ\varphi)(t)}{\mathbb{R}} \arrow[d, "\beta = \mathrm{Id}"] \\
& \mathbb{R}^n \arrow[r, "f_\alpha = f\circ\alpha^{-1}"'] & \underset{(f_\alpha\circ\varphi_\alpha)(t)}{\mathbb{R}}
\end{tikzcd}$$

$$\begin{tikzcd}
\overset{\xi = [\varphi]_x}{T_x U} \arrow[r, "f_{*x}"] \arrow[d, "\kappa_\alpha"'] & \overset{[f\circ\varphi]_{f(x)}}{\mathbb{R}} \arrow[d, "T(\mathrm{Id})"] \\
\mathbb{R}^n_{\tilde{x}} \arrow[r] & \mathbb{R}
\end{tikzcd}$$

$\xi_\alpha = d(\alpha\circ\varphi)_0(1) = (\alpha\circ\varphi)'(0)$ $\varphi_\alpha'(0)$

$(f_\alpha\circ\varphi_\alpha)'(0)$ important

$$\boxed{D(f_\alpha)_{\varphi_\alpha(0)}(\vec{\varphi}_\alpha'(0)) \quad [\quad]\,[\,]}$$

So for $\xi = [\varphi]_x$

They define $\xi(f) :=$ "D_φ"$(f) := (f\circ\varphi)'(0)$ or $d(f\circ\varphi)_0$

$\boxed{\text{See additional remarks on sheet ⑤a} \rightarrow}$

$\boxed{\text{P.383-384}}$ *Lie Derivatives* - The idea is compose a fcn f on mfd M (or a vf Y, or a diff form ω, etc...) with a curve ⊛ on M and take the <u>exterior</u> deriv of that. So it has a meaning indep of any chart, but of course we actually do the computations in a chart.

⊛ The curve is the flow of a vf X

This becomes more meaningful when $X = \frac{\partial}{\partial x}$ coming up

<u>Notation</u>
$$\left.\begin{array}{l} (Xf)(x) \\ \text{or } D_X f(x) \\ \text{or more } \mathcal{L}_X f(x) \end{array}\right\} := \underset{t\to 0}{\lim} \frac{1}{t}\left[\varphi_t^* f - \varphi_0^* f\right] = \lim \frac{1}{t}\left[f\circ\varphi_t(x) - f\circ\varphi_0(x)\right]$$

Fix x

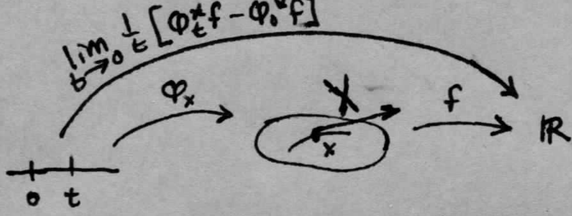
$\frac{d}{dt}(\varphi_t^* f)\Big|_{t=0}$ since x fixed

$$= \lim \frac{1}{t}\left[f\circ\varphi_x(t) - f\circ\varphi_x(0)\right]$$

$$= (f\circ\varphi_x)'(0) \quad \text{since } f\circ\varphi_x : \mathbb{R}\to\mathbb{R}$$

$$= d(f\circ\varphi_x)_0(1)$$

↑ Note this is also $\tilde{d}f_x([\varphi]_x)$ which will be defined soon.



Fix base pt x, vf X has flow φ, stream line φ_x

$$\begin{aligned} \mathcal{L}_X f(x) &= (f\circ\varphi_x)'(0) \\ &= (f_\alpha\circ\varphi_\alpha)'(0) \\ &= D(f_\alpha)_{\tilde{x}}(\vec{\varphi}_\alpha'(0)) \\ &= D(f_\alpha)_{\tilde{x}}(X_\alpha(\tilde{x})) \end{aligned}$$

← So really the Lie deriv of a fcn is a chain rule composition in ordinary calculus

To be <u>directional</u> deriv, we need $\hat{w}$ unit v

In the special case of straight line flow $X = W$ (fixed)

$\varphi_x(t) = x + tW$

$\varphi_x'(t) = \varphi_x'(0) = W$

$$\mathcal{L}_X f(x) = D(f_\alpha)_{\tilde{x}}(W) = \delta(f_\alpha)_{\tilde{x}}(W) = \lim_{t\to 0}\frac{1}{t}\left[f_\alpha(x + t\vec{w}) - f(x)\right]$$

directional deriv

Gateaux deriv of Avez DC sheets

So Lie is more general, $\varphi_\alpha(t)$ doesn't have to be $\varphi_\alpha(t) = x + tW$ for fixed W

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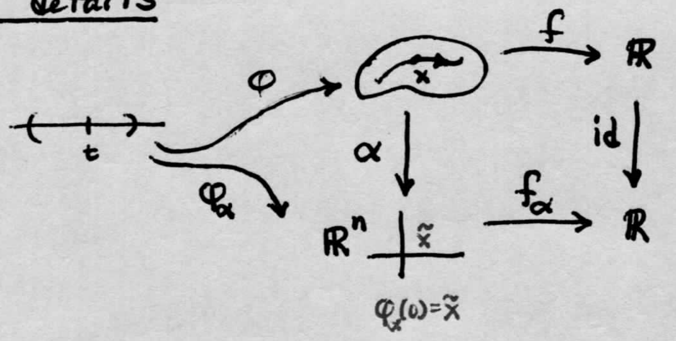
⑤a

p.376 additional details

we know



$\xi = [\varphi]_x$ $\qquad [f\circ\varphi]_y$

$$\begin{tikzcd}
T_xU \arrow[r, "f_{*x}"] \arrow[d, "K_\alpha"'] & T_{f(x)}R \cong R \arrow[d, "K_{id}"] \\
\mathbb{R}^n \arrow[r, "(f_\alpha)_* = D(f_\alpha)_{\tilde{x}}"] & \mathbb{R}^n
\end{tikzcd}$$

$$\vec{\xi}_\alpha = d(\alpha\circ\varphi)_0(1) = \vec{\varphi}_\alpha'(0)$$

$$(f_\alpha\circ\varphi_\alpha)'(0) = D(f_\alpha)_{\tilde{x}}(\vec{\varphi}_\alpha'(0))$$

we said $\xi(f)_x := (f\circ\varphi)'(0)$
$\qquad = (f_\alpha\circ\varphi_\alpha)'(0)$

$$\boxed{(4.5)\ \text{claim:}\quad \xi(f) = f_{*x}(\xi)}$$

push forward of vector ξ by $\overset{\mathbb{R}\text{-valued}}{\text{fcn}}$ f = Lie deriv of f wrt ξ

$\qquad \|$
$[f\circ\varphi]_{f(x)}$
$\qquad \| \longleftarrow$ identifying $T_{f(x)}\mathbb{R}$ with $\mathbb{R}$ downstairs
$(f_\alpha\circ\varphi_\alpha)'(0)$

This is quite natural in any $\mathbb{R}^n$
The single defining property of an equivalence class of curves is this vector.

SAME RESULT

Push forward ξ and apply to f

Pull back f and apply ξ

$$\boxed{(4.6)\ \text{Claim}\quad \xi(\psi^*f) = \psi_{*x}(\xi)(f)}$$

$$\begin{tikzcd}
 & & \mathbb{R} & & \\
{} \arrow[r, "\varphi"] & M \arrow[ur, "\xi(\psi^*f)"] \arrow[rr, "\psi"] & & N \arrow[ul, "(\psi_{*x}\xi)(f)"'] \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$\|$ $\qquad\qquad\qquad\qquad \|$

"D_φ"(ψ^*f)
$=$ "D_φ"$(f\circ\psi)$
ordinary $\mathbb{R}^n$ calculus "d = D" $= d((f\circ\psi)\circ\varphi)_0(1)$
$= (f\circ\psi\circ\varphi)'(0)$ $\xleftrightarrow{\text{Same}}$

$[\psi\circ\varphi](f)$
$= d(f\circ(\psi\circ\varphi))_0(1)$
$= (f\circ\psi\circ\varphi)'(0)$

□

This is a key part of showing Lie deriv of fcn commutes with pull back:

$$\psi^*(\mathcal{L}_Xf) = \mathcal{L}_{\psi^*X}(\psi^*f)$$

p.385 sheet ⑦

Compare this with the 'material deriv' of Marsden&Tromba p.177:

They define $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ depends on t unlike L&S
vf $F : \mathbb{R}^3 \to \mathbb{R}^3$ $\Big\}$ "$\frac{D}{Dt}f$" $:= \frac{\partial f}{\partial t} + \overrightarrow{\nabla_x f_{(x)}} \cdot \vec{F}$ not nec evaluated at t=0 but if we do

show:

$$\frac{D}{Dt}f = \frac{d}{dt} f(\varphi(x,t),t) = D_1 f_{(\varphi(x,t),t)} \cdot \underbrace{D_t \vec{\varphi}(x,t)}_{\frac{\partial}{\partial t}\vec{\varphi}_x(t)} + \underbrace{D_2 f(\varphi,t)}_{\frac{\partial f}{\partial t}}$$

$\dot{\varphi}_x(t) = F(\varphi_x(t))$

$$= \frac{\partial f}{\partial \varphi^{\textcircled{1}}}\dot{\varphi}^{\textcircled{1}} + \frac{\partial f}{\partial \varphi^{2}}\dot{\varphi}^{\textcircled{2}} + \frac{\partial f}{\partial \varphi^{3}}\dot{\varphi}^{\textcircled{3}} + \frac{\partial f}{\partial t} = \nabla_1 f_{(x)} \cdot F_{(x)} + \frac{\partial f}{\partial t}(x,0)$$

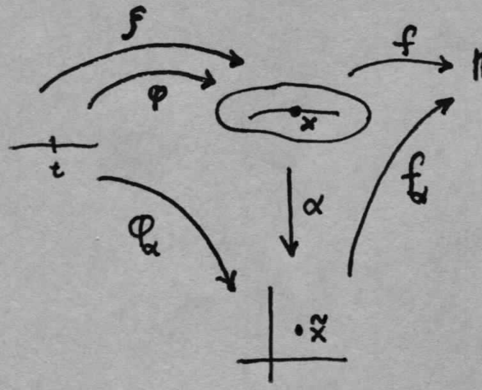
In this context,
L&S would have $\mathcal{L}_F f_{(x)} = (f \circ \varphi_x)'(0) = Df_{\varphi(x,0)}\,\dot{\varphi}_x(0)$ same thing, just no $\frac{\partial f}{\partial t}$ term

p.384 ▷ $\mathcal{L}_X f$ is linear in vf X: $\boxed{\mathcal{L}_{aX+bY} f_{(x)} = a\mathcal{L}_X f_{(x)} + b\,\mathcal{L}_Y(f)_{(x)}}$

$$\mathcal{L}_{aX+bY} f_{(x)} = (aX+bY)_{(x)} f \overset{p.380}{=} (aX_{(x)} + bY_{(x)})f = (a[\varphi]_x + b[\sigma]_x) f$$

p.374 This is defined downstairs as $\kappa_\alpha^{-1}(a\vec{\varphi}_\alpha'(0) + b\vec{\sigma}_\alpha'(0))$

" $\xi_\alpha'(0)$



$$= [\xi]_x f \text{ for some curve } \xi$$
$$= d(f \circ \xi)_0$$
$$= D(f_\alpha)_{\xi_\alpha(0)}(\xi_\alpha'(0))$$
$$= D(f_\alpha)_{\tilde{x}}(a\,D(\varphi_\alpha)_0 + b\,D(\sigma_\alpha)_0)$$
$$= a\,D(f_\alpha)_{\tilde{x}}\,D(\varphi_\alpha)_0 + b\,D(f_\alpha)_{\tilde{x}}\,D(\sigma_\alpha)_0$$
$$= a(\mathcal{L}_X f)_{(x)} + b(\mathcal{L}_Y f)_{(x)}$$

▷ Now we want to define $\mathcal{L}_X Y$ Lie deriv of vf. First some preliminaries:

$T\psi_x = \psi_{*x}$

(a) Define pullback of vf $\psi^* X$
ψ must be a diffeo
$(\psi^* X)_{(x)} := T\psi^{-1} \circ X \circ \psi_{(x)}$

$$\begin{tikzcd}
 & {[\gamma^X]_{\psi(x)}} \\
TM \arrow[r, "T\psi"] & TN \\
M \arrow[u, "\psi^* X"] \arrow[r, "\psi"] & N \arrow[u, "X"'] \\
x \arrow[r, mapsto] \arrow[uu, bend left=60, mapsto] & \psi(x) \arrow[uuu, bend right=60, mapsto]
\end{tikzcd}$$

Cf p.513 ch 13

$$(T\psi)_x^{-1} : T_{\psi(x)}N \longrightarrow T_x M$$
$$[\gamma]_{\psi(x)} \longmapsto [\psi^{-1} \circ \gamma]_x$$

Since ψ is a diffeo, ψ^{-1} exists and is Smooth

$\Rightarrow (\psi^{-1} \circ \gamma) : \mathbb{R} \to M$ This is a curve thru x

12/25/2022

$D(\psi_{\alpha\beta}^{-1})\circ X_\beta\circ\underbrace{(\beta\circ\psi\circ\alpha^{-1})}_{\psi_{\alpha\beta}}$

(b) Show ψ^*X is as smooth of a vf as X is:

see the CCW path ↺ around diagram

(6.5) $(\psi^*X)_\alpha(v) = (T\alpha\circ T\psi^{-1}\circ T\beta^{-1})\circ X_\beta\circ (\beta\circ\psi\circ\alpha^{-1})(v)$

$$\begin{tikzcd}
U\times\mathbb{R}^n & & & W\times\mathbb{R}^n \arrow[lll, bend right, "(T\psi)^{-1}"'] \\
& TM \arrow[ul, "T\alpha"'] \arrow[r, "T\psi"] & TN \arrow[ur, "T\beta"] & \\
& M \arrow[u, "\psi^*X"] \arrow[r, "\psi"] \arrow[dl, "\alpha"] & N \arrow[u, "X"'] \arrow[dr, "\beta"] & \\
U \arrow[uuu, bend left, "(\psi^*X)_\alpha"] \arrow[rrr, "\psi_{\alpha\beta}:=\beta\circ\psi\circ\alpha^{-1}"] & & & W \arrow[uuu, bend right, "X_\beta"']
\end{tikzcd}$$

$\cdot v$

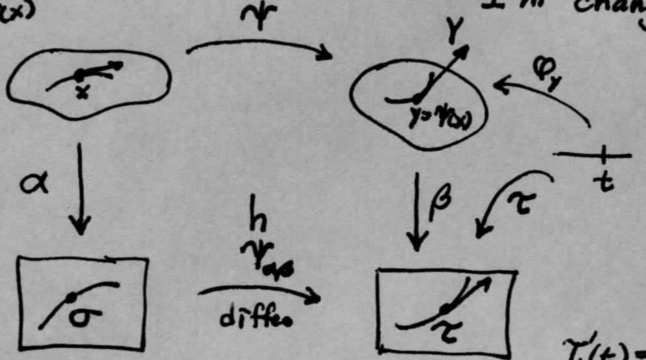
p.385 exercise

$$\begin{tikzcd}
M \arrow[r, "\psi"] & N \arrow[r, phantom, ""] \arrow[ur, "Y"] & {}
\end{tikzcd}$$

If ψ is a diffeo between mfds, and Y is a vf on N with flow φ, show $\psi^{-1}\circ\varphi$ is the flow of ψ^*Y

I'm changing notation a bit from the book

For a fixed base pt $y=\psi(x)$



$\tau'(t)=Y_\beta(\tau(t))$

We claim the pullback flow is

$$(x,t)\longmapsto \psi^{-1}(\varphi(\psi(x),t))$$

How do we know this is valid?

Look downstairs in charts

$\sigma(t):=h^{-1}\circ\tau(t)$ $\qquad h\circ\sigma=\tau$

then
$$\begin{aligned}\sigma'(t) &= D(h^{-1})_{\tilde y}(\tau'(t))\\ &= Dh^{-1}_{\tilde y}(Y_\beta(\tau(t)))\\ &= (Dh^{-1}_{\tilde y}\circ Y_\beta\circ h)(\sigma(t))\\ &= (h^*Y_\beta)(\sigma(t))\end{aligned}$$
□

(c)

▷ $(\psi_2\circ\psi_1)^*Y = \psi_1^*\psi_2^*Y$

(d)

▷ Now show $\boxed{\psi^*(\mathcal{L}_Xf) = \mathcal{L}_{\psi^*X}(\psi^*f) \quad \text{Lie deriv commutes with pull back.}}$

$$\begin{aligned}\mathcal{L}_{\psi^*X}\psi^*f &= (\psi^*X)(\psi^*f) && \text{just chaning notation}\\ &= (\underbrace{\psi_*^{-1}\circ X\circ\psi}_{Y})(\psi^*f) && \text{def of } \psi^*X\\ &= (\psi_{*x}Y)(f) && \text{eq(4.6) sheet ⑤a}\\ &= \cancel{\psi_*\circ\psi_*^{-1}\circ} X_{(\psi(x))}(f)\\ &= X_{(\psi(x))}(f(\psi(x)))\\ &= (\mathcal{L}_Xf)(\psi(x))\\ &= (\psi^*\mathcal{L}_Xf)_{(x)}\end{aligned}$$
□

$\mathcal{L}_Xf: M\longrightarrow\mathbb{R}$

Real valued fcn

$$\begin{tikzcd}
& & TN \\
M \arrow[urr, "\psi^*X"] \arrow[r, "\psi"] \arrow[drr, "\psi^*f"'] \arrow[ddr, bend right, "\psi^*(\mathcal{L}_Xf) = \mathcal{L}_{\psi^*X}(\psi^*f)"'] & N \arrow[ur, "X"'] \arrow[dr, "f"] \arrow[dd, bend left=60, "\mathcal{L}_Xf"] & \\
& & \mathbb{R} \\
& {} &
\end{tikzcd}$$

L&S p.385

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▷ Now we can define the Lie Deriv of another vf Y:

$$\mathcal{L}_X Y := \lim_{t\to 0} \frac{1}{t}\left(\varphi_t^* Y - \overset{Id}{\varphi_0^*} Y\right)$$

$$\begin{tikzcd} M \arrow[r, "Y"] \arrow[r, bend left, "X"] \arrow[r, bend right, "\mathcal{L}_X Y"'] & TM \end{tikzcd}$$

L&S give an example where $X = \hat{e}_1$ and $Y = x\hat{e}_2$ in $\mathbb{R}^2$ which traditionally I skip.

Thm $\mathcal{L}_X Y = -\mathcal{L}_Y X$ Thus we can write this in 'Commutator' bracket notation

$$\mathcal{L}_X Y = [X, Y] = -[Y, X]$$

anti-symm

Lie Bracket

pf.

$$\begin{tikzcd}
\mathcal{U}' \times \mathbb{R}^n & & & \mathcal{U} \times \mathbb{R}^n \arrow[lll, "D(\varphi_\alpha^t)^{-1}"'] \\
& TU' \arrow[ul, "T\alpha"'] \arrow[r, "T(\varphi_t)"'] & TU \arrow[ur, "T\alpha"'] & \\
& U \supset U' \arrow[u, "\varphi_t^* Y"'] \arrow[r, "\varphi_t"] \arrow[d, "\alpha"] & U \arrow[u, "Y"] \arrow[d, "\alpha"'] & \\
& \mathcal{U} \ni \tilde{x} \arrow[r, "\varphi_\alpha^t"] \arrow[uuul, bend left, "(\varphi_t^* Y)_\alpha"] & \mathcal{U} \arrow[uuur, bend right, "Y_\alpha"'] &
\end{tikzcd}$$

Let's compute $\mathcal{L}_X Y$ downstairs: $(\mathcal{L}_X Y)_\alpha = \lim_{t\to 0} \frac{1}{t}\left((\varphi_\alpha)^* Y_\alpha - Y_\alpha\right) = \frac{d}{dt}\left((\varphi_\alpha)_t^* Y_\alpha\right)\Big|_{t=0}$

From the diagram $(\varphi_t^* Y)_\alpha = \underbrace{D(\varphi_\alpha^t)_{\tilde{x}}^{-1}}_{A_t^{-1}} \circ \underbrace{Y_\alpha \circ \varphi_\alpha^t}_{z_t}$ so $z_0 = Y_\alpha(\tilde{x})$

So we compute $\frac{d}{dt}(A_t^{-1} z_t)\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\underbrace{(A_t^{-1} z_t - \overset{I}{A_0^{-1}} z_0)}$

$$A_t^{-1} z_t - z_0$$

$$A_t(A_t^{-1} z_t - z_0)$$

because $\lim(P(t)Q(t)) = \lim P \cdot \lim Q$, $P(t) = A_t$, $\lim_{t\to 0} A_t = I$

$$z_t - A_t z_0$$

$$= \lim_{t\to 0} \left(\frac{z_t - z_0}{t}\right) - \left(\frac{A_t z_0 - z_0}{t}\right)$$

add & subtract z_0

$$z'\big|_{t=0} - (A_t z_0)'\big|_{t=0} = z_0' - A_0' z_0$$

Lemma ⓐ

Lemma ⓑ

$$D(Y_\alpha)_{\tilde{x}}(X_\alpha(\tilde{x})) - D(X_\alpha)_{\tilde{x}}(Y_\alpha(\tilde{x}))$$

ord calculus Frechet

cont'd →

$(\mathcal{L}_X Y)_\alpha(\tilde{x}) =$ [redacted]

This comes first

$$D(Y_\alpha)_{\tilde{x}}(X_\alpha(\tilde{x})) - D(X_\alpha)_{\tilde{x}}(Y_\alpha(\tilde{x})) \Longleftarrow D(Y_\alpha)_{\tilde{x}}(X_\alpha(\tilde{x})) - D(X_\alpha)_{\tilde{x}}(Y_\alpha(\tilde{x}))$$

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Lemma ⓐ : $\frac{d}{dt} Z_t \Big|_{t=0} = \frac{d}{dt}\left(Y_\alpha(\varphi_t^\alpha)\right)\Big|_{t=0}$

Calculation

$$Z_0' = D(Y_\alpha)_{\tilde{x}}\left(\frac{\partial}{\partial t}\varphi_\alpha(\tilde{x},t)\Big|_{t=0}\right)$$
$$= D(Y_\alpha)_{\tilde{x}}\,(X_\alpha(\tilde{x}))$$

Sheet ②: $\frac{\partial}{\partial t}\varphi_\alpha(\tilde{x},t)\Big|_{t=0} = X_\alpha(\tilde{x})$

$\dot{\varphi}_{\tilde{x}}^\alpha(0) = X_\alpha(\varphi_{\tilde{x}}(0))$

Lemma ⓑ : $A_0' = \frac{d}{dt} A_t \Big|_{t=0} = \frac{\partial}{\partial t} D(\varphi_t^\alpha) = D\left(\frac{\partial}{\partial t}\varphi_t^\alpha\right) = D(X_\alpha)_{\tilde{x}}$

Then $A_0' Z_0 = D(X_\alpha)_{\tilde{x}}(Y_\alpha(\tilde{x}))$ since $Z_0 = Y_\alpha(\tilde{x})$

☐

This comes first

remember I use 'D' as ordinary $\mathbb{R}^n$ Frechet deriv

$$\Rightarrow (\mathcal{L}_X Y)_\alpha(\tilde{x}) = \underline{D(Y_\alpha)_{\tilde{x}}(X_\alpha(\tilde{x})) - D(X_\alpha)_{\tilde{x}}(Y_\alpha(\tilde{x}))} = -(\mathcal{L}_Y X)_\alpha(\tilde{x})$$

In view of the the anti-symm, we introduce the notation $[X,Y] := \mathcal{L}_X Y$

$[Y,X] = -[X,Y]$ $\boxed{\text{Lie Bracket}}$

p. 389

Let's do some examples in $M = \mathbb{R}^n$

(ex 1) $X(x) = \vec{u}$ const vfs then $DX_x = 0 \Rightarrow DX_x(\vec{v}) = 0$

$Y(x) = \vec{w}$ $\qquad DY_x = 0 \Rightarrow DY_x(v) = 0$

$$[X,Y] = 0 - 0 = 0$$

(ex 2) $X(x) = Ax$ for linear $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\qquad DX_x = A$

$Y(x) = \vec{w}$ const $\qquad DY_x = 0$

$$[X,Y]_x(v) = \underbrace{DY_x(X(x))}_{\text{comes first}} - DX_x(Y(x))$$
$$= 0\,Ax - Aw \quad = -Aw$$

(ex 3) $X(x) = Ax$ $\quad DX_x = A$

$Y(x) = Bx$ $\quad DY_x = B$

$$\mathcal{L}_X Y = [X,Y]_x(v) = DY_x(X(x)) - DX_x(Y(x))$$
$$= BAv - ABv = (BA - AB)v$$

p. 389-390

L&S also show

- $\mathcal{L}_{[X,Y]} f = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$ (6.13)
- $\mathcal{L}_X(aY + bZ) = a\mathcal{L}_X Y + b\mathcal{L}_X Z$
- $\psi^*[X,Y] = [\psi^* X, \psi^* Y]$ (6.14)
- $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ Jacobi identity (6.16)

$$\begin{tikzcd} X \arrow[rr, bend left] & & Y \arrow[dl, bend left] \\ & Z \arrow[ul, bend left] & \end{tikzcd}$$

L&S ch 9.7 Linear Differential Forms

1/8/2023

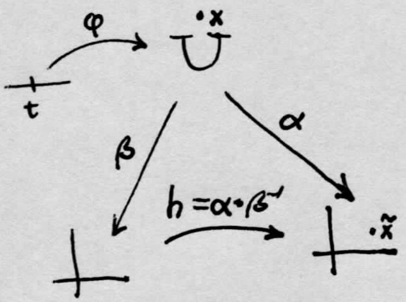
⑩

p.390-391

Given M, T_x^*M is all $\overset{\text{Bdd}}{\text{linear}}$ fcnals $\ell : T_xM \longrightarrow \mathbb{R}$ $\quad \overset{T_x^*M}{\text{Co tangent space}}$, ℓ is also called 'co vector'

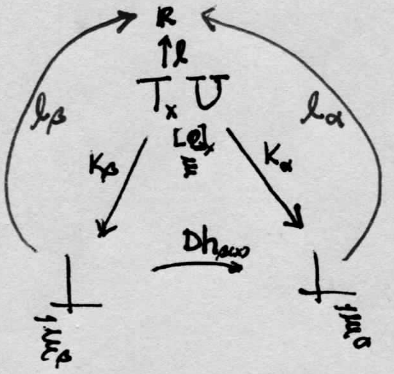
$\xi = [\varphi]_x \longmapsto$

Let's look at this in a chart (U, α) just like everything else. Take a given $\ell : T_xU \longrightarrow \mathbb{R}$



$\beta \circ \varphi = h^{-1} \alpha \varphi$

$\underset{\vec{\xi}_\beta}{\vec{\varphi}'_\beta(0)} = Dh^{-1}_{\tilde{x}} \underset{\vec{\xi}_\alpha}{\vec{\varphi}'_\alpha(0)}$



So we have $\ell_\alpha(\xi_\alpha) = \ell(\xi)$

what if we had another overlapping chart β? Does $\ell_\beta(\xi_\beta) = \ell_\alpha(\xi_\alpha)$?

Yes, by the familiar chart overlap rules.

we know $\xi_\beta = Dh^{-1}(\xi_\alpha)$

Thus
$$
\begin{aligned}
\ell_\beta(\xi_\beta) &= \ell_\alpha(Dh_{\beta\alpha0}\xi_\beta)\\
&= \ell_\alpha(\cancel{Dh} \cdot \cancel{Dh^{-1}}\xi_\alpha)\\
&= \ell_\alpha(\xi_\alpha) \quad \checkmark
\end{aligned}
$$

L&S introduce some 'weird' notation using inner prod for fcn evaluation.

$\langle \xi, \ell \rangle := \ell(\xi)$ I will write $\langle\!\langle \xi, \ell \rangle\!\rangle$ for this.

For justification Cheney Apl M ch 3 Reisz Rep thm: Given linear fcnal $\theta : \underset{\text{Hilbert sp}}{H} \longrightarrow \mathbb{R}$ or $\mathbb{C}$

$\exists!$ vector $v_\theta \in H \ni \theta(\cdot) = \langle \cdot, v_\theta \rangle$

$\uparrow$ 2nd position

So here, L&S are putting ℓ itself, instead of some vector $v_\ell \in T_xU$: $\langle\!\langle \xi, \ell \rangle\!\rangle$ vs. $\langle \xi, v_\ell \rangle$ which would be an actual IP

▷ Now for $f : M \longrightarrow \mathbb{R}$ Define $\underset{\sim}{d} f_x : T_xM \longrightarrow \mathbb{R}$

$\xi \longmapsto \xi(f)$

My symbol: above the ~ means 'upstairs' at mfd

on sheet ⓹ⓐ we showed

$\xi(f) = f_{*x}(\xi)$ if we identify

$(f\circ\varphi)'(0) \longleftrightarrow [f\circ\varphi]_{f(x)}$

this identification is natural, since the value on the LHS is the single condition defining the Equiv class on RHS.

$$
\begin{tikzcd}
\xi = [\varphi]_x \arrow[rr, bend left, mapsto] \arrow[dd, bend right, mapsto] & & (f\circ\varphi)'(0) \\
& T_xU \arrow[r, "\underset{\sim}{d}f_x"] \arrow[d, "K_\alpha"'] & \mathbb{R} \\
\vec{\varphi}'_\alpha(0) & \mathbb{R}^n \arrow[ur] &
\end{tikzcd}
$$

$(df_x)_\alpha = D(f_\alpha)_{\tilde{x}}(\cdot)$

Looking ahead:

$\hookrightarrow [D_1(f_\alpha) \ \dots \ D_n(f_\alpha)] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \sum D_i(f_\alpha) v^{ⓘ}$

and for $\pi_i(v) = v^i$

and $dx^i_\alpha(v) = \pi_i(v)$

$\Downarrow$

$$(df_x)_\alpha(\vec{v}) = \left(\sum D_i(f_\alpha)\, dx^i_\alpha\right)(v)$$

we call $df : TM \longrightarrow$

a 'co vector field' or

'linear differential form'

See ch 9.8 →

L&S ch 9.7

1/8/2023

⑪

Let's discuss the pullback of a diff 1-form ω:

$$\begin{tikzcd}
T^*M & T^*N \arrow[l, "(\psi_*)^*"'] \\
\underset{x}{M} \arrow[u, "\psi^*\omega"] \arrow[r, "\psi"'] & N \arrow[u, "\omega"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{R} & \\
T_xM \arrow[ur, "(\psi^*\omega)_x(\cdot)"] \arrow[rr, "\psi_{*x}"] & & T_{\psi(x)}N \arrow[ul, "\omega_{\psi(x)}(\cdot)"']
\end{tikzcd}$$

From Schaums LA ch 11
Given vector spaces X, Y
and a linear map $L: X \to Y$
Let $\omega: Y \to \mathbb{R}$ be a linear fcnal

$$\begin{tikzcd}
 & \mathbb{R} & \\
X \arrow[ur, "\omega\circ L = L^*\omega"] \arrow[rr, "L"] \arrow[d, dotted] & & Y \arrow[ul, "\omega"'] \arrow[d, dotted] \\
X^* & & Y^* \arrow[ll, "L^*"']
\end{tikzcd}$$

a dual map L^* is induced

$$(7.5)\quad (\psi^*\omega)_x^{(\cdot)} = (\psi_{*x})^*\omega_{\psi(x)}^{(\cdot)}$$

Their discussion is a bit confusing; I know the answer must be $\boxed{(\psi^*\omega)_x(\xi) = \omega_{\psi(x)}(\psi_{*x}(\xi))}$

$$(7.7)\quad \psi^*(\omega+\eta) = \psi^*\omega + \psi^*\eta$$

$$(7.8)\quad \varphi^*(f\omega) = \varphi^*f\,\varphi^*\omega = (f\circ\varphi)\,\varphi^*\omega$$

$$(7.9)\quad (\psi\circ\varphi)^*\omega = \varphi^*\psi^*\omega$$

(7.10) $\underset{\sim}{d}$ commutes with pullback: $\underset{\sim}{d}(\psi^*f)_x = \psi^*(df_x)$

pf. $\underset{\sim}{d}(\psi^*f)_x(\xi) \overset{?}{=} (\psi^*\underset{\sim}{d}f_x)(\xi) \qquad \xi = [\varphi]_x$

eq(4.6) sheet (5a) $= \xi(f\circ\psi)$

$= (f\circ\psi\circ\varphi)'(0)$

$= \underset{\sim}{d}f_x(\psi_{*x}(\xi))$

$= (\psi_{*x}(\xi))(f)$

$= [\psi\circ\varphi](f)$

$= (f\circ\psi\circ\varphi)'(0)$

SAME

$$\begin{tikzcd}
 & \mathbb{R} & & \\
 & & & \mathbb{R} \\
M \arrow[uur, "\xi(\psi^*f)"] \arrow[rr, "\psi"'] & & N \arrow[uul, "(\psi_{*x}\xi)(f)"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

sheet (5a)

(7.11) A Leibniz prod rule: $\mathcal{L}_X(f\omega) = (\mathcal{L}_Xf)\omega + f\mathcal{L}_X\omega$ ☐

pf. Just like always the Lie Deriv of a form ω is $\boxed{\mathcal{L}_X\omega := \lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega - \overset{id}{\varphi_0^*}\omega)}$

φ flow of vf X

$$\begin{tikzcd}
 & T^*N \\
M \arrow[r, "\varphi"] & N \arrow[u, "\omega"']
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{L}_X(f\omega) = \lim_{t\to 0}\frac{1}{t}(\varphi_t^*(f\omega) - f\omega) &= \lim\frac{1}{t}\left(\varphi_t^*f\,\varphi_t^*\omega - f\varphi_t^*\omega + f\varphi_t^*\omega - f\omega\right) \\
&= \lim\frac{1}{t}\left([\varphi_t^*f - f]\varphi_t^*\omega + f[\varphi_t^*\omega - \omega]\right) \\
&= \lim\left[\frac{\varphi_t^*f - f}{t}\right]\lim_{t\to 0}\varphi_t^*\omega + f\lim\left[\frac{\varphi_t^*\omega - \omega}{t}\right] \\
&= (\mathcal{L}_Xf)\,\omega + f\,(\mathcal{L}_X\omega)
\end{aligned}$$

Same trick as used to prove ordinary prod rule

☐

(7.12) $\underset{\sim}{d}$ commutes with Lie Deriv: $\mathcal{L}_X(\underset{\sim}{d}f) = \underset{\sim}{d}(\mathcal{L}_Xf)$

Finish

Let's define the upstairs objects $\{\frac{\partial}{\partial x^i_\alpha}\}$ which make a basis for T_xU
(They are the upstairs version of std basis $\{\hat{e}_i\}$. Later, $\{\underset{\sim}{d}x^i_\alpha\}$ will be the dual basis, the upstairs analogue of $\hat{e}_i^T$ = dual $e_i^* = \pi_i$)

Fix chart (U,α). First define my symbol $[e_i]_{\underset{\sim}{x}}$

$$\begin{tikzcd}
t \arrow[r, bend left, "\varphi"] \arrow[dr, bend right, "\varphi_\alpha"'] & \overset{\cdot x}{U} \arrow[r, "X"] \arrow[d, "\alpha"'] & TU \arrow[r, dotted] \arrow[d, "T\alpha"] & \overset{\text{For particular } x}{T_xU} \arrow[d, "K_\alpha \approx \alpha_{*x}"] & {[e_i]_x} \arrow[d, bend left] \\
 & \underset{\cdot x}{\mathcal{U}} \arrow[r, "X_\alpha"] & \mathcal{U}\times\mathbb{R}^n \arrow[r, dotted] & \mathbb{R}^n & \hat{e}_i
\end{tikzcd}$$

$X_\alpha = T\alpha \circ X \circ \alpha^{-1}$

Then $T\alpha^{-1}(X_\alpha) = X(\underset{\sim}{x}) = [e_i]_{\underset{\sim}{x}}$

Fix x $\quad K_\alpha^{-1}(\hat{e}_i) = X(\underset{\sim}{x}) = [e_i]_{\underset{\sim}{x}}$

We decree $X_\alpha(x) = \hat{e}_i$ std basis vector

Thus upstairs $X(\underset{\sim}{x}) = [e_i]_{\underset{\sim}{x}}$
and $[e_i]_x$ is the thing that has
$K_\alpha([e_i]) = \hat{e}_i$

For any vector $\xi = [\varphi]$
$K_\alpha([\varphi]) = \overrightarrow{(\alpha\circ\varphi)'(0)} = \vec{\varphi}_\alpha'(0)$

For what curve φ_α is $\varphi_\alpha'(0) = \hat{e}_i$?
Obviously $\varphi_\alpha(t) = t\hat{e}_i + x$
and then $\varphi_\alpha(0) = x$
I will give this curve the name e_i^α

This jibes with $\varphi_\alpha'(t) = X_\alpha(\varphi_\alpha(t)) = \hat{e}_i$
$\varphi_\alpha(t) = t\hat{e}_i + x = e_i^\alpha(t)$

We can't give an explicit form for $[e_i]_{\underset{\sim}{x}}$
other than to say $K_\alpha^{-1}(e_i^\alpha) = e_i$

▷ Now we must show $\{[e_i]_{\underset{\sim}{x}}\}$ forms a basis for T_xU

Take any $[\varphi]_x \in T_xU$ $\quad K_\alpha([\varphi]) = \vec{\varphi}_\alpha'(0) = \begin{bmatrix} \dot{\varphi}_\alpha^{(1)}(0) \\ \vdots \\ \dot{\varphi}_\alpha^{(n)}(0) \end{bmatrix} = \sum_{i=1}^n \dot{\varphi}_\alpha^{(i)} \hat{e}_i$

α_{*x}
K_α is an iso
See ch 9 sheet ①

Then $K_\alpha^{-1}(\sum \dot{\varphi}_\alpha^{(i)}(0)\hat{e}_i) = \sum \dot{\varphi}_\alpha^i(0) \underbrace{K_\alpha^{-1}(\hat{e}_i)}_{[e_i]_{\underset{\sim}{x}}}$

Thus arb $[\varphi]_{\underset{\sim}{x}} = \sum \dot{\varphi}_\alpha^{(i)}(0)[e_i]_{\underset{\sim}{x}}$ LC of $\{[e_i]\}$

□

▷ How do we justify the $\frac{\partial}{\partial x^i_\alpha}$ notation for $[e_i]_x$?

$$\begin{tikzcd}
t \arrow[r, bend left, "e_i"] \arrow[dr, bend right, "e_i^\alpha"'] & U \arrow[r, "f"] \arrow[d, "\alpha"'] & \mathbb{R} \\
 & \mathcal{U} \arrow[ur, "f_\alpha"'] &
\end{tikzcd}$$

$(f\circ e_i)(t) \overset{!}{=} (f_\alpha \circ e_i^\alpha)(t) = f_\alpha(x+t\hat{e}_i)$ since $e_i^\alpha(t) = t\hat{e}_i + x$ and $e_i^\alpha(0) = x$

Then $(f\circ e_i)'(0) = (f_\alpha \circ e_i^\alpha)'(0) = \frac{d}{dt}(f_\alpha(x+t\hat{e}_i))\Big|_{t=0} = D_i(f_\alpha)(x)$
$= \frac{\partial f_\alpha}{\partial x^i_\alpha}(x) = (\frac{\partial}{\partial x^i_\alpha})f_\alpha(x)$

Temporarily we will write co-ord $x^{(i)}$ as x^i_α to emphasize it comes from chart (U,α).

Then $([e_i]_x)(f) = (f\circ e_i)'(0) = \frac{\partial}{\partial x^i_\alpha} f_\alpha(x)$

Thus we use the slang "$\frac{\partial}{\partial x^i_\alpha}$" = vector $[e_i]_x$

L&S ch 9.8 1/28/2023

⑬

▷ For $f : M \longrightarrow \mathbb{R}$ we know $\mathcal{L}_X f : M \longrightarrow \mathbb{R}$

$$x \longmapsto X_{(x)} f = [\varphi]_x f = (f\circ\varphi)'(0)$$

Now we want to establish $\mathcal{L}_X f = \sum_{i=1}^{n} X_\alpha^i \frac{\partial f}{\partial x_\alpha^i}$

Since $\left\{\frac{\partial}{\partial x_\alpha^i}\right\}$ is a basis, $X = X_\alpha^1 \frac{\partial}{\partial x^1} + \dots + X_\alpha^n \frac{\partial}{\partial x^n}$

$$\mathcal{L}_X f(x) = \Big(\sum \underbrace{\boxed{X_\alpha^i}}_{\text{call this } a_i} \frac{\partial}{\partial x_\alpha^i}\Big)(f) = \big(a_1[e_1] + \dots + a_n[e_n]\big)(f)$$

$$= a_1 [e_1](f) + \dots + a_n [e_n](f)$$

$$= a_1 (f\circ e_1)'(0) + \dots + a_n (f\circ e_n)'(0)$$

$$= a_1 \frac{\partial f_\alpha}{\partial x_\alpha^1}(x) + \dots + a_n \frac{\partial f_\alpha}{\partial x_\alpha^n}(x) \quad \text{see prev sheets}$$

$$= \sum_{i=1}^{n} X_\alpha^i \frac{\partial f}{\partial x_\alpha^i}$$

□

▷ Now we establish $\{dx_\alpha^i\}$ is the upstairs analog of $\hat{e}_i^T = e_i^* = \pi_i$ and a basis for $T_x^* U$, in fact the dual basis to $\left\{\frac{\partial}{\partial x_\alpha^i}\right\}$

Plan Going Forward:

① we will define $\underset{\sim}{d} x_\alpha^i$

② we will show $\underset{\sim}{d} x_\alpha^i\left(\frac{\partial}{\partial x_\alpha^j}\right) = \delta_{ij}$

③ Using Linear Algebra, we now show $\{dx_\alpha^i\}$ is a basis for $T_x^* U$

④ Thus we will establish $\omega = \sum a_i dx_\alpha^i$ and $\underset{\sim}{d} f_x(\cdot) = \sum_{i=1}^{n} \frac{\partial f_\alpha}{\partial x_\alpha^i} dx_\alpha^i(\cdot)$

① Now we know

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\alpha"'] & \mathbb{R} \\
\mathcal{U} \arrow[ur, "f_\alpha = f\circ\alpha^{-1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
{[\varphi]_{\underset{\sim}{x}}} \arrow[rr, mapsto, bend left] \arrow[ddd, bend right] & & (f\circ\varphi)'(0) \\
T_x U \arrow[r, "\underset{\sim}{d} f_{\underset{\sim}{x}}"] \arrow[d, "K_\alpha"', "\alpha_{*x}"] & \mathbb{R} & \\
\mathbb{R}^n \arrow[ur, "D(f_\alpha)_x"'] & & \\
\vec{\varphi}_\alpha'(0) & &
\end{tikzcd}$$

$$(f\circ\varphi)'(0) = D(f_\alpha)_x\left(\vec{\varphi}_\alpha'(0)\right) = [D_1(f_\alpha) \dots D_n(f_\alpha)] \begin{bmatrix} \dot{\varphi}^{①} \\ \vdots \\ \dot{\varphi}^{ⓝ} \end{bmatrix}$$

Now specialize to $f = \alpha^{ⓘ}$ and call this fcn x_α^i — it is a std abuse of notation to have this be both a co-ord pt and a fcn

$$\begin{tikzcd}
U \arrow[r, "x_\alpha^i"] \arrow[d, "\alpha"'] & \mathbb{R} \\
\mathcal{U} \arrow[ur, "\pi_i"'] &
\end{tikzcd}$$

$(x_\alpha^i)_\alpha = \pi_i$

$(x_\alpha^i)_\alpha(x) = \pi_i$

$D(\pi_i)_x(\vec{h}) = \pi_i(\vec{h})$

$$\begin{tikzcd}
{[\varphi]_x} \arrow[rr, mapsto, bend left] & & (\alpha^i\circ\varphi)'(0) \\
T_x U \arrow[r, "\underset{\sim}{d} x_\alpha^i"] \arrow[d, "K_\alpha"'] & \mathbb{R} & \\
\mathbb{R}^n \arrow[ur, "D\pi_i = \pi_i"'] & &
\end{tikzcd}$$

$$\pi_i(x) = [0 \dots 1 \dots 0] \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = x^i$$

Thus $D(\pi_i)_x(h) = \pi_i(h) = h^i$

② $\underset{\sim}{d}(x_\alpha^i)_{\underset{\sim}{x}}([\varphi]_x) = D(x_{\alpha\alpha}^i)_x\left(\vec{\varphi}_\alpha'(0)\right) = \pi_i\left(\vec{\varphi}_\alpha'(0)\right) = \dot{\varphi}_\alpha^{ⓘ}(0)$

Now observe $\left\langle\!\left\langle \frac{\partial}{\partial x_\alpha^i}, dx_\alpha^j \right\rangle\!\right\rangle = dx_\alpha^i([e_i]) = (x_\alpha^j \circ e_i)'(0)$

Kronecker

Downstairs $D(x_{\alpha\alpha}^j)(\hat{e}_i) = \pi_j(\hat{e}_i) = [0 \cdots 1 \cdots 0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \delta_{ij}$

=

③ Let's detour into linear algebra to prove a thm that will show $\{dx_\alpha^i\}$ is a basis

Let X, Y be vectr spaces
$L: X \to Y$ linear map

$$\begin{tikzcd} & \mathbb{R} & \\ X \arrow[ur, "L^*\omega"] \arrow[rr, "L"] \arrow[d, dotted] & & Y \arrow[ul, "\omega"'] \arrow[d, dotted] \\ X^* & & Y^* \arrow[ll, "L^*"'] \end{tikzcd}$$

An iso exists here, once we choose a basis

$\left\{\begin{matrix}\text{all fcnals}\\ \text{on } X\end{matrix}\right\}$ $\left\{\begin{matrix}\text{all fcnals}\\ \text{on } Y\end{matrix}\right\}$

$$\begin{tikzcd} T_x^*U & T_xU \arrow[l, dotted, "\underset{\sim}{\Theta}"'] \arrow[d, "K_\alpha\ \text{iso}"', "(\alpha_{*x})"] \arrow[r, "\omega_x(\cdot)"] & \mathbb{R} \\ (\mathbb{R}^n)^* \arrow[u, "K_\alpha^*"] & \mathbb{R}^n \arrow[l, dotted, "\Theta"] \arrow[ur, "(\omega_x)_\alpha(\cdot)"'] & \end{tikzcd}$$

This map is induced because K_α exists

This is just the transpose map

Thm Let $\{e_i\}$ $i=1,\dots,n$ be a basis for Y (vectr sp) and $\exists$ n fcnals $\{d_j(\cdot)\} \ni d_j(e_i) = \delta_{ij}$ $\Rightarrow$ $\{d_j(\cdot)\}$ $j=1\dots n$ is a basis for Y^*

pf. To be a basis, we need $\{d_j\}$ is LI set and spans Y^*

(LI) we want to show $\sum \gamma_i d_i = 0 \Leftrightarrow$ each $\gamma_i = 0$
we only need to show $(\Rightarrow)$: Let $\sum \gamma_i d_i(\cdot) = 0$

Fix j. apply both sides to e_j: $\left(\sum \gamma_i d_i(e_j)\right) = 0(e_j) = 0$

$\Rightarrow \gamma_j \cdot 1 = 0$

Do this for each j.

(Span) Choose arb $\omega \in Y^*$. $\omega: Y \to \mathbb{R}$
Choose arb $v \in Y$ $\quad v = \sum \gamma_i e_i = \sum d_i(v) e_i$ because

$d_i(v) = d_i(\gamma_1 e_1 + \dots + \gamma_n e_n) = \gamma_i \cdot 1$

$$\text{Thus } \omega(v) = \omega\left(\sum d_i(v) e_i\right) = \sum d_i(v)\,\omega(e_i) = \underbrace{\left(\sum \omega(e_i) d_i(\cdot)\right)}_{\text{Thus } \omega \text{ is a LC of } \{d_i\}}(v)$$

□

Observe that another way to phrase this thm is:
The map $\lambda: \mathbb{R}^n \longrightarrow Y^*$
$\{\gamma_1, \dots, \gamma_n\} \longmapsto \sum \gamma_i d_i(\cdot)$

so (LI) was showing λ is <u>One-to-One</u> ($\ker(\lambda) = \{0\}$)
and (Span) was showing λ maps <u>Onto</u> Y^*

▷ so take "Y" $= T_xU$ and "d_j" $= \underset{\sim}{d}x_\alpha^j$ and we see $\{\underset{\sim}{d}x_\alpha^j(\cdot)\}$ is a basis fr T_x^*U.

④ Thus for any $\omega \in T_x^*U$, $\omega = \sum a_i\, dx_\alpha^i(\cdot) = \sum \omega([e_i])\, dx_\alpha^i(\cdot)$ from above thm (see above diagram) since $\{[e_i]\}$ is the dual basis to $\{dx_\alpha^i\}$, $\underset{\sim}{\Theta}\left(\sum a_i [e_i]_x\right) = \sum a_i\, dx_\alpha^i(\cdot)$

$K_\alpha^* = \underset{\sim}{\Theta} \cdot K_\alpha^{-1} \Theta^{-1}$ Thus K_α^* is a linear iso.

By design $K_\alpha^*(\pi_i) = dx_\alpha^i$ Thus $K_\alpha^*([a_1 \dots a_n]) = K_\alpha^*\left(\sum a_i \pi_i(\cdot)\right) = \sum a_i\, dx_\alpha^i(\cdot)$

$$\boxed{\begin{matrix} K_\alpha\left(\frac{\partial}{\partial x^i}\right) = \hat{e}_i \\ dx_\alpha^i = K_\alpha^*(\pi_i) \end{matrix}}$$

$\sum a_i [e_i]$

$$\begin{tikzcd} T_xU \arrow[r, "\omega_x(\cdot)"] \arrow[d, "K_\alpha"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "(\omega_x)_\alpha(\cdot)"'] & \end{tikzcd}$$

$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$(\omega_x)_\alpha(\cdot) = \sum a_i (\omega_\alpha(e_i))$
$= [\omega_\alpha^1 \cdots \omega_\alpha^n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$$\begin{tikzcd} T_xU \arrow[r, "df_x"] \arrow[d, "K_\alpha"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "D(f_\alpha)_x"'] & \end{tikzcd}$$

$D(f_\alpha)_x = [D_1(f_\alpha) \dots D_n(f_\alpha)]$
$= \left[\frac{\partial f_\alpha}{\partial x^1} \dots \frac{\partial f_\alpha}{\partial x^n}\right]$

$$\Rightarrow \boxed{\underset{\sim}{d}f_{\underline{x}}(\cdot) = \frac{\partial f_\alpha}{\partial x^1} dx_\alpha^1 + \dots + \frac{\partial f_\alpha}{\partial x^n} dx_\alpha^n}$$