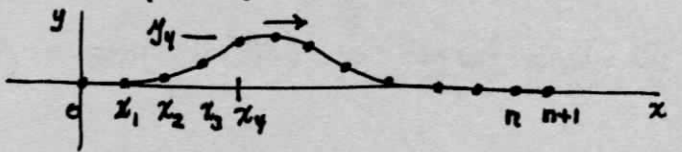
Fowles AM
p. 300 – 305

Loaded String (Analytical solution)

Consider a loaded string with n mass pts (0 and $n+1$ are fixed boundary pts).



equidistant separation d on x axis (rest position of string)

Consider only purely transverse waves.
We have the recurrence relation

(*) $$m\ddot{y}_i = -K(y_i - y_{i+1}) + -K(y_i - y_{i-1}) = K(y_{i+1} - 2y_i + y_{i-1})$$

K is Hooke's law const.

where $y_0 = 0$, $y_{n+1} = 0$

Then we get the system of ODEs: (take $n=4$)

$$m\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\\ \ddot{y}_3\\ \ddot{y}_4\end{bmatrix} = K\begin{bmatrix}-2&1&0&0\\1&-2&1&0\\0&1&-2&1\\0&0&1&-2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$$

Form $\ddot{y} = Ay$

A is Real & Symmetric
⇒ ~~n distinct EWs~~ and n O.G. EVs.

Following Strang LAAIA p. 284
We want to diagonalize A, but doing that directly is not feasible — instead we use the recurrence relation (*). → See Appendix sheet →

We assume a solution of the form (periodic motion of each mass pt (= node)).

really $y^{(i)}(t)$ →

$$y_i(t) := S^{i}_{j}\rho_j\cos(\omega_j t - \varepsilon_j)$$

2 parameters to satisfy ICs.

This form is easiest to work with for some things but not solving for ICs.

$$= S^i_j\rho_j[\underbrace{\cos(\omega_j t)\cos(\varepsilon_j)}_{\alpha_j := \rho_j\cos(\varepsilon_j)} + \sin(\omega_j t)\underbrace{\sin(\varepsilon_j)}_{\beta_j := \rho_j\sin(\varepsilon_j)}]$$

$|\alpha_j| \le 1$
$|\beta_j| \le 1$

Then we get [redacted]

(**) $$-m(-S^i_j\omega_j^2\cancel{\rho_j\cos(\omega_j t - \varepsilon_j)}) = K(S^{i+1}_j\cancel{\rho_j\cos} - 2S^i_j\cancel{\rho_j\cos} + S^{i-1}_j\cancel{\rho_j\cos})$$

$j = 1, \dots, n$

This is saying
$\vec{y}_j(t) = \rho_j\vec{S}_j\cos(\omega_j t - \varepsilon_j)$
A single "mode."

This part specifies the extreme configurations that the mode y_j oscillates between ie ($\rho_j\vec{S}_j$ and $-\rho_j\vec{S}_j$).

This part does the oscillation with t

$$\Rightarrow \begin{bmatrix}-2&1&0&0&0&\cdots\\1&-2&1&0&0&\cdots\\0&1&-2&1&0&\cdots\\\vdots&&&\ddots&&\\&&&&1&-2\end{bmatrix}\begin{bmatrix}\vdots\\S^{i-1}_j\\S^{i}_j\\S^{i+1}_j\\\vdots\end{bmatrix} = \frac{-m\omega_j^2}{K}\begin{bmatrix}\vdots\\S^i_j\\\vdots\end{bmatrix}$$

EW λ_j

Now we let $S^i_j = \sin(i\varphi_j)$
for some angle φ_j to be determined.

cont'd →

plugging $S_j^i = \sin(i\varphi_j)$ into the recurrence relation ★★

$$\sin((i-1)\varphi_j) - 2\sin(i\varphi_j) + \sin((i+1)\varphi_j) \overset{!}{=} \frac{-m\omega_j^2}{K}\sin(i\varphi_j)$$

$$\overset{\shortparallel}{\sin(i\varphi_j - \varphi_j)}$$

trig: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$$\sin(i\varphi_j)\cos\varphi_j - \cancel{\sin\varphi_j\cos(i\varphi_j)} - 2\sin(i\varphi_j) + \sin(i\varphi_j)\cos(\varphi_j) + \cancel{\sin\varphi_j\cos(i\varphi_j)}$$

$$= \frac{-m\omega_j^2}{K}\sin(i\varphi_j)$$

$$\Rightarrow 2\cancel{\sin(i\varphi_j)}\cos\varphi_j - 2\cancel{\sin(i\varphi_j)} = \frac{-m\omega_j^2}{K}\cancel{\sin(i\varphi_j)}$$

It gives the same result for φ_j at the end pts

$$S_j^{2} - 2S_j^{1} \overset{!}{=} \frac{-m\omega_j^2}{K}S_j^{1}$$

$$\Rightarrow \quad 2(\cos\varphi_j - 1) = \frac{-m\omega_j^2}{K}$$

$$\Rightarrow \quad \boxed{\omega_j^2 = -2\frac{K}{m}(\cos\varphi_j - 1)}$$

To solve for φ_j, we use the fixed end pt conditions (see Corben & Steele p. 127 for non-fixed end pts)

$0 = y_0 = S_j^{0}\sin(0\varphi_j) = 0$ so no help to us.

$0 = y_{n+1} = S_j^{n+1}\sin((n+1)\varphi_j) \Rightarrow \sin((n+1)\varphi_j) \overset{!}{=} 0$

$\Rightarrow (n+1)\varphi_j \overset{!}{=} p\pi$ for some $p \in \mathbb{Z}$

w.l.o.g. take $p = j$

$$\boxed{\varphi_j = \frac{j\pi}{n+1}}$$

Now we can solve for ω_j

$$\omega_j^2 = -2\omega_0^2(\cos\varphi_j - 1) \quad \text{where } \omega_0 := \sqrt{K/m}$$

natural freq. of harmonic oscillator

$$= 2\omega_0^2\, 2\left(\frac{1-\cos\varphi_j}{2}\right) \quad \text{trig}$$

$$\overset{\text{trig}}{=} 4\omega_0^2\sin^2\left(\frac{\varphi_j}{2}\right)$$

$$\Rightarrow \quad \omega_j = 2\omega_0\sin\left(\frac{j}{2}\frac{\pi}{n+1}\right) \qquad j = 1, \ldots, n.$$

How do we know we have found all the EVs?
For $j = 1, \ldots, n$ we know the first elt, thus the whole vector, is distinct. ~~since there can't be more than n EVs, we know we have them all!~~ FALSE

Eigen Function

Thus we have $\vec{y}_j(t) = \begin{bmatrix} \sin(1 \cdot j \frac{\pi}{n+1}) \\ \vdots \\ \sin(n j \frac{\pi}{n+1}) \end{bmatrix} \rho_j \cos(\omega_j t - \varepsilon_j)$

$\omega_j = 2\omega_0 \sin\left(j \frac{\pi}{2(n+1)}\right)$

This is called "Normal Mode" j

eigen fcn

And the general soln (see appendix sheet)

$$\vec{y}(t) = \begin{bmatrix} | & | & & | \\ s_1 & s_2 & \cdots & s_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \rho_1 \cos(\omega_1 t - \varepsilon_1) \\ \vdots \\ \rho_n \cos(\omega_n t - \varepsilon_n) \end{bmatrix}$$

$\vec{y} = \sum_{j=1}^{n} \rho_j \vec{s}_j \cos(\omega_j t - \varepsilon_j)$

$\alpha_j = \rho_j \cos(\varepsilon_j)$

$\beta_j = \rho_j \sin(\varepsilon_j)$

$$= \begin{bmatrix} | & | & & | \\ s_1 & s_2 & \cdots & s_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \cos(\omega_1 t) + \beta_j \sin(\omega_1 t) \\ \vdots \\ \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) \end{bmatrix}$$

Thus to incorporate the initial conds

$$\vec{y}(0) = \begin{bmatrix} | & & | \\ s_1 & \cdots & s_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\dot{y}(0) = \begin{bmatrix} | & & | \\ s_1 & \cdots & s_n \\ | & & | \end{bmatrix} \begin{bmatrix} \omega_1 \beta_1 \\ \vdots \\ \omega_n \beta_n \end{bmatrix}$$

$S^T S = R$

$R^{-1} S^T S = I$

$R^{-1} S^T = S^{-1}$

call this matrix R

$$S^T S = \begin{bmatrix} r_1^2 & & \\ & r_2^2 & \\ & & \ddots & \\ & & & r_n^2 \end{bmatrix}$$

Thus $S^{-1} = \begin{bmatrix} \frac{1}{r_1^2} & & \\ & \ddots & \\ & & \frac{1}{r_n^2} \end{bmatrix} S^T$

Now since A is Real, symmetric, S is O.G. $\Rightarrow$ ~~$S^{-1} = S^T$~~

NOT O.N.!

NOT TRUE! $Ix = 1x$ for all $x \in \mathbb{R}^n$

Thus $\vec{\alpha} = R^{-1} S^T y(0)$

$\overrightarrow{\omega\beta} = R^{-1} S^T \dot{y}(0)$

If $Aw = \eta w$ for $w \neq \vec{s}_j$ some j

$w = \sum \beta_j \vec{s}_j$ since $\{s_j\}$ basis

$Aw = A \sum \beta_j s_j = \sum \beta_j \lambda_j s_j \overset{!}{=} \eta \sum \beta_j \vec{s}_j$

η/n

$\Rightarrow \sum (\lambda_j - \frac{\eta}{n}) \beta_j \vec{s}_j = 0$

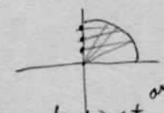
$\Rightarrow \eta = n\lambda_j$ for all j, but λ_j distinct $\Rightarrow\Leftarrow$

Aside: we know we have found all the EVs of A with our $\vec{s}_j$ $j = 1, \ldots, n$ because

$\lambda_j = \frac{-m}{K} \omega_j^2$

$\omega_j = 2\sqrt{\frac{K}{m}} \sin\left(j \frac{1}{n+1} \frac{\pi}{2}\right)$ $j = 1, \ldots$

we know $\sin(\theta) = \sin(\pi - \theta)$ but these values are all distinct because the angle is always between 0 and $\pi/2$



Thus $\omega_1, \ldots, \omega_j$ distinct and pos $\Rightarrow \omega_1^2, \ldots, \omega_n^2$ distinct

$\Rightarrow \lambda_1, \ldots, \lambda_n$ distinct

$\Rightarrow$ the associated $\vec{s}_1, \ldots, \vec{s}_n$ are distinct, LI and a basis for $\mathbb{R}^n$.

So we have all the EVs in $\{s_j\}$. □

Strang LAAIA
P. 284-286

Appendix to Loaded String

6/1/2002

Real symmetric $\Rightarrow A = Q^T \Lambda Q$ by Spectral Thm Strang p. 307 p. 296

If $\quad \ddot{u} = Au$

then a solution is given by $\vec{u}(t) = e^{i\omega t}\vec{x}$ where $\vec{x}$ is EV of A.

$\ddot{u} = (i\omega)^2 \cancel{e^{i\omega t}} x \overset{!}{=} A \cancel{e^{i\omega t}} x$

$\Rightarrow -\omega^2 x = Ax$

$\Rightarrow -\omega^2 \overset{!}{=} \lambda$

we get $\lambda = -\omega^2$ for EWs of A

$$Ax = \lambda x$$
$$= -\omega^2 x$$

Thus $\lambda = -\omega^2$, or what is more relevant to us

$\omega = \pm\sqrt{-\lambda}$ Thus we had better have λ neg if we want ω Real.

A <u>basis</u> for the space of all solutions is given by:

For each EW λ_j there are 2 solutions $\omega_+ = +\sqrt{-\lambda_j}$ and $\omega_- = -\sqrt{-\lambda_j}$

call these ω_j and $-\omega_j$

then
$$u(t) = \sum_{j=1}^{n} \left(c_j e^{i\omega_j t} + d_j e^{-i\omega_j t}\right) \vec{x}_j$$
This is the general solution

Boyce & diPrima p. 106

If $e^{i\omega t}$ and $e^{-i\omega t}$ are a basis for the Complex solution space to

$$\ddot{u} = L(u)$$

Then we know that $\sin(\omega t)$ and $\cos(\omega t)$ are a basis for the Real solution space.

$$u(t) = \sum_{j=1}^{n} \left(\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\right) \vec{x}_j$$

$\vec{x}$ is Real by Spectral Thm

then
$$u(t) = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \cos(\omega_1 t) + \beta_1 \sin(\omega_1 t) \\ \vdots \\ \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) \end{bmatrix}$$

Then at $t=0$ we can solve for α, β

$$u(0) = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Remember that this matrix is O.G.

and
$$\dot{u}(0) = \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} \omega_1 \beta_1 \\ \vdots \\ \omega_n \beta_n \end{bmatrix}$$