Linear Systems of ODEs: $\dot{x} = Ax \quad A: \mathbb{R}^n \to \mathbb{R}^n$

We mostly follow A&P ODE ch2, but they present things out of logical order. So we will summarize the key results here.

Given a linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ (If we specified a basis, e.g. the std basis, we would have our matrix A)

we define $e^T := \sum_{k=0}^{\infty} \frac{1}{k!} T^k \qquad e^T: \mathbb{R}^n \to \mathbb{R}^n$ linear

Perko DEADS p.12

T is Bdd Linear since $\mathbb{R}^n$ is finite dim $\|Tx\| \leq M\|x\| \quad \forall x$

Thm the series $\sum_{k=0}^{\infty} \frac{1}{k!} T^k$ conv abs and uniformly.

pf. Let $\|T\|_{op} = a$

$\left\|\frac{1}{k!}T^k\right\| \leq \frac{1}{k!}\|T\|^k = \frac{1}{k!}a^k$ and this converges by Weierstrauss M test. □

Define $e^{tA} := \sum^{\infty} \frac{t^k}{k!} A^k =: f(t)$

Thm $f'(t) = Af(t)$

That is to say $\frac{d}{dt}(e^{tA}) = Ae^{tA} \overset{\text{since they commute}}{=} e^{tA}A \qquad A\left[\sum \frac{t^k}{k!}A^k\right] = \left[\sum \frac{t^k}{k!}A^k\right]A$

pf.

This is done in Avez DC Thm 5.6 but that pf seems to have some problems so I followed Hirch & Smale DEDSALA p.89-90 and Perko DEADS p.17

First a result from Rudin POMA p.149 on interchanging limits:
Let E be a metric sp.
$(t_i) \to x$
Seq of fcns $(f_n) \to f$ unif over E

$\Rightarrow \lim_{i\to\infty} \lim_{n\to\infty} f_n(t_i) = \lim_{n\to\infty} \lim_{i\to\infty} f_n(t_i)$

$$\frac{d}{dt}e^{tA} = \lim_{h\to 0} \frac{1}{h}\left[e^{(t+h)A} - e^{tA}\right] = \lim_{h\to 0} \frac{1}{h}\left[e^{tA}e^{hA} - e^{tA}\right]$$

$$= \lim_{h\to 0} e^{tA} \frac{1}{h}\left[e^{hA} - I\right]$$

looks like we need $A \mapsto e^A$ is continuous Avez Thm 5.2
and possibly $AB = BA \Rightarrow e^{A+B} = e^A e^B$ Avez Thm 5.4

$$= e^{tA} \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to\infty}\left(I + hA + \frac{h^2}{2!}A^2 + \dots + \frac{h^k}{k!}A^k\right) - I\right]$$

$$= e^{tA} \lim_{h\to 0} \frac{1}{h}\left[\cancel{I} + \lim_{k\to\infty}\left(hA + \sum_{j=2}^{k} \frac{h^j}{j!}A^j\right) - \cancel{I}\right]$$

$$= e^{tA} \lim_{h\to 0} \left[\frac{1}{\cancel{h}}\cancel{h}A + \lim_{k\to\infty} \sum_{j=2}^{k} \frac{h^{j-1}}{j!}A^j\right]$$

interchange

$$= e^{tA}A \lim_{k\to\infty} \sum_{j=2} \frac{0^{j-1}}{j!}A^j$$

$$= e^{tA}A$$

□

3/12/2016

(00)

Key Thm $A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\dot{x} = Ax$, $x(0) = x_0$ $\Big\} \Longrightarrow$ The unique soln is $x(t) = e^{tA}x_0$

pf: (Hirch & Smale) By preceding thm $f(t) = e^{tA}$ satisfies $f'(t) = Af(t)$ so it is a soln.
To show there are no other soln's, let $x(t)$ be an arb soln of $\dot{x} = Ax$.

claim: $e^{-tA}x(t) = \text{const} = \vec{c}$

Trick: define $y(t) := e^{-tA}x(t)$

Then
$$y'(t) = \frac{d}{dt}\left(e^{-tA}\right)x(t) + e^{-tA}x'(t)$$
$$= -Ae^{-tA}x(t) + e^{-tA}(Ax(t)) \quad \text{by hypoth}$$
$$= e^{-tA}\left[-Ax(t) + Ax(t)\right]$$
$$= 0 \quad \text{for any } t.$$

$\Rightarrow y(t) = \vec{c}$

$\Rightarrow \vec{c} = e^{-tA}x(t)$ or $e^{tA}\vec{c} = x(t)$ and when we plug in $t=0$ we get $c \overset{!}{=} x(0) = x_0$

QED

▷ So now that we know e^{tA} is the soln, we need to actually be able to compute it in real problems.

Avez DC Thm 5.3

Thm If $A = MJM^{-1} \Rightarrow e^A = Me^JM^{-1}$

Perko DEADS p.12

pf For fixed n, $\sum_{k=0}^{n} \frac{1}{k!}(MJM^{-1})^k = I + MJM^{-1} + \frac{1}{2!}MJ\cancel{M^{-1}}\cancel{M}JM^{-1} + \frac{1}{3!}MJ\cancel{M^{-1}}\cancel{M}J\cancel{M^{-1}}\cancel{M}JM^{-1} + \dots$
$$= I + MJM^{-1} + \frac{1}{2!}MJ^2M^{-1} + \dots$$
$$= M\left[I + J + \frac{1}{2!}J^2 + \dots + \frac{1}{k!}J^k + \dots \frac{1}{n!}J^n\right]M^{-1}$$

Take $\lim_{n\to\infty}$: $\lim_{n\to\infty} M\left[I + \dots + \frac{1}{n!}J^n\right]M^{-1}$ Take lim inside since M and M^{-1} are cont.
$$= M\left[\sum_{k=0}^{\infty}\frac{1}{k!}J^k\right]M^{-1}$$
$$= Me^JM^{-1}$$
□

now we jump back to A&P ch 2 discussion ⟶

Remark what is the flow of the vf $\dot{x} = Ax$?
From Key Thm $\varphi_t(x) = e^{tA}x$ for any pt $x \in \mathbb{R}^n$

stream line curve $\dot{\varphi}_x(t) =$ [redacted] $A\varphi_x(t)$

$$\varphi_{s+t} = \varphi_s \circ \varphi_t$$

we specialize $\dot{x} = X(x)$ $x \in \mathbb{R}^n$ to the linear system $\dot{x} = Ax$

If we make a linear COV $x = My$ then $\dot{x} = M\dot{y}$ (M indep of t and linearity of $\frac{d}{dt}$)

$$\Rightarrow \underset{M\dot{y}}{\dot{x}} = AMy \Rightarrow \dot{y} = \underbrace{M^{-1}AM}_{B}\, y$$

Our goal is to choose M such that $\dot{y} = By$ is uncoupled
i.e. B is diagonal, or at least as close to diagonal as possible (Jordan normal form)

§2.2 Similiarity Types for 2×2 Real matrices

Prop 2.2.1 Jordan Form for 2×2
If A is a 2×2 matrix with Real entries ⇒ ∃ Real, nsing M ∋ $J = M^{-1}AM$ has one of the following forms:

ⓐ $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 > \lambda_2$

ⓑ $\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}$

(ⓐ and ⓑ: These could also be grouped together.)

ⓒ $\begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$

(ⓐ and ⓒ: In the sense of Jordan form, these are the same but behavior of ODE is quite different)

ⓓ $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ $\beta > 0$ $= \begin{bmatrix} \lambda & \\ & \bar{\lambda} \end{bmatrix}$ $\lambda = \alpha + i\beta$

Pf

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Find EWs:

$$0 \overset{!}{=} \det(A - \lambda I) = \begin{vmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$= \lambda^2 \underbrace{- \operatorname{tr}(A)}_{\mathcal{B}}\lambda + \underbrace{\det A}_{C}$$

quadratic formula: $\lambda = \dfrac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4aC}}{2a} = \frac{1}{2}\operatorname{tr}(A) \pm \frac{1}{2}\sqrt{\underbrace{\operatorname{tr}^2(A) - 4\det(A)}_{\text{call this } \Delta}}$

Thus $\lambda_1 = \frac{1}{2}\left[\operatorname{tr}(A) + \sqrt{\Delta}\right]$

$\lambda_2 = \frac{1}{2}\left[\operatorname{tr}(A) - \sqrt{\Delta}\right]$

Case (a) Real, distinct EWs $[\Delta > 0]$

Let $\vec{u}_1, \vec{u}_2$ be EVs. $Au_1 = \lambda_1 u_1$ $Au_2 = \lambda_2 u_2$

Let $M = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$ Then $AM = \begin{bmatrix} Au_1 & Au_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}}_{J}$ □

Case (b) Equal EWs $[\Delta = 0]$

$\lambda_1 = \lambda_2 = \frac{1}{2}\operatorname{tr}(A) = \lambda$

Subcase 1 λ has 2 LI EVs: $Au = \lambda u$ and $Av = \lambda v$
Then any $w \in \mathbb{R}^2$ is $w = \alpha u + \beta v$. Thus $Aw = \alpha Au + \beta Av = \alpha\lambda u + \beta\lambda v = \lambda w$
So A is already diag! $A = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} = J$ because $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ □

cont'd →

Subcase 2 EW λ has only one EV u

This is actually case (c) p.43-44

We want to find an M such that $M^{-1}AM = J = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$

where M is Real and invertible (nsng).

Step 1 Take $B = \begin{bmatrix} \vdots & \vdots \\ u & m \\ \vdots & \vdots \end{bmatrix}$ where $\vec{m}$ is any vector $\ni m \neq \alpha u$ (Thus B is invertible)

Then $AB = [Au \vdots Am] = [\lambda u \vdots Am] = [\lambda B\hat{e}_1 \vdots Am]$ ← since $u = B\hat{e}_1$

$= B[\lambda e_1 \vdots B^{-1}Am]$

$\Rightarrow B^{-1}AB = [\lambda e_1 \vdots B^{-1}Am] = \begin{bmatrix} \lambda & c \\ 0 & d \end{bmatrix}$ for some $c, d \in \mathbb{R}$

Step 2 We know A and $B^{-1}AB$ have the same EWs

If $C = B^{-1}AB$ then $BCB^{-1} = A$

(to see this: If $Ax = \lambda x$, let

$\Rightarrow BCB^{-1}x = Ax = \lambda x$

$\Rightarrow C(B^{-1}x) = \lambda(B^{-1}x)$ $\square$)

So $B^{-1}AB$ has λ with multiplicity 2.

But we also observe $B^{-1}AB$ is triangular and triang matrices have their EWs on main diag. Hence $d \doteq \lambda$.

$\Rightarrow B^{-1}AB = \begin{bmatrix} \lambda & c \\ & \lambda \end{bmatrix}$ This is almost the Jordan form, we just need $c = 1$.

Step 3

We know $c \neq 0$ because, if it did, we'd have 2 LI EVs — The cols of B: $\vec{u}$ and $\vec{m}$ and we ruled that out by hypoth on A.

Define $D := \begin{bmatrix} 1 & \\ & 1/c \end{bmatrix}$ Then $D^{-1} = \begin{bmatrix} 1 & \\ & c \end{bmatrix}$

Then $M := BD$ will do the trick:

$M^{-1}AM = D^{-1}B^{-1}ABD = \begin{bmatrix} 1 & \cdot \\ \cdot & c \end{bmatrix}\begin{bmatrix} \lambda & c \\ \cdot & \lambda \end{bmatrix}\begin{bmatrix} 1 & \cdot \\ \cdot & 1/c \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} = J$ $\square$

Case (d) Complex EWs $[\Delta < 0]$

EWs occur in conj pairs since A is Real.

If $\lambda \in \mathbb{C}$ is an EW: $Ax = \lambda x$ conj both sides: $\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$= A\bar{x} = \bar{\lambda}\bar{x}$

Following Hirch & Smale DEDSALA p.68

Let $\lambda = \alpha + i\beta$ $\beta > 0$

We want to prove the existence of a Real matrix M such that

$M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

<u>Step 1</u> First extend the domain of A $A: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

Then (since $\det(A - \lambda I) = 0$) $\exists$ $\mathbb{x} \in \mathbb{C}^2 \ni A\mathbb{x} = \lambda\mathbb{x}$ $\mathbb{x} = \vec{u} + i\vec{v}$ $u, v \in \mathbb{R}^2$

Remark: we shall see

$M = \begin{bmatrix} \vdots & \vdots \\ v & u \\ \vdots & \vdots \end{bmatrix}$ note the order

cont'd →

Step 2 Show that $u \neq 0$ $v \neq 0$. In fact $\{u,v\}$ is LI over $\mathbb{R}$ scalars

$A(u+iv) = (\alpha+\beta i)(\vec{u}+i\vec{v}) =$ [redacted]$\vec{u}$ − [redacted]$\vec{v}$ + $i($[redacted]$\vec{v}$ + [redacted]$\vec{u})$
$\qquad \alpha \quad \beta \quad \alpha \quad \beta$

$\mathbb{R}e:\ Au =$ [redacted]$\vec{u}$ − [redacted]$\vec{v}$ $\quad (\alpha, \beta)$

$\mathbb{I}m:\ Av =$ [redacted]$\vec{v}$ + [redacted]$\vec{u}$ $\quad (\alpha, \beta)$

if $u = 0 \overset{\mathbb{R}e}{\Longrightarrow} 0 = -$[redacted]$v$ (β) $\Rightarrow$ [redacted] $= 0$ (β) and thus $\lambda =$ [redacted] (α) $\notin \mathbb{C} \Rightarrow\Leftarrow$

if $v = 0 \overset{\mathbb{I}m}{\Longrightarrow} 0 = \beta u \Rightarrow \beta = 0 \Rightarrow\Leftarrow$

if $\gamma u + \delta v = 0$ for scalars $\gamma, \delta \in \mathbb{R} \Rightarrow v = \left(-\frac{\gamma}{\delta}\right) u$ (call this k)

then $\left.\begin{aligned} Au &= \alpha u - \beta(ku) \\ A(ku) &= \alpha k u + \beta u \end{aligned}\right\} \Rightarrow \left.\begin{aligned} Au &= (\alpha-\beta k)u \\ kAu &= (\alpha k+\beta)u \end{aligned}\right\} \Rightarrow \begin{aligned} Au &= (\alpha-\beta k)u \\ Au &= (\alpha+\tfrac{\beta}{k})u \end{aligned}$

$\Rightarrow (\not\alpha - \beta k) = \not\alpha + \frac{\beta}{k} \Rightarrow -\not\beta k = \frac{\not\beta}{k} \Rightarrow -k^2 = 1$
$\Rightarrow k^2 = -1$ or $k = \pm i$ $\Rightarrow\Leftarrow$ since $k \in \mathbb{R}$.

Step 3 Thus we have 2 LI vectors u, v and $\dim(\mathbb{R}^2) = 2$ so they form a basis. (V)
Express A wrt this basis (actually <u>switch</u> order $\{v, u\}$)

$$\left.\begin{aligned} Au &= \alpha u - \beta v = \{v\ u\}\begin{bmatrix}-\beta \\ \alpha\end{bmatrix} \\ Av &= \alpha v + \beta u = \{v\ u\}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \end{aligned}\right\} \Rightarrow A\underbrace{\{v\ u\}}_{M} = \underbrace{\{v\ u\}}_{M}\underbrace{\begin{bmatrix}\alpha & -\beta \\ \beta & \alpha\end{bmatrix}}_{K}$$

[This same pf holds for a 2-dim vector subspace $V \subseteq \mathbb{R}^n$]

QED

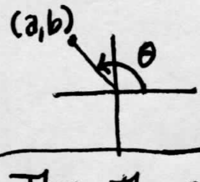
Aside Let's give a few results about the matrix $\begin{bmatrix}\alpha & -\beta \\ \beta & \alpha\end{bmatrix}$ from (Hirch & Smale DEDSALA p. 55–56)

For my convenience, I am going to rename $\alpha \longleftrightarrow a$, $\beta \longleftrightarrow b$ for this section.

Consider $T_{ab}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear map. Matrix $[T] = A = \begin{bmatrix}a & -b \\ b & a\end{bmatrix}$ $b \neq 0$

$\det(A - \lambda I) = \Delta = \lambda^2 - 2a\lambda + (a^2+b^2)$ $\quad \mathbb{C}$ EWs $\lambda = a+bi$ and $\tilde{\lambda} = a-bi$

Geometrically we will see that $A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a <u>rotation</u> and a <u>stretch</u>



For any specified pt (a,b), the angle θ is well defined $-\pi < \theta \leq \pi$ (no worries about branch cuts)
$r = \sqrt{a^2+b^2}$

Thm The effect of T_{ab} is a counter-clockwise rotation of θ rads, composed with a stretching (or contraction) of the length by the factor r:

$$T_{ab}(x) = r R_\theta(x) = R_\theta(rx) \quad \text{where } R_\theta := \text{rotation by } \theta \text{ ccw (linear map)}$$

The std basis rep of $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

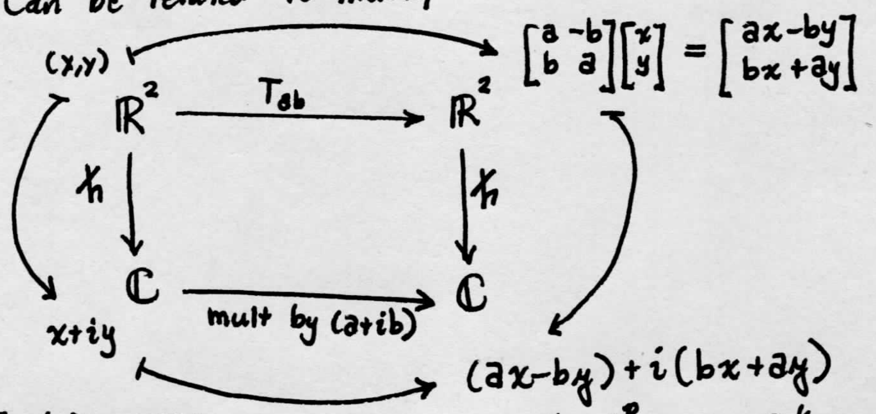
In polar co-ords: $a = r\cos\theta$
$\qquad b = r\sin\theta$

$$A = \begin{bmatrix}a & -b \\ b & a\end{bmatrix} = \begin{bmatrix}r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta\end{bmatrix} = r\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} = rR_\theta$$

Then by linearity
$Ax = rR_\theta(x) = R_\theta(rx)$
or $\begin{bmatrix}r & \\ & r\end{bmatrix}\begin{bmatrix}R_\theta\end{bmatrix}$ if you prefer

□

Thm T_{ab} can be related to multiplication in $\mathbb{C}$ as follows:



Thus $T_{ab}(x) \rightsquigarrow zx$

This gives us, from facts about $\mathbb{C}$ mult, $(T_{ab})^p(x) = z^p h(x) = (re^{i\theta})^p h(x)$
$= r^p e^{ip\theta} h(x)$ so we can avoid all the sin/cos addition formulas.

Now that we have the cannonical forms $J = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}, \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$
and that we know $e^{tA} = M e^{tJ} M^{-1}$,
Let's compute e^{tJ}:

▷ $J = \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix}$ $\quad e^{tJ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k = \sum \frac{t^k}{k!} \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix}^k = \sum \begin{bmatrix} \frac{t^k}{k!}\lambda^k & \\ & \frac{t^k}{k!}\mu^k \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & \\ & e^{\mu t} \end{bmatrix}$

Easy to see how this generalizes to $n \times n$ matrices.

▷ $J = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$

Prob (27) shows us an easy way to do this (and more general than $n=2$ case)
Let A be an $n \times n$ matrix with EW λ_0 repeated n times. Thus $\det(A - \lambda I) = (\lambda - \lambda_0)^n = 0$

Define $N := A - \lambda_0 I$ $\quad$ N will turn out to be nilpotent.
$A = \lambda_0 I + N$ $\quad$ and since ~~any matrices commute~~ the special matrix $\lambda_0 I$ commutes with any matrix, $\lambda_0 I N = N(\lambda_0 I)$

$e^{tA} = e^{t\lambda_0 I} e^{tN}$

We know $e^{t\lambda_0 I} = \begin{bmatrix} e^{\lambda_0 t} & & \\ & \ddots & \\ & & e^{\lambda_0 t} \end{bmatrix}$ $\quad$ So we only need to compute e^{tN}

Observe that $N^n = 0$: By the Cayley-Hamilton thm, we get the 0 matrix if we plug any matrix into its own characteristic eq. For our A, this is $(A - \lambda_0 I)^n = 0$ which is $N^n = 0$.

$\Rightarrow e^{tN} = \sum_{k=0}^{\infty} \frac{t^k}{k!} N^k = \left[I + tN + \frac{t}{2} N^2 + \dots + \frac{t^{n-1}}{(n-1)!} N^{(n-1)} \right]$

$\Rightarrow e^{tA} = e^{\lambda_0 t} \left[I + tN + \dots + \frac{t^{n-1}}{(n-1)!} N^{(n-1)} \right]$

Then for our specific $J = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$ $\quad n=2$ and we get

$$e^{tJ} = e^{\lambda t} \left[\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \right] = e^{\lambda t} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$$ ▫

▷ $J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ To compute e^{tJ}, first note

$$J = \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} + \begin{bmatrix} & -\beta \\ \beta & \end{bmatrix} = \alpha I + \beta H \quad \text{where } H := \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$$

Since the special matrix αI commutes with all matrices, it commutes with βH.

$$\Rightarrow e^{J} = e^{\alpha I + \beta H} = e^{\alpha I} e^{\beta H} \quad \text{and} \quad e^{\alpha I} = e^{\alpha} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Now observe that H behaves just like multiplying by Complex i (see Thm on p. 4 sheets) $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$$H^2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} = -I$$

$H^3 = -IH = -H$

$H^4 = -H^2 = -(-I) = I$

$H^5 = H$ and the cycle repeats

Note that I and H have no components in common so sums don't interfere with each other.

$$\Rightarrow e^{\beta H} = \sum^{\infty} \frac{\beta^k}{k!} H^k = I + \beta H - \frac{\beta^2}{2!} I - \frac{\beta^3}{3!} H + \frac{\beta^4}{4!} I + \frac{\beta^5}{5!} H - \frac{\beta^6}{6!} I - \frac{\beta^7}{7!} H + \dots$$

$$= \underbrace{\left(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \dots\right)}_{\cos\beta} I + \underbrace{\left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \frac{\beta^7}{7!} + \dots\right)}_{\sin\beta} H$$

$$= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

Putting it all together:

$$e^{tJ} = e^{t(\alpha I + \beta H)} = e^{\alpha t I} e^{\beta t H} = \underbrace{\begin{bmatrix} e^{\alpha t} & \\ & e^{\alpha t} \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}}_{\text{rotation}}$$

□

▷ Now let's list some <u>stability</u> definitions and results from Perko DEADS Ch 1.9 for Linear Systems $\dot{x} = Ax$ $A: \mathbb{R}^n \to \mathbb{R}^n$ $A\vec{0} = \vec{0}$ origin is FP

If $\lambda = \alpha + i\beta$ is a $\mathbb{C}$ EW, then $\vec{w} = \vec{u} + i\vec{v}$ is the corresponding EV.

<u>Thm</u> $\mathbb{R}^n$ can be decomposed as $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$ where each subspace is <u>invariant</u> under the flow $\varphi_t(\cdot) = e^{tA}(\cdot)$. [For non-linear systems we speak of stable, center, and unstable mfds in nbhd of FP]

where $E^s := \text{Span}\{\text{all } u_j, v_j \mid \text{EVs for any } \lambda \text{ with } \underline{\text{neg}} \text{ Real part } \alpha\}$ Stable subsp. $e^{\alpha t} \to 0$ as $t \to \infty$ and sol'ns move to FP

$E^u := \text{Span}\{\text{all } \vec{u}_j, \vec{v}_j \mid \text{EVs for any } \lambda \text{ with } \underline{\text{pos}} \text{ Real part } \alpha\}$ Unstable subspace $e^{\alpha t} \to 0$ as $t \to -\infty$ time flows back

$E^c := \text{Span}\{u, v \mid \text{Real}(\lambda) = 0 \text{ pure imaginary}\}$ Centre Mfd periodic orbits

If all λ's have <u>non-zero</u> Real part, we call 0 a <u>Hyperbolic FP</u>

all $\text{Real}(\lambda) > 0 \Rightarrow$ Source

all $\text{Real}(\lambda) < 0 \Rightarrow$ Sink

▷ Now let's back up and discuss phase portraits for 2×2 systems in A&P ODE Ch 2.3. Following A&P, we solve these with our bare hands, rather than using the solns that we just found.

$\dot{x} = Ax$ is simple [better term: non-degenerate] if A is nsing [$Ax=0 \Leftrightarrow x=0$]

Then we only have one FP at origin (If A had rank 1, we'd get a whole line of FPs)

▷ Case $J = \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix}$

Then $\frac{dx}{dt} = \lambda x$

$\frac{dy}{dt} = \mu y$

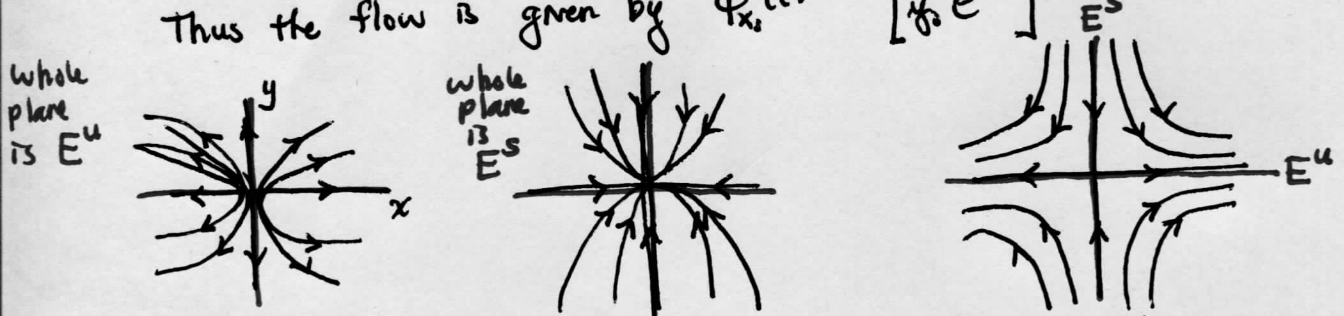
$\frac{1}{x} dx = \lambda dt \Rightarrow \ln|x| = \lambda t + c_0 \Rightarrow |x(t)| = e^{\lambda t + c_0} = e^{\lambda t} e^{c_0} = c_1 e^{\lambda t}$

$\Rightarrow x(t) = \pm c_1 e^{\lambda t} = c_2 e^{\lambda t}$ where $c_2 \in \mathbb{R}$ and can be pos or neg.

Likewise $y(t) = \gamma e^{\mu t}$ $\gamma \in \mathbb{R}$

$x_0 = c_2$
$y_0 = \gamma$

Thus the flow is given by $\varphi_{x_0}(t) = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$

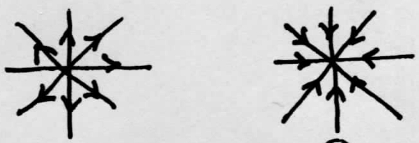


Source
$0 < \mu < \lambda$
pos

sink
$\lambda < \mu < 0$
neg

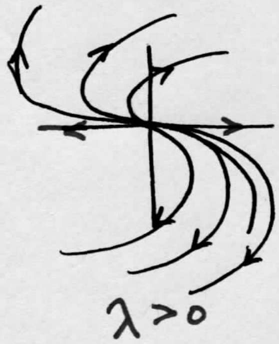
Saddle
$\mu < 0 < \lambda$ diff signs

▷ $J = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}$ is a special case of above

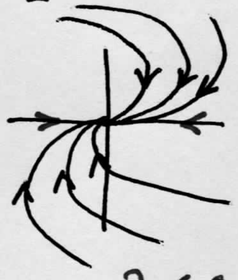


▷ $J = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$ Here just read off soln's found before: sheet (4)

$e^{Jt} = e^{\lambda t} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \Rightarrow$ $x(t) = e^{\lambda t} x_0 + y_0 t e^{\lambda t}$
$y(t) = y_0 e^{\lambda t}$



$\lambda > 0$



$\lambda < 0$

Domain?

$\triangleright\ J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad \beta > 0$

$\dot{x} = \alpha x - \beta y$

$\dot{y} = \beta x + \alpha y$

Make Polar COV

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\cancel{2} r\dot{r} = \cancel{2} x\dot{x} + \cancel{2} y\dot{y}$

$\tan\theta = \frac{y}{x}$

$\frac{1}{\cos^2\theta}\dot{\theta} = \frac{1}{x^2}(x\dot{y} - y\dot{x})$

Then plug in from ODE:

$r\dot{r} = x(\alpha x - \beta y) + y(\beta x + \alpha y)$

$= \alpha x^2 - \cancel{\beta yx} + \cancel{\beta yx} + \alpha y^2$

$= \alpha(x^2 + y^2)$

$= \alpha r^2$

$\Rightarrow \dot{r} = \alpha r$

$\frac{1}{\cos^2\theta}\dot{\theta} = \frac{1}{x^2}\left[x(\beta x + \alpha y) - y(\alpha x - \beta y)\right]$

$\beta x^2 + \cancel{\alpha yx} - \cancel{\alpha yx} + \beta y^2$

$= \frac{\beta}{x^2}(x^2 + y^2)$ $\cancel{r^2}$

$\frac{1}{\cancel{\cos^2\theta}}\dot{\theta} = \frac{\beta}{\cancel{r^2\cos^2\theta}}$

$\dot{\theta} = \beta$

$\boxed{\theta(t) = \beta t + \theta_0}$

$\Rightarrow \frac{dr}{dt} = \alpha r \Rightarrow \frac{1}{r}dr = \alpha dt$

$\ln|r| = \alpha t + c_0$

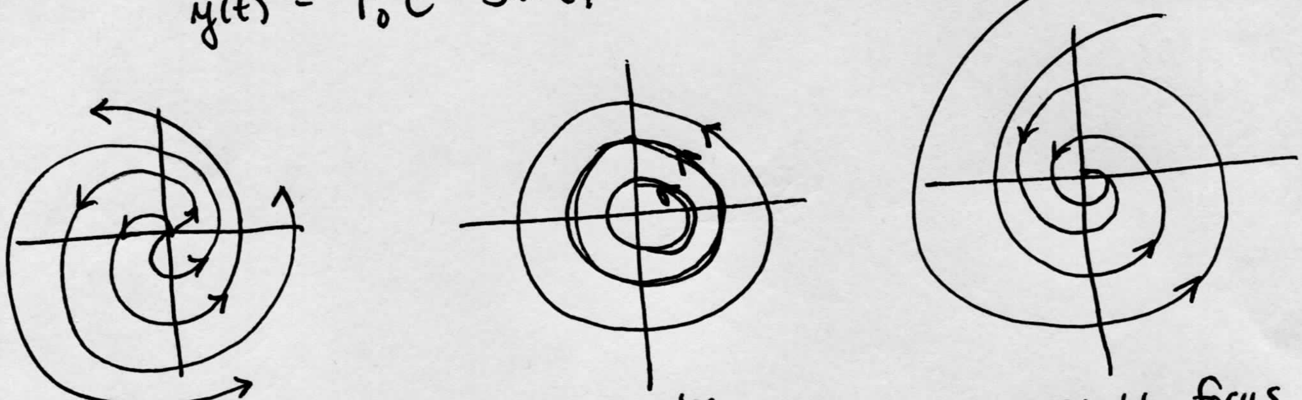
$|r| = c_1 e^{\alpha t}$

$\boxed{r(t) = r_0 e^{\alpha t}}$ and we know $r > 0$ so drop abs val

Then $x(t) = r(t)\cos\theta(t)$

$= r_0 e^{\alpha t}\cos(\beta t + \theta_0)$

$y(t) = r_0 e^{\alpha t}\sin(\beta t + \theta_0)$



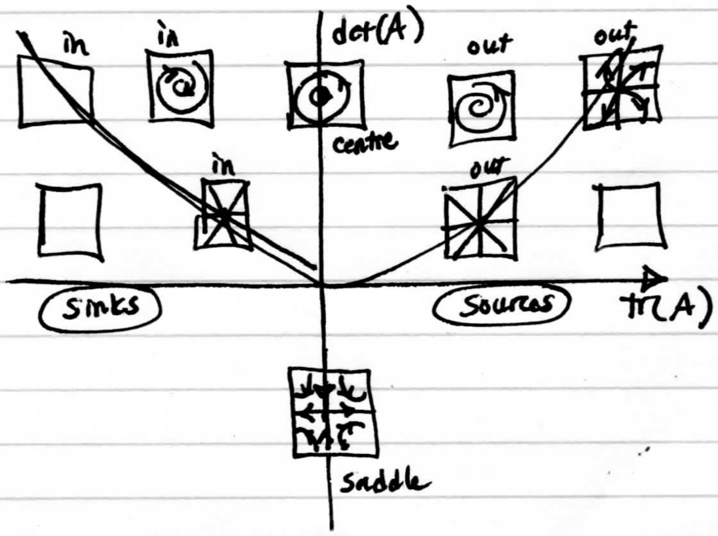
unstable focus $\alpha > 0$

Centre $\alpha = 0$ — **Periodic orbits**

Stable focus $\alpha < 0$

Quickly Summarizing the rest of Ch 2

p. 53
Hirsch & Smale
p. 96



p. 57 We group the phase portrait together into EQ classes if there is a homeo between them. These classes correspond to the previous classes of Jordan form.

p. 61 Affine Systems $\dot{x} = Ax + h$

3/17/2016

H&S p.99-100

b

Non-homog $\dot{x} = Ax + \vec{B}(t)$ (★)

look for soln $X(t) = e^{tA} f(t)$ vs $e^{tA} x_0$ — variation of constants

$$x'(t) = e^{tA} A f(t) + e^{tA} f'(t)$$
$$= A \underbrace{e^{tA} f(t)}_{X(t)} + e^{tA} f'(t)$$

$$A x(t) + b \overset{!}{=} A x(t) + e^{tA} f'(t) \Rightarrow f'(t) = e^{-tA} b(t)$$

$$\Rightarrow f(t) = \int_0^t e^{-As} b(s)\, ds + k \quad \text{for some const } k \in \mathbb{R}$$

$$\Rightarrow X(t) = e^{tA}\left[\int_0^t e^{-sA} b(s)\, ds + k\right]$$

▷ Check that this is a soln

$$X'(t) = Ae^{tA}\left[\int^t e^{-sA} b(s)\, ds + k\right] + e^{tA}\left[e^{-tA} b(t)\right]$$
$$= A \underbrace{e^{tA}\left[\int^t e^{-sA} b(s)\, ds + k\right]}_{X(t)} + b(t)$$

$$= A x(t) + b(t) \quad \square$$

▷ Every soln must be of this form:

Let $y : t \mapsto \mathbb{R}^n$ be another soln: $\dot{y} = Ay + b$

Then

$$\dot{x} - \dot{y} = A(x - y) \quad \text{homog}$$
$$\Rightarrow x - y = e^{tA} k_0 \quad \text{for some } k_0 \in \mathbb{R}$$
$$\Rightarrow y = x - e^{tA} k_0 = e^{tA} f(t) - e^{tA} k_0 = e^{tA}(f(t) - k_0)$$

Lets rewrite $X(t) = e^{tA}\left[\int_0^t e^{-sA} b(s) + k\right] =: u(t) + \underbrace{e^{tA} k}_{\text{soln to homog}}$

where $u(t) = e^{tA} \int_0^t e^{-sA} b(s)\, ds$