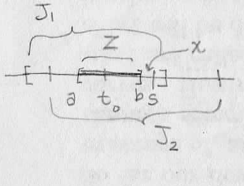
Lemma 1.1 : $g_1 : J_1 \longrightarrow W$ (interval), $g_2 : J_2 \longrightarrow W$ 2 solns to $\begin{cases} \dot{\alpha} = F(\alpha, t) \\ \alpha(t_0) = \alpha_0 \end{cases} \Rightarrow g_1(t) = g_2(t) \quad \forall t \in J_1 \cap J_2$

$\$$ not. Then $\exists\ s \in J_1 \cap J_2 \ni g_1(s) \neq g_2(s)$. wlog $\$\ s > t_0$.

Let $C := \{ \text{all } t > t_0 \mid g_1(t) \neq g_2(t) \}$ $\quad C \neq \emptyset$ since $s \in C$.



Let $x = \inf C$.

Note $x > b$ since by Thm $\exists$ interval $[a, b]$ where $g_1(t) = g_2(t)$ by uniqueness $[g_1(t), g_2(t) \in B(\alpha_0, r)]$.

CLAIM : $x \notin C$.

Let $(t_n) \nearrow x$. Then $g_1(t_n) = g_2(t_n)$

$\Rightarrow \lim g_1(t_n) = \lim g_2(t_n)$

$= g_1(\lim t_n) = g_2(\lim t_n)$ by cont

$\Rightarrow x \notin C$.

Let $g_1(x) = \alpha_1 = g_2(x)$.

So g_1, g_2 are 2 solns to ODE that pass thru $\langle x, \alpha_1 \rangle$.

We can apply thm in a small ball $B(\alpha_1, \delta)$ and conclude $\exists$ interval Q containing x where $g_1 \equiv g_2$.

$\Rightarrow x \neq \inf(C)$

$\Rightarrow\Leftarrow$

We can only conclude $C = \emptyset$

$\square$

My Reworking of L&S pf of Thm 1.1 — Some details were vague or misleading. I took some ideas from Avez DC p.82

The steps of this pf are fundamentally same as Avez p. 82; the difference is here we explicitly treat the case $\dot{\alpha} = F(t, \alpha)$. We do this by showing we have a Lip cond in the 2^{nd} variable, unif over all t.

PRELIMINARIES: Let $F : I \times \mathcal{a} \longrightarrow W$, $(t, \alpha) \longmapsto$; I open interval, $\mathcal{a}^{open} \subseteq W^{Banach}$; F C^1 smooth, time dependent v.f. ($\mathbb{R} \times W$)

consider the pt $\langle t_0, \alpha_0 \rangle$.
choose a closed interval $L := [t_0 - \rho, t_0 + \rho] \subset I$, and a ball $\overline{B}(\alpha_0, r) \subseteq \mathcal{a}$.

$DF_{(t,\alpha)}$ is cont since F is C^1

If W is fin dim, $\overline{B}$ is cpt and hence $\sup_{(t,\alpha) \in L \times \overline{B}} \|F(t,\alpha)\|_W \le m < \infty$

$$c := \sup_{(t,\alpha) \in L \times \overline{B}} \|DF_{(t,\alpha)}\|_{op} < \infty$$

And since $\|D_2 F_{(t,\alpha)}\|_{op} \le \|DF_{(t,\alpha)}\|_{op}$ (t fixed) $\Rightarrow \sup_{(t,\alpha)} \|D_2 F\| \le c$.

Then since $\overline{B}(\alpha_0, r)$ is convex, we can apply the MVT

$$\|F(t,\alpha) - F(t,\beta)\|_W \le \sup_{\xi \in \overline{B}} \|D_2 F_{(t,\xi)}\|_{op} \|\alpha - \beta\| \le c \|\alpha - \beta\| \quad \underline{\text{indep of } t \in L} \quad \forall \alpha, \beta \in \overline{B}$$

So we have established a Lip cond in $\overline{B}$.

THM REFORMULATED:

Given ODE $\begin{cases} \dot{\alpha} = F(t,\alpha) \\ \alpha(t_0) = \alpha_0 \end{cases}$ where $F : I \times \mathcal{a} \longrightarrow W$ ($\mathbb{R} \times W$) C^1 smooth.

I open interval in $\mathbb{R}$, $\mathcal{a}^{open} \subseteq W^{Banach}$, L closed interval in I.

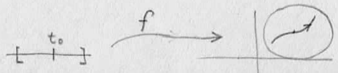
$L \times \overline{B}(\alpha_0, r) \subset I \times \mathcal{a}$

* $\|F\|_\infty \le m$

* $\|F(t,\alpha) - F(t,\beta)\|_W \le c \|\alpha - \beta\|_W \quad \forall t \in L, \ \forall \alpha, \beta \in \overline{B}$.

These automatically

THEN: $\exists$ a unique soln $f : \underbrace{\left[t_0 - \frac{r}{2(m+cr)},\ t_0 + \frac{r}{2(m+cr)}\right]}_{\text{this interval is } J} \longrightarrow \overline{B}(\alpha_0, r)$



GoTo ↗

pf

step 1 Convert to integral eq $f(t) = \int_{t_0}^{t} F(s, f(s))\,ds + \alpha_0$

define const map $\overline{\alpha}_0 : J \longrightarrow \overline{B}(\alpha_0, r)$, $t \longmapsto \alpha_0$

step 2 Find a set G that contains all possible soln's of ODE and upon which we can apply the Contraction mapping thm.

Take $G := \overline{B}(\overline{\alpha}_0, r) = \{\text{all } f \in BC(J \to W) \mid \|f - \overline{\alpha}_0\|_{sup} \le r\}$

This won't quite work; we need also $f(t_0) = \alpha_0$.

Following Avez p.82 $G := \left\{ \text{all } f : J \to \overline{B}(\alpha_0, r) \;\middle|\; \begin{matrix} f \text{ cont} \\ f(t_0) = \alpha_0 \end{matrix} \right\}$. This is a closed subset of $C(J \to \overline{B})$, hence complete metric sp.

step 3 Show $K : G \longrightarrow G$, $f \longmapsto Kf$ where $(Kf)(t) = \int_{t_0}^{t} F(s, f(s))\,ds + \alpha_0$

$(Kf)(t_0) = \alpha_0$.

We show K does not move the center of $B(\overline{\alpha}_0, r)$ "too far"

we want to appeal to COR 1 p. 229; we also need K is a contraction.

$$\|K(\overline{\alpha}_0) - \overline{\alpha}_0\| = \sup_{t \in J} \left\| \int_{t_0}^{t} F(s, \alpha_0)\,ds \right\| \le \sup_{t \in J} \int_{t_0}^{t} \underbrace{\|F(s,\alpha_0)\|}_{\le m} ds \le m\delta$$

$\delta = \text{len}(J)$

So by cor 1 p.229 we need $\delta m < (1 - \delta c) r$ see step 4

step 4 Show K is a contraction on G — then we are done by Contraction Map. Thm

$$\|K(f_1) - K(f_2)\|_\infty = \sup_{t \in J} \left\| \int_{t_0}^{t} [F(s, f_1(s)) - F(s, f_2(s))]\,ds \right\|$$

$$\le \delta \left[\sup_{s \in J} \|F(s, f_1(s)) - F(s, f_2(s))\| \right]$$

$$\overset{Lip}{\le} \delta c \left[\sup_{s \in J} \|f_1(s) - f_2(s)\| \right] = \delta c \|f_1 - f_2\|_\infty$$

so we need $\delta c < 1$, but we get this for free from stronger cond in step 3.

□