p. 530

- In A&M 2nd ch 3.5 - 3.6 "$\mathcal{L}$" (L&S) = "$\lambda_{\mathfrak{m}}$" (A&M) = $\mathbb{F}L$ Legendre transform, but no mention is made of a Riemann metric (except maybe p. 223)

  "X" = X_H

  "Y" = X_E

  L&S use it to give an explicit form of H

I will continue to call "$\mathcal{L}$" as "$\lambda_{\mathfrak{m}}$" like p. 521 sheets.

Book says

$$\begin{array}{ccc} T(TM) & \xrightarrow{(\lambda_{\mathfrak{m}})_*} & T(T^*M) \\ Y \uparrow & & \uparrow X \\ TM & \xrightarrow{\lambda_{\mathfrak{m}}} & T^*M \end{array}$$

but this is just the pullback of p. 384 BOPP.

$$\begin{array}{ccc} TA & \xrightarrow{T\psi} & TB \\ \psi^*X \uparrow & & \uparrow X \\ A & \xrightarrow[\text{diffeo}]{\psi} & B \end{array}$$

For this we really should have TU

$$\begin{array}{ccc} TM & \xrightarrow{\lambda_{\mathfrak{m}}} & T^*M \\ \downarrow T\alpha & & \downarrow T^*\alpha \\ \mathcal{U}\times\mathbb{R}^n & \xrightarrow{(T\alpha^*)\lambda_{\mathfrak{m}}(T\alpha)^{-1}} & \mathcal{U}\times\mathbb{R}^{n*} \\ \langle q_1 \dots q_n, \dot q_1 \dots \dot q_n\rangle & & \langle q_1 \dots q_n, p_1 \dots p_n\rangle \end{array}$$

$$\psi^*X = (T\psi)^{-1}\circ X \circ \psi$$

Now rename "ψ" as "$\lambda_{\mathfrak{m}}$"

"ψ^*X" as "Y" = $\lambda_{\mathfrak{m}}^* X$

"$T\psi$" = ψ_* = $(\lambda_{\mathfrak{m}})_*$

$$\boxed{\begin{array}{l} \mathcal{L}_{T\alpha} = (\lambda_{\mathfrak{m}})_{T\alpha} : \mathcal{U}\times\mathbb{R}^n \longrightarrow \mathcal{U}\times\mathbb{R}^{n*} \\ \qquad \langle q, \dot q\rangle \longmapsto \langle q, \sum g_{ij}(q)\rangle \end{array}}$$

$$\Rightarrow \quad Y = (\lambda_{\mathfrak{m}})_*^{-1}\circ X \circ \lambda_{\mathfrak{m}} \quad \Rightarrow \quad \underline{(\lambda_{\mathfrak{m}})_* Y = X\circ\lambda_{\mathfrak{m}}}.$$

Recall: Given $\Xi_x \in TM$, we define $l_x^{\Xi}(\cdot) = \mathfrak{M}_x(\Xi_x, \cdot)$

Following p. 521 sheets

$$\Xi_q = \dot q_1 \frac{\partial}{\partial q^1}(q) + \dots + \dot q_n \frac{\partial}{\partial q^n}(q)$$

$$\mathfrak{M}^{T\alpha}_{(q,\dot q)}(\Xi,\Xi) = \sum_i \sum_j g_{ij}(q)\, \underbrace{dq^i(\Xi)}_{\dot q^i}\, \underbrace{dq^j(\Xi)}_{\dot q^j}$$

$$\begin{array}{rcl} \lambda_{\mathfrak{m}} : T_qM & \longrightarrow & T_q^*M \\ \Xi & \longmapsto & l_q^{\Xi}(\cdot) = \sum_i \sum_j g_{ij}(q)\dot q_i\, dq^j(\cdot) \end{array}$$

But we also know $l_q^{\Xi} = \sum_k p^k(l_q)\, dq^k$

By comparison we see $p^i(l_q^{\Xi}) = \sum_j g_{ij}(q)\dot q_j$

$H : T^*M \longrightarrow \mathbb{R}$

p. 393

$$\begin{aligned} H(l_q) &= \langle\!\langle \lambda_{\mathfrak{m}}^{-1}(l), l \rangle\!\rangle_q - L(\lambda_{\mathfrak{m}}^{-1}(l)) \\ &= l_q(\Xi_q) - L(\Xi_q) \\ &= \mathfrak{M}_q(\Xi,\Xi) - \left(\tfrac{1}{2}\|\Xi\|^2 - U(q)\right) \\ &= \|\Xi\|^2 - \tfrac{1}{2}\|\Xi\|^2 + U = K + U. \end{aligned}$$

$$\begin{array}{ccccccc} \mathbb{R} & \xleftarrow{L} & TU & \xrightarrow{\lambda_{\mathfrak{m}}} & T^*U & \xrightarrow{H} & \mathbb{R} \\ & \nwarrow L_{T\alpha} & \downarrow T\alpha & & \downarrow T^*\alpha & \nearrow H_{T\alpha} & \\ & & \mathcal{U}\times\mathbb{R}^n & \xleftarrow{G} & \mathcal{U}\times\mathbb{R}^{n*} & & \\ & & \langle q, \dot q\rangle & & \langle q, p\rangle & & \end{array}$$

Some details of where objects actually live are unclear.

$$\begin{aligned} H_{T\alpha}(l_{q_{T\alpha}}) &= \langle\!\langle \lambda_{\mathfrak{m}}^{-1}(l), l \rangle\!\rangle - L(\lambda_{\mathfrak{m}}^{-1}(l)) \\ &= \mathfrak{M}_q^{T\alpha}(\Xi,\Xi) - L(\Xi) \\ &= \sum_i \sum_j g_{ij}(q)\dot q_i \dot q_j - L(\Xi) \\ &= \sum_i p_i \dot q_i - L(\Xi) \end{aligned}$$

Downstairs version of the Legendre transform

$$G = T\alpha \cdot \lambda_{\mathfrak{m}}^{-1}\cdot (T^*\alpha)^{-1} \qquad \begin{array}{rcl} G : \mathcal{U}\times\mathbb{R}^{n*} & \longrightarrow & \mathcal{U}\times\mathbb{R}^n \\ \langle q, p\rangle & \longmapsto & \langle q, \dot q\rangle \end{array}$$

Then $\dot q_i = G_{n+i}(q,p)$

$$L = L(q_1, \dots, q_n, \underbrace{G_{n+1}(q,p)}_{\dot q_1}, \dots, \underbrace{G_{n+n}(q,p)}_{\dot q_n})$$

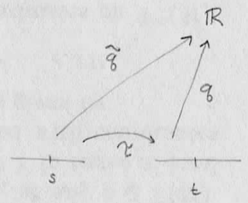
(★)

$$\begin{aligned} \frac{\partial}{\partial q_\alpha^i} H_{T\alpha} &= \frac{\partial}{\partial q_\alpha^i}\left[\sum_{j=1}^n G_{n+j}(q,p)\, p_j - L\right] \\ &= \sum_{j=1}^n \frac{\partial G_{n+j}}{\partial q^i} p_j - D_i L - \underbrace{\sum_{j=1} (D_{n+j}L)_{(q,G(q,p))}}_{= p_j \text{ by } (8.3)} \cdot \frac{\partial G_{n+j}}{\partial q^i}(q,p) \\ &= -(D_i L)_{(q,G(q,p))} \\ &= -\frac{\partial L}{\partial q_i} \end{aligned}$$

M is an open subset of $\mathbb{R}^n$

$$\mathcal{M}_{id}(\xi, \xi) = \sum_i \sum_j g_{ij}\, \dot{q}_i\, \dot{q}_j \qquad g_{ij} = \mathrm{Const}_{ij}$$

$$\bar{U}(\alpha x^1, \dots, \alpha x^n) = \alpha^p\, \bar{U}(x^1, \dots, x^n)$$



Let us consider a solution curve $q = q(t)$.

$$\tau : \mathbb{R} \longrightarrow \mathbb{R}$$
$$s \longmapsto \tfrac{1}{\beta} s = t$$

$\tilde{q} = q \circ \tau$ $\qquad \dfrac{d\tilde{q}_i}{ds} = \dfrac{dq_i}{dt}\dfrac{dt}{ds} = \dfrac{dq_i}{dt}\dfrac{1}{\beta}$

Then $\alpha \dfrac{d\tilde{q}_i}{ds} = \dfrac{d}{ds}(\alpha \tilde{q}_i) = \dfrac{\alpha}{\beta}\dfrac{dq_i}{dt}$

does not depend on q by hypoth.

Then plug into $L(\alpha\tilde{q}, \frac{d}{ds}(\alpha\tilde{q})) = K - U$

since by p. 531 (8.7) $\frac{d}{dt} q_i = \dot{q}_i$

$$\overset{(8.2)}{=} \frac{1}{2}\sum_i\sum_j g_{ij}\, \underbrace{\frac{d}{ds}(\alpha\tilde{q}_i)\,\frac{d}{ds}(\alpha\tilde{q}_j)} - \bar{U}(\alpha\tilde{q})$$

$$= \overset{\alpha^p}{\frac{\alpha^2}{\beta^2}}\,\frac{1}{2}\sum_i\sum_j g_{ij}\,\frac{d}{dt}q_i\,\frac{d}{dt}q_j - \alpha^p\,\bar{U}(q)$$

$$= \alpha^p\, L\!\left(q, \frac{dq}{dt}\right)$$

$\square$