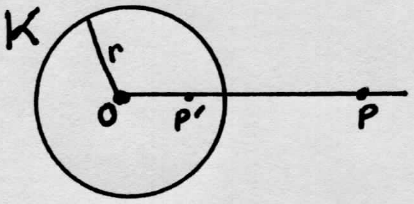
10/27/2024
Nick
①

Inversion in a Circle

Inversion of a pt wrt a circle (or 'reflection' thru a circle) does not preserve Euclidean distance. [It does preserve angles, although their sense is reversed] we shall see this later.



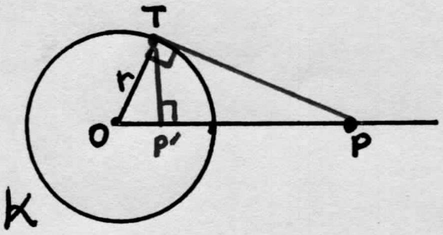
a pt P outside circle $K(O,r)$ has an inverse P' inside K, and P' has inverse P. These points lie on a radial line and are connected by the relation $|OP|\cdot|OP'| = r^2$

The prototype is $x \longmapsto \frac{1}{x}$ on $\mathbb{R}$. The 'circle' is $(-1, 1)$ here.
To generalize to $(-r, r)$ take $x \longmapsto r^2\left(\frac{1}{x}\right)$ Then $x x' = x\left(\frac{r^2}{x}\right) = r^2$

Or think of $\mathbb{R}^2$: Given radial vector $\hat{u}$, $\vec{P} = \alpha\hat{u}$ and thus $\vec{P}' = \beta\hat{u}$
what is β? $\|\vec{P}\|\,\|\vec{P}'\| = r^2 \Rightarrow \alpha\beta = r^2 \Rightarrow \beta = \frac{r^2}{\alpha}$

▷ Construct inverse of P by Euclidean geometric techniques:

P.486
Jacobs

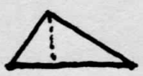


Given O and P and circle K

① Construct tangent line $\overline{PT}$ to K as per sheet ㉜ Line meets K at pt T
Then $OT \perp PT$ by Thm 67 and 68 sheet 32

② Drop a line down from T that is perp ⊥ to $\overline{OP}$ [Thru a pt not on a line ∃! perp to line] where this line meets $\overline{OP}$ is the pt P' that we seek.

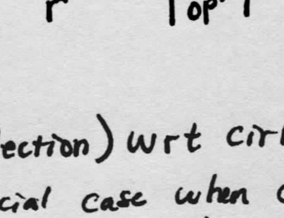
③ How do we know $|OP|\cdot|OP'| = r^2$? $\overline{P'T}$ is altitude to hypot.
Δ_{OTP} is a right Δ with hypot $\overline{OP}$.



[Altitude wrt hypot
Thm 6 sheet ㉙
COR 2] $\Rightarrow \frac{|OP|}{|OT|} = \frac{|OT|}{|OP'|}$ Each leg is geo mean between hypot and O.G. proj of leg onto hypot.

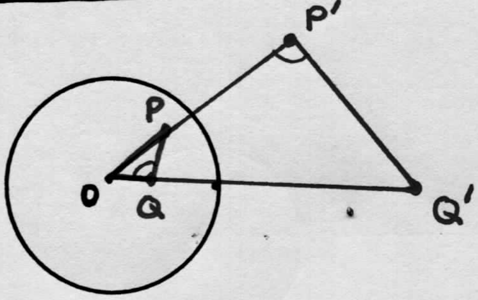
But $|OT| = r \Rightarrow \frac{|OP|}{r} = \frac{r}{|OP'|} \Rightarrow |OP|\,|OP'| = r^2$ □

- Let us denote inversion (or reflection) wrt circle K as ϕ_K: Plane ∪ {∞} ⟶ Plane ∪ {∞}
- Reflection thru a line is a special case when circle's radius = ∞
- ϕ_K is One-to-One (consider any ray from O), and Onto (although tricky at ∞)
- $\phi_K^2 = Id$ $\phi_K = \phi_K^{-1}$



↻ maps to ↺
or reversing

Key Lemma for Circle Inversions



Given any 2 pts P, Q where O-P-Q not a line,
THEN $\Delta_{OPQ} \sim \Delta_{OQ'P'}$

Specifically $\overline{OP}$ corresponds to $\overline{OQ'}$
$\overline{OQ}$ $\overline{OP'}$
$\overline{PQ}$ $\overline{Q'P'}$

$\measuredangle_{Q'} = \measuredangle_P$
$\measuredangle_{P'} = \measuredangle_Q$

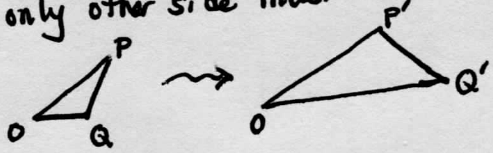
Pf. (1) angle $\measuredangle_O$ is common to both triangs

(2) we know $|OP||OP'| = r^2 = |OQ||OQ'|$ def of inversion

$$\Rightarrow \frac{OP \cdot OP'}{OQ'} = \frac{r^2}{OQ'} = OQ \quad \Rightarrow \quad \frac{OP}{OQ'} = \underbrace{\frac{r^2}{OP'\,OQ'}}_{\lambda} = \frac{OQ}{OP'}$$

$OP = \lambda OQ'$ $\qquad$ $OQ = \lambda OP'$

(3) By SAS for similiar Triangs,
using correspondence $OP \rightsquigarrow OQ'$
$OQ \rightsquigarrow OP'$
and the common angle $\measuredangle_O$ is between
Then the only other side must be $PQ \longrightarrow P'Q'$

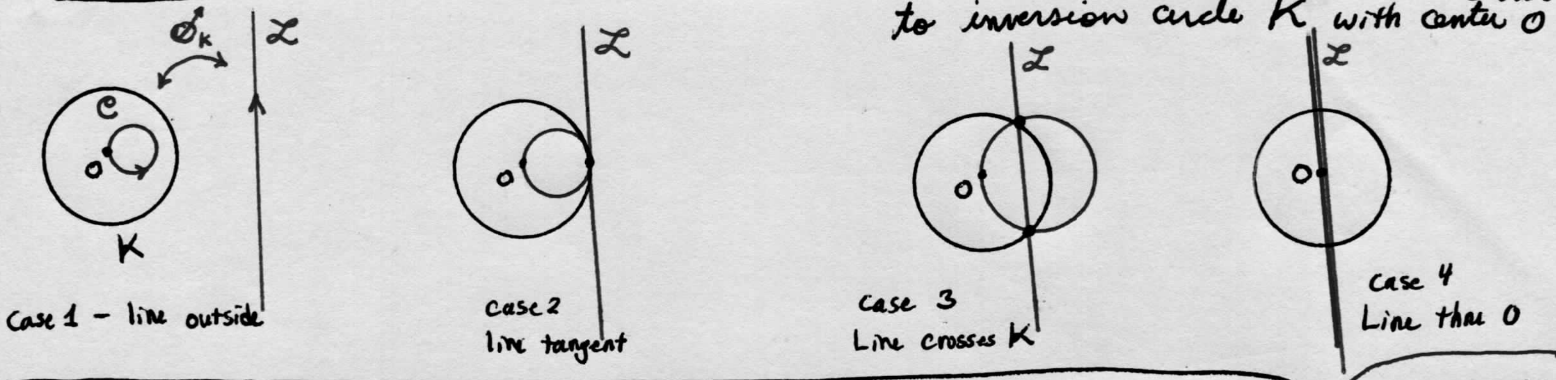


(4) We also get $\measuredangle_Q = \measuredangle_{P'}$
because $QP \rightsquigarrow Q'P'$
$QO \rightsquigarrow P'O$
So the angles between these corresponding sides are $\measuredangle_Q = \measuredangle_{P'}$
and likewise for the remaing angle.

$\square$

Reflections of Lines and Circles

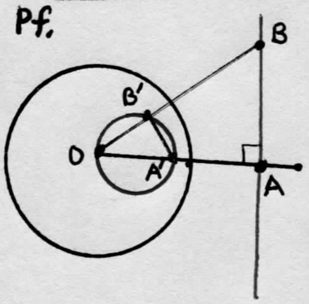
Consider a line $\mathcal{L}$ that moves closer to inversion circle K with center O



Thm Lines reflect to circles meeting O, and vice versa.

Ingredients:
- Key Lemma
- Conv Thales' Diam Thm
- Thales Diam Thm

Pf.



$\boxed{\text{Case 1}}$ Given $\mathcal{L}$, it has pt A, the closest pt to O (because $\exists$ uniq $\perp$ segment from $\mathcal{L}$ to O)

$\sigma_K(A) = A'$ inside K $\quad |OA| \cdot |OA'| = r^2$

Consider segment $\overline{OA'}$

Draw uniq circle C which has diam $\overline{OA'}$. Clearly C meets O.

We will show $\sigma(\mathcal{L}) \subseteq C$ and $C \subseteq \sigma(\mathcal{L})$ – Thus $C = \sigma(\mathcal{L})$

NOTE: $\sigma(C) = \sigma^2(\mathcal{L}) = Id(\mathcal{L}) = \mathcal{L}$

▷ $\sigma(\mathcal{L}) \subseteq C$:

Choose another pt $B \in \mathcal{L}$, reflect B to B'

B' does not lie on $\overline{OA'A}$ because it is on a different ray

Apply $\boxed{\text{Key Lemma}}$ $\triangle_{OB'A'} \sim \triangle_{OAB}$ and $\measuredangle_{OB'A'} = \measuredangle_{OAB} = 90°$

Since $\overline{OA} \perp \mathcal{L}$

By $\boxed{\text{Converse Thales' Diam thm} \ \text{Sheet 36b}}$ B' must lie on circle C, since it is the uniq circle with diam $\overline{OA'}$.

▷ $\sigma(\mathcal{L}) \supseteq C$:

Choose $B' \in C$. B' reflects to B. Is B on $\mathcal{L}$?

By Thales Diam Thm $\measuredangle_{OB'A'} = 90°$

By Key Lemma $\measuredangle_{OAB} = 90 \Rightarrow B \in \mathcal{L}$ $\square$

$\boxed{\text{Case 2}}$ Same pf applies

$\boxed{\text{Case 3}}$ See next page ⟶

$\boxed{\text{Case 4}}$ The top half of $\mathcal{L}$ is mapped to top half by inside of circle mapped out, flipped by the pt on the circle K. Same for bottom half.

C is Line (is a circle of radius ∞)

con'td ⟶

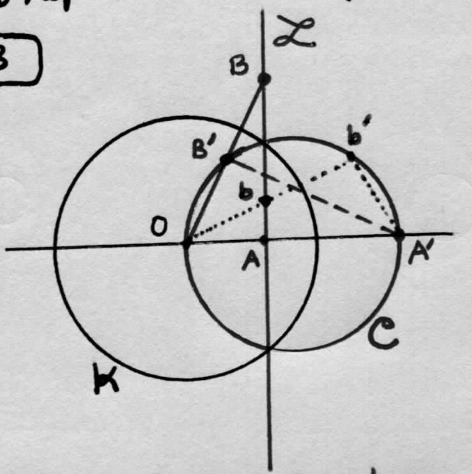
Lines reflect to circles, contd

Really just the same pf as Case 1

10/31/2024

④

Case 3



For a pt b inside K, it reflects to b'

Apply $\boxed{\text{Key Lemma}}$ $\Delta_{OAb} \sim \Delta_{Ob'A'}$

$\measuredangle_A = 90 \Rightarrow \measuredangle_{b'} = 90°$

$\overline{OA'}$ is still diam of image circle

Since $\measuredangle_{b'} = 90$, apply $\boxed{\text{Conv Thales Diam}}$

For a pt $B \in \mathcal{L}$ but outside K, SAME PF!

$\Delta_{OAB} \sim \Delta_{OB'A'}$ and $\underset{90}{\underset{\parallel}{\measuredangle_A}} = \measuredangle_{B'} \Rightarrow$ Thales

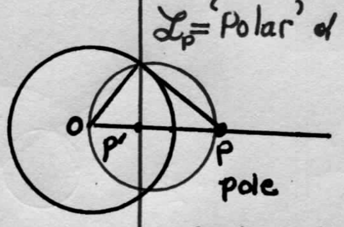
QED

Duality between a pt and a Line,

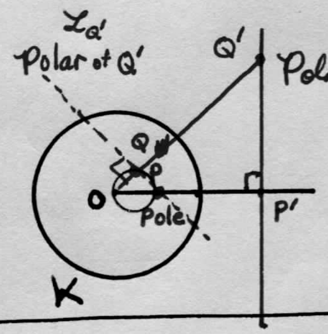
▷ Pole and Polar

Smart 'Mod Geos' p.202

The terms 'pole' and 'polar' have different meanings in
- Finite Geo of Desargues p.21
- Projective Geo p.265



A 'Pole' pt P lies on a circle touching O



Thm 6.9 Smart MG p.202: If $Q' \in \underset{\mathcal{L}_P}{(\text{Polar of } P)}$ wrt K $\Rightarrow$ $P \in \underset{\mathcal{L}_{Q'}}{(\text{Polar of } Q')}$

pf. From $\boxed{\text{prev thm}}$, Q lies on a circle C with diam $\overline{OP}$

From $\boxed{\text{Thales Diam}}$ $\measuredangle_{OQP} = 90$

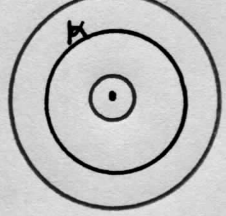
$\Rightarrow \overline{OQ'} \perp \overline{QP}$ $\quad \overleftrightarrow{QP}$ is the Polar of Q'

□

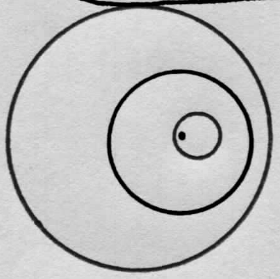
▷ Circles reflect to Circles

Let small circle move out from being concentric with K:

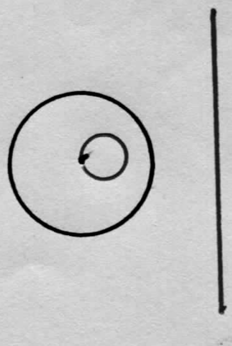
Case 1 concentric circles with K

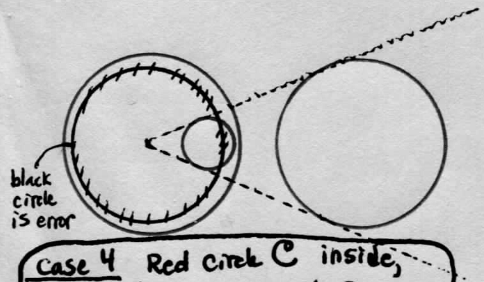


Case 2 O encircled, but off center

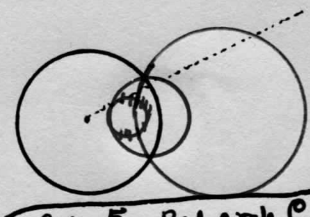


Case 3 red circle meets O. This was covered on prev sheet,

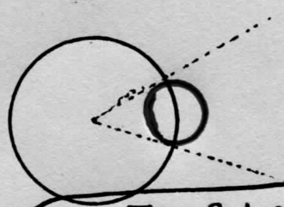




Case 4 Red circle C inside, does not encircle O



Case 5 Red circle C Crosses K



Case 5a Red circle C Cross K orthogonally

The following thm covers these cases
(Main arg based on Kozai & Libeskind 'Circle Inversions and Applications to Euclidean Geo'

Thm 1.3 Circles not meeting O reflect to Circles not meeting O

Given inversion circle K with center O, C a circle not touching O $\Rightarrow$

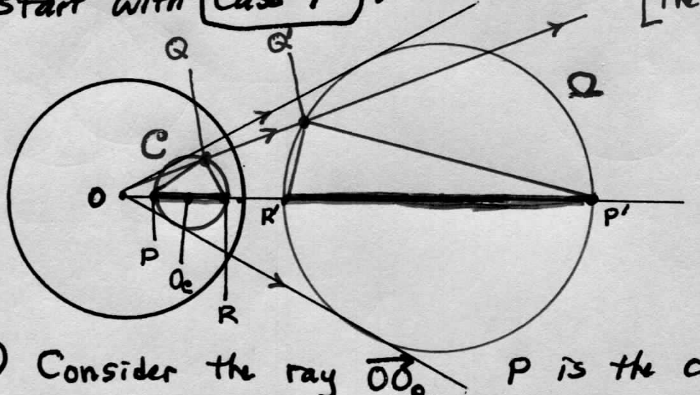
- $C' := \vec{\varnothing}_K(C)$ is again a circle, not meeting O
- If C does not encircle O, pts on the 'front' of C are flipped to pts on the 'back' of C' and vice-versa. 2 pts 'fixed' relatively speaking
- The center pt O_C is NOT mapped to $O_{C'}$, except in cases 1 - concentric, 5a - O.G. ∩

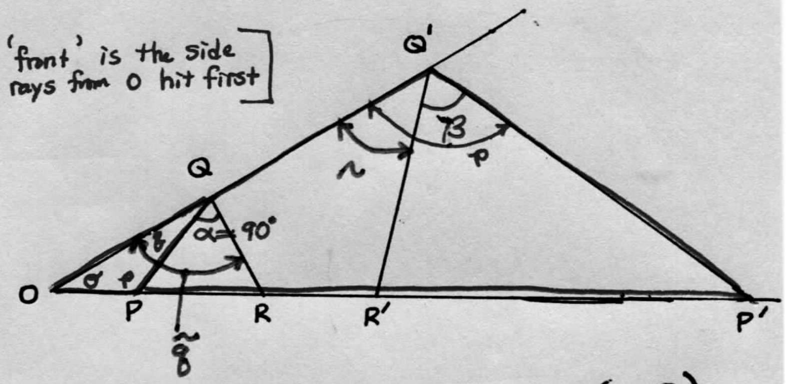
The circle is a special shape that gets flipped to another circle under inversion.

pf. Case 1 is obvious, a priori.
Case 3 was already covered.

Let's start with Case 4:

[The 'front' is the side the rays from O hit first]

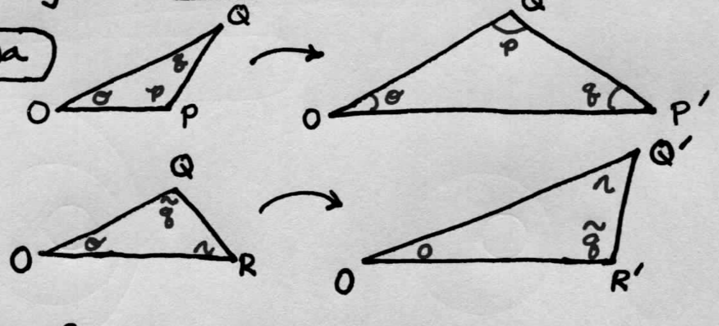




(a) Consider the ray $\overrightarrow{OO_C}$. P is the closest pt to O and R the furthest (on C). $\overline{PR}$ is diam of C, and it maps to $\overline{R'P'}$. Let this segment be the diam of a circle we call Ω. We must show $C' = \Omega$.

(b) Show $\vec{\varnothing}_K(C) \subseteq \Omega$
choose another pt $Q \in C$ and show Q' makes a vertex angle of $\beta = 90°$ with the segment $\overline{R'P'}$. Then by the Converse of Thales' Diam Thm, $Q' \in \Omega$

From the Key Lemma



write in the equal angles from Lemma
use lower case letters for angles.

We want to solve for β $\quad \beta = p - r \qquad \alpha = \tilde{q} - q$

We know

$$\theta + p + q = \pi = 180 \quad \text{from } O\triangle_{P'}^{Q'} \quad (1)$$

$$\theta + r + \tilde{q} = \pi \quad \text{from } O\triangle_{P'}^{Q'} \quad (2nd)$$

Subtract: $\quad p - r = q - \tilde{q} = 90°$ since $\overline{PR}$ is diam (Thales)

$\Rightarrow \beta = 90$

$\Rightarrow Q' \in \Omega$ by Conv Thales' Diam

con'd →

(c) show $\phi_k^{\uparrow}(C) \supseteq \Omega$

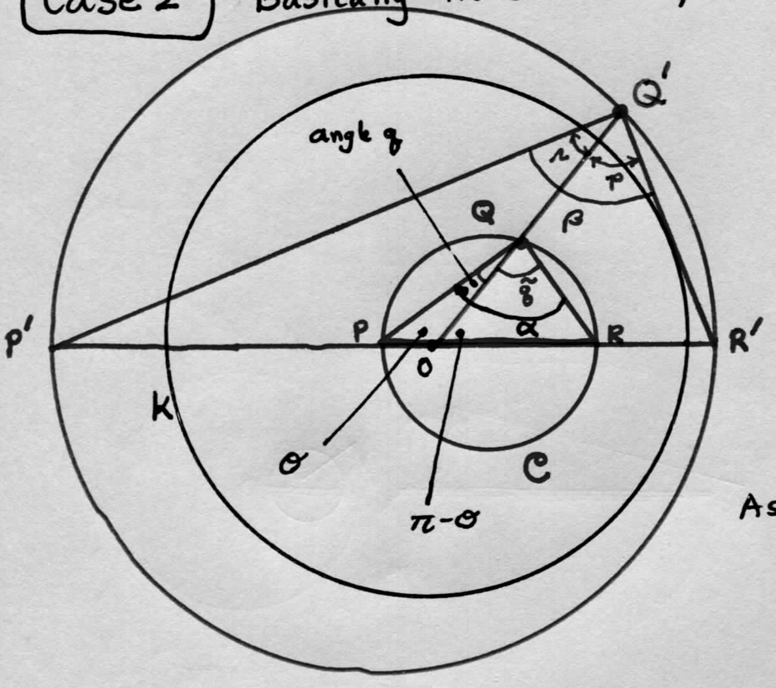
Basically the same arg but reversed. Choose $Q' \in \Omega$

then $\beta = 90° \Rightarrow p - r = 90° \Rightarrow q - \tilde{q} = 90°$

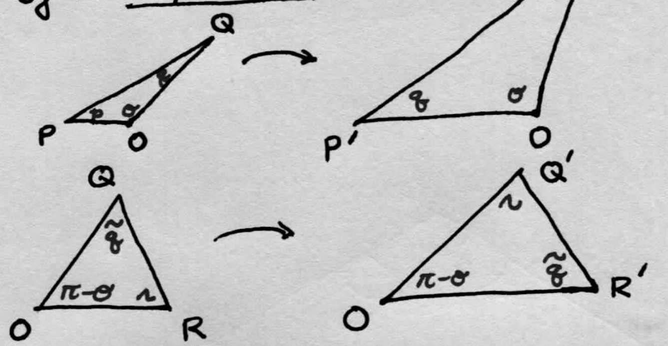
But $\alpha = q - \tilde{q}$ and this is what we seek: $\alpha = 90°$ so Thales says $Q \in C$ $\square$

(d) so $\phi_k^{\uparrow}(C) = \Omega$ $\square$

$\boxed{\text{Case 2}}$ Basically the same idea, with a few modifications.



By the Key Lemma



Here $\alpha = q + \tilde{q} = \frac{\pi}{2}$ and $\beta = p + r$

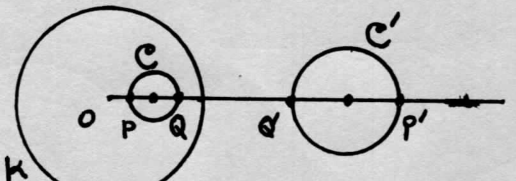
As before, we sum the angles in the 2 image triangs:

$$
\begin{array}{rl}
\theta + p + q & = \pi \\
+ \quad \pi - \theta + r + \tilde{q} & = \pi \\
\hline
\pi + \underbrace{(p+r)}_{\beta} + \underbrace{(q+\tilde{q})}_{\alpha = \frac{\pi}{2}} & = 2\pi
\end{array}
\Rightarrow \beta = \frac{\pi}{2}
$$

$\square$

$\boxed{\text{Case 5}}$ Same idea again, similiar to $\boxed{\text{case 4}}$. These results also follow from the Dilation of Circles Thm, I'm going to skip writing it here. And $\boxed{\text{Case 5a}}$ will have its own pf, comming up.

▷ The last thing to show is that $\phi_k^{\uparrow}(O_C) \neq O_{C'}$. Let's work in co-ords along pos x axis



$0 < P < Q$

$P' = \frac{r^2}{P}$ $\quad Q' = \frac{r^2}{Q}$

C has center $O_C = \frac{P+Q}{2}$

C' has center $O_{C'} = \frac{Q'+P'}{2} = \frac{1}{2}\left(\frac{r^2}{Q} + \frac{r^2}{P}\right)$

Show $\phi_k^{\uparrow}(O_C) < O_{C'}$ except in special cases.

$$\phi_k^{\uparrow}\left(\frac{P+Q}{2}\right) \overset{?}{<} \frac{Q'+P'}{2} = \frac{1}{2}\left(\frac{r^2}{Q} + \frac{r^2}{P}\right)$$

$$= \frac{r^2}{\frac{P+Q}{2}}$$

$$= 2\cancel{r^2}\left(\frac{1}{P+Q}\right) < \frac{\cancel{r^2}}{2}\left(\frac{1}{P} + \frac{1}{Q}\right)$$

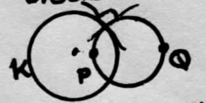
$$\frac{2}{P+Q} < \frac{1}{2}\frac{(P+Q)}{PQ}$$

$$4PQ < (P+Q)^2 = P^2 + 2PQ + Q^2$$

$$0 < P^2 - 2PQ + Q^2 = (P-Q)^2$$

since this ineq is true, the whole chain is true (unless $P = Q \Rightarrow$ circle is a pt)

There is one other special case where C crosses K Orthogonally (we will discuss this comming up)



$C = C'$

Since C is fixed, obviously the center must stay fixed

$\Rightarrow O_C \in K$

$P' = Q$ so $\frac{P'+Q'}{2} = \frac{P+Q}{2}$

$Q' = P$

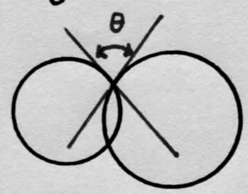
$\square$

QED

11/2/2024

7

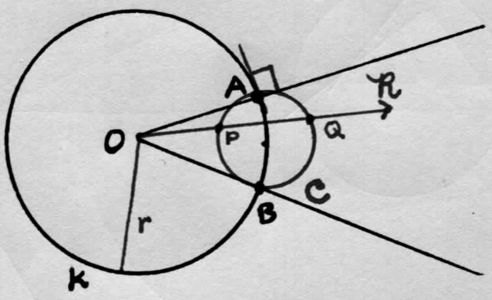
Def If 2 circles meet at a pt P, the <u>angle</u> between them is the angle between their unique tan lines at P (the one $\le 90°$)



Hartshorne G:EAB p.336

Thm Circle C meets inversion Circle K O.G. $C \cap K$	$\Leftrightarrow$	$C' = C$ i.e. C is invariant as a <u>set</u> (but pts are flipped from the front of C to the back)	$\Leftrightarrow$	C contains at least one pair of pts that are inverted to each other: $P, P' \in C$

Pf.



($\Rightarrow$) $C \cap K$ O.G.

Let R be a ray from O thru C (secant line) which meets C at 2 pts P and Q

By $\boxed{\text{Euclid III,36}}$ (my sheet 36a) <u>Power of pt O wrt C</u>:

$|OA|^2 = |OP|\,|OQ|$ and $|OA| = r$ since radius of K

$\Rightarrow |OP|\,|OQ| = r^2$ so P, Q inverses wrt K

Thus every pt $P \in C$ has its reflection $P' \in C$

$\Rightarrow \varnothing_K(C) = C$ (but pts are flipped front to back).

of course the radial lines $\overrightarrow{OA}$ are tangent to C and $\perp$ to tangent line to K at A

($\Leftarrow$) on R, P reflects to P', one pt must be outside K and one inside (say P).

Since C is a continuous (continuum) set, C must meet K at 2 pts, A & B

Radial segment OA $|OA| = r$

Since P, P' inverses $OP \cdot OP' = r^2 \; (= |OA|^2)$

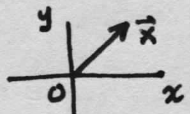
By $\boxed{\text{Euclid III,37}}$ (sheet 36a) $\overline{OA}$ is actually tangent to C at A

Thus C meets K $\perp$ at A (and likewise at B) $\square$

▷ Def: <u>Dilation</u>

Easiest to understand in terms of position vectors in $\mathbb{R}^2$

Let $\rho \ge 1$ Then the linear map $\vec{x} \mapsto \rho\vec{x}$ stretches everything a proportional amount based on the dist of pt x from origin – a Dilation

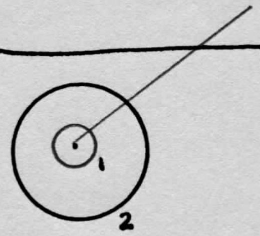


$0 < \rho < 1$ $\;x \mapsto \rho x$ shrinks everything (contraction) but still call it a Dilation

ρ negative – This flips pts radially thru O and then does a dilation of magnitude $|\rho|$

we still accept this as a <u>Dilation</u> here.

Dilation Lemma The composition of 2 circle inversions $\vec{\varnothing}_1, \vec{\varnothing}_2$ wrt same center O is a dilation of the plane



pf. Since the inversions have the same center O, it is enough to consider any radial line (call it the x axis)

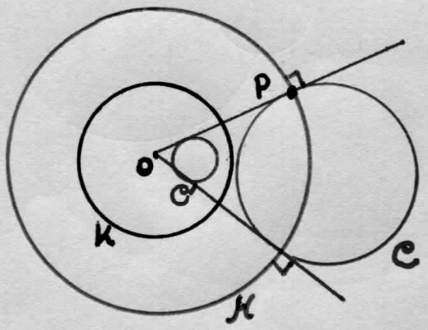
$\vec{\varnothing}_1 : x \longmapsto \frac{r^2}{x}$ and $\vec{\varnothing}_2 : x \longmapsto \frac{R^2}{x}$

Then $\vec{\varnothing}_2 \circ \vec{\varnothing}_1 (x) = \frac{R^2}{r^2/x} = \left(\frac{R^2}{r^2}\right)x$ and the scale factor $\lambda = \frac{R^2}{r^2}$ is the same for every radial line. □

Hartshorne p.337

Thm $\vec{\varnothing}_k\big|_C$ is a dilation wrt O of circle C (not other pts!)
(This provides another pf that $\vec{\varnothing}_k(C) = C'$ another circle)

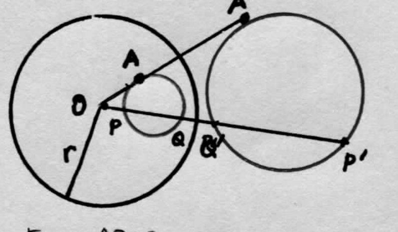
Pf. Case where C does not encircle O.



We can draw tan line OP where $P \in C$
Draw big circle $\mathcal{H}$ with center O and radius $|OP|$
$\mathcal{H}$ crosses C O.G. $\Rightarrow \vec{\varnothing}_{\mathcal{H}}(C) = C$
observe $C' = \vec{\varnothing}_k(C) = \underbrace{\vec{\varnothing}_k \circ \vec{\varnothing}_{\mathcal{H}}}_{\text{dialation by Lemma}}(C)$

Thus C' is a dilation of C, and so a circle. □

For comparison, here is Smart MG p.195-196 pf



Cf. Eves ASOG p.125

$OP \cdot OP' = r^2 = OQ \cdot OQ'$ by inversion ★

Power of pt O wrt circle C $\quad OA^2 = OP \cdot OQ = k$ const ★★

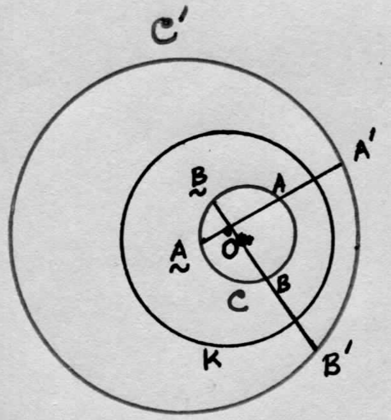
$\frac{★}{★★} \Rightarrow \frac{\cancel{OP} \cdot OP'}{\cancel{OP}\, OQ} = \frac{r^2}{\underbrace{OP \cdot OQ}_{k}} = \frac{\cancel{OQ} \cdot OQ'}{OP \cdot \cancel{OQ}}$

$\Rightarrow OP' = \overset{\lambda}{\left(\frac{r^2}{k}\right)} \underset{\text{"Homothety"}}{OQ}$ and $OQ' = \frac{r^2}{k} OP$

This is in fact a Dilation: pt Q on back of C corresponds to pt P' on back of C'
By the same scale factor λ, pt P on the front corresp to Q' on front □

Hartshorne prob 37.4

Case where C encircles O



Secant line
Power of pt O wrt circle C : $OA \cdot O\underset{\sim}{A} = \lambda = OB \cdot O\underset{\sim}{B}$

using signed lengths, since A and $\underset{\sim}{A}$ are in opposite directions, $\lambda < 0$

By inversion rule $OA \cdot OA' = r^2$

Then $OA = \frac{\lambda}{O\underset{\sim}{A}}$ and $OA = \frac{r^2}{OA'} \Rightarrow \frac{\lambda}{O\underset{\sim}{A}} = \frac{r^2}{OA'}$

$\Rightarrow OA' = \overset{\text{negative}}{\left(\frac{r^2}{\lambda}\right)} O\underset{\sim}{A}$

So OA' is not proportional to OA, but rather $O\underset{\sim}{A}$ on other side

This is the type of dilation $\vec{x} \longmapsto (-k)\vec{x}$

□