10/7/2023

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Hydon p.32

ex 2.13

$$\frac{dy}{dx} = \frac{1-y^2}{xy} + 1 \qquad (2.60)$$

Let's try $\xi = \alpha(x)$, $\eta = \beta(x)\, y + \gamma(x)$ for some α, β, γ

Does (2.60) have any such Syms?

$$\eta_x + (\eta_y - \xi_x)\,\omega - \xi_y\,\omega^2 = \xi\omega_x + \eta\,\omega_y \qquad (2.57)$$

Linearized cond

$$\Rightarrow \beta' y + \gamma' + (\beta - \alpha')\left(\frac{1-y^2}{xy} + 1\right) - 0\cdot\omega^2 \stackrel{!}{=} \alpha\Big[$$

$$\omega = \frac{1-y^2}{xy} + 1$$

$$\omega = \left(\frac{1-y^2}{y}\right)x^{-1} + 1 \Rightarrow \omega_x = -\left(\frac{1-y^2}{y}\right)x^{-2}$$

$$\frac{lh' - hl'}{l^2}$$

$$\frac{y(-2y) - (1-y^2)1}{y^2}$$

$$\frac{-2y^2 - 1 + y^2}{y^2}$$

$$\frac{-(1+y^2)}{y^2}$$

$$\omega_y = -\frac{(1+y^2)}{y^2 x}$$

$$\Rightarrow \beta' y + \gamma' + (\beta - \alpha')\left(\frac{1-y^2}{xy} + 1\right) \stackrel{!}{=} \alpha\left[\frac{-(1-y^2)}{yx^2}\right] + (\beta y + \gamma)\left[\frac{-(1+y^2)}{y^2 x}\right]$$

$$(\beta - \alpha')\left(\frac{1}{xy} - \frac{y}{x} + 1\right) = -\alpha\left[\frac{1}{x^2 y} + \frac{y}{x^2}\right] - (\beta y + \gamma)\left[\frac{1}{xy^2} + \frac{1}{x}\right]$$

$$= -\frac{\alpha}{x^2}\frac{1}{y} - \frac{\alpha}{x^2}y - \frac{\beta}{x}\frac{1}{y} - \frac{\beta y}{x} - \frac{\gamma}{x}\frac{1}{y^2} - \frac{\gamma}{x}$$

Term by term:
Powers of y

$\frac{1}{y^2}$: $0 \stackrel{!}{=} \frac{-\gamma}{x} \Rightarrow \gamma = 0$

$\frac{1}{y^1}$: $\frac{\beta - \alpha'}{x} \stackrel{!}{=} \frac{-\alpha}{x^2} - \frac{\beta}{x}$

y^0: $\cancel{\gamma'}_0 + \beta - \alpha' \stackrel{!}{=} \cancel{\frac{-\gamma}{x}}_0 \Rightarrow \beta = \alpha'$

$$\frac{\alpha' - \alpha'}{x} = \frac{-\alpha}{x^2} - \frac{\alpha'}{x}$$

$$0 = \frac{\alpha}{x^2} + \frac{\alpha'}{x}$$

$$\Rightarrow \alpha' + \frac{\alpha}{x} = 0$$

$\boxed{\text{soln } \alpha = \frac{c_1}{x}} \Rightarrow \beta = \frac{-c_1}{x^2}$

Hydon p.34 $\bar{Q}(x,y) := Q(x,y,\omega(x,y))$ 10/8/2023

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$\boxed{\text{From p.20}}$ $\bar{Q}(x,y) := \eta(x,y) - \omega(x,y)\,\xi(x,y)$

a soln curve $y=f(x)$ is invariant under sym flow $\Leftrightarrow \bar{Q}(x,y)=0$ $\forall\, y=f(x)$

The Lie Syms are <u>trivial</u> $\Leftrightarrow \bar{Q}(x,y) \equiv 0$ i.e. $\eta \equiv \omega\xi$

if $\bar{Q}_y \neq 0 \Rightarrow$ it is possible to determine curves $y=f(x) \ni Q(x,f(x)) \equiv 0$

Recall <u>Imp Fcn thm</u> $\left.\begin{array}{l} F(x_0,y_0)=0 \\ D_2F \neq 0 \end{array}\right\} \Rightarrow \exists$ fcn $y=f(x)$ in a nbhd of $\langle x_0,y_0\rangle$

[redacted] $\ni F(x,f(x))=0\ \forall x \in I$ and

$$\frac{dF}{dx} = D_1F + D_2F\,y' \Rightarrow y' = \frac{-D_1F}{D_2F}$$

p.34 [redacted]

"Every curve $y=f(x)$ satisfying $\left[\bar{Q}=0 \text{ when } y=f(x),\ \bar{Q}_y \neq 0\right]$ (2.63)

is a soln of (2.1) $y' = \omega(x,y)$"

Differentiate (2.63) wrt x: $\bar{Q}_x + f'(x)\bar{Q}_y = 0$ (2.64)

(2.58) $\bar{Q}_x + \omega\bar{Q}_y = \omega_y\bar{Q}$

I didn't really follow this!

This gives $f'(x) = \omega(x,f(x))$

Ch 2.5 Sym and Std Methods

Cannonical co-ord are assoc with particular Lie Sym group S_ε

ex 2.15 Homogeneous ODE

$$\frac{dy}{dx} = F(y/x) \quad (2.65)$$

This ~~has~~ is invariant under $S_\varepsilon(x,y) = \begin{bmatrix} e^\varepsilon & \\ & e^\varepsilon \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^\varepsilon x \\ e^\varepsilon y \end{bmatrix}$

Standard soln is to introduce COV

$r = y/x$

$s = \ln|x|$ these are cannonical co-ords (for $x \neq 0$)

<u>Case 1</u> $F(r) = r$ the syms are trivial $r = c$

$y = cx$

<u>Case 2</u> $\dfrac{ds}{dr} = \dfrac{1}{F(r)-r}$

General soln $\ln|x| = \displaystyle\int^{y/x} \frac{dr}{F(r)-r} + c$

Hydon p.34 10/10/2023

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ex 2.15 Homogeneous ODE

$\frac{dy}{dx} = F(y/x)$ this ODE invariant under Sym $S_\varepsilon(x,y) = \begin{bmatrix} e^\varepsilon & \\ & e^\varepsilon \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^\varepsilon x \\ e^\varepsilon y \end{bmatrix}$

$X = e^\varepsilon x$

$Y = e^\varepsilon y$

then $dX = e^\varepsilon dx \Rightarrow e^{-\varepsilon} dX = dx$

$dY = e^\varepsilon dy \quad\; e^{-\varepsilon} dY = dy$

$\frac{dy}{dx} = F(y/x) = F\left(\frac{e^\varepsilon y}{e^\varepsilon x}\right) = F(Y/X)$

$\overset{\shortparallel}{\frac{\cancel{e^{-\varepsilon}} dY}{\cancel{e^{-\varepsilon}} dX}} \Rightarrow \frac{dY}{dX} = F(Y/X)$

But if we wanted to do it the "proper" way:

$$\frac{Y_x + Y_y\, w(x,y)}{X_x + X_y\, w(x,y)} \overset{!}{=} w(X,Y)$$

$$\frac{0 + \cancel{e^\varepsilon} F(y/x)}{\cancel{e^\varepsilon} + 0} \overset{!}{=} F(Y/X) \Rightarrow F(y/x) = F(Y/X)$$

▷ ok, now lets try to form the r, s co-ord system from S_ε

$$\sigma_x(\varepsilon) = \begin{bmatrix} e^\varepsilon x \\ e^\varepsilon y \end{bmatrix} \Rightarrow \sigma_x'(0) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

From p.24 or sheet 6 eq (a): $\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{y}{x}$

$\Rightarrow \frac{1}{y} dy = \frac{1}{x} dx \rightarrow y = cx$

eq (b): $ds = \frac{1}{x} dx \Rightarrow s = \ln|x|$

if $r(x,y) = x^{-1} y$

then $r(x, y(x)) = \frac{1}{\cancel{x}} c\cancel{x} = c$ const

Thus $r = x^{-1} y = y/x$

$s = \ln|x|$

$s_x = 1/x$

$s_y = 0$

$r_x = -x^{-2} y = \frac{-y}{x^2}$

$r_y = 1/x$

So we must form
$$\frac{ds}{dr} = \frac{s_x + w(x,y)\overset{0}{s_y}}{r_x + w(x,y)\, r_y} = \frac{1/x + 0}{-\frac{y}{x^2} + F(r)\frac{1}{x}}\,\frac{x}{x} = \frac{1}{-\frac{y}{x} + F(r)} = \frac{1}{F(r) - r}$$

provided $F(r) \neq r$ so we consider that case separately.

case 1 $F(r) = r$

case 2 $\frac{ds}{dr} = \frac{1}{F(r) - r}$

Hydon p.35 $y' = G(x) - F(x)y$ 10/10/2023

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ex 2.16 General linear ODE $y' + F(x)y = G(x)$ (2.66)

in other words corresponding "homog" ODE $u' + F(x)u = 0$ is separable.

one nonzero soln $u = u_0(x) := e^{-\int F(x)dx}$

Princ of Linear Superpos: If $y = y(x)$ is a soln, so is $y(x) + \varepsilon u_0(x)$

This is equivalent to $S_\varepsilon(x,y) = \begin{bmatrix} x \\ y+\varepsilon u_0 \end{bmatrix} = \sigma_x(\varepsilon)$

$\sigma_x'(0) = \begin{bmatrix} 0 \\ u_0(x) \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$

eq ⓐ: $\frac{1}{\eta}dy = \frac{1}{\xi}dx$ but $\xi = 0$!

Going to have to revise p. 24-25

so we find $\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} x \\ y/u_0 \end{bmatrix}$

$r_x = 1 \quad r_y = 0$

$s_x = y(-u_0^{-2}u_0') \quad s_y = \frac{1}{u_0}$

$$\frac{ds}{dr} = \frac{s_x + W(x,y)s_y}{r_x + W(x,y)r_y}$$

$$= \frac{\frac{-y}{u^2}u' + (G(x) - F(x)y)\frac{1}{u}}{1 + (\quad)0}$$

$$\frac{ds}{dr} = \frac{-y}{u^2}u' + (G(x) - F(x)y)\frac{1}{u}$$

$$= \frac{-y}{\cancel{u^2}}(-\cancel{u}F) + \frac{G}{u} - \frac{Fy}{u}$$

$$= \cancel{\frac{yF}{u}} + \frac{G}{u} - \cancel{\frac{Fy}{u}}$$

$$= \frac{G}{u}$$

wants $\frac{ds}{dr} = \frac{G(r)}{u_0(r)}$

$u = e^{-\int F} = e^H$

$u' = e^H H' = e^{-\int F}F = uF$

$\begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

here we have $\xi = 0$, $\eta = u_0(x)$

$\begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix}\begin{bmatrix} 0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(a) $r_x \cdot 0 + r_y u_0 = 0$

$\Rightarrow r_x$ is arb and $r_y = 0$

lets choose simplest fcn of x

$r(x,y) = x$

r is fcn of x only

(b) $s_x \cdot 0 + s_y u_0 = 1$

$\Rightarrow s_x$ arb $\quad s_y = \frac{1}{u_0}$

so just take $\Rightarrow s = \frac{1}{u_0}y$

$$\frac{ds}{dr} = \frac{G(r)}{u(r)}$$

$$\int ds = \int \frac{G(r)}{u(r)}dr + c$$

$$s = \frac{y}{u_0} = \int^x \frac{G(r)}{u(r)}dr + c$$

$$y(x) = u_0(x)\int^x \frac{G(r)}{u_0(r)}dr + c \qquad u_0 = e^{-\int F dx}$$

$$y = e^{-\int F dx}\int G(r)e^{\int F dx}dr + c$$